INTRODUCTION TO ALGEBRAIC TOPOLOGY

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6. More homological algebra

In this section we will deal with algebraic constructions leading to the definitions of homology and cohomology groups with coefficients given in the previous sections. At the end we use introduced notions to state and prove so called universal coefficient theorems for singular homology and cohomology groups.

6.1. Functors and cofunctors. Let A and B be two categories. A *functor* $t : \mathsf{A} \to \mathsf{B}$ assigns to every object x in A an object $t(x)$ in B and to every morphism $f : x \to y$ in A a morphism $t(f) : t(x) \to t(y)$ such that $t(\mathrm{id}_x) = \mathrm{id}_{t(x)}$ and $t(fg) = t(f)t(g)$.

A *contravariant functor* or briefly *cofunctor* $t : \mathsf{A} \to \mathsf{B}$ assigns to every object x in A an object $t(x)$ in B and to every morphism $f : x \to y$ in A a morphism $t(f) : t(y) \to t(x)$ in B such that $t(\mathrm{id}_x) = \mathrm{id}_{t(x)}$ and $t(fg) = t(g)t(f)$.

Let R be a commutative ring with a unit element. The category of R-modules and their homomorphisms will be denoted by R-Mod. R-GMod will be used for the category of graded R-modules, R-Ch and R-CoCh will stand for the categories of chain complexes and the category of cochain complexes of R-modules, respectively. For $R = \mathbb{Z}$ the previous categories are Abelian groups Ab, graded Abelian groups GAb, chain complexes of Abelian groups Ch and cochain complexes of Abelian groups CoCh, respectively.

Homology H is a functor from the category R-Ch to the category R-GMod. Let t be a functor from R-Mod to R-Mod which induces a functor $t : \mathsf{Ch} \to \mathsf{Ch}$, and let s be a cofunctor from R-Mod to R-Mod, which induces a cofunctor from Ch to CoCh. The aim of this section is to say something about the functor $H \circ t$ and the cofunctor $H \circ s$. Model examples of such functors will be $t(-) = - \otimes_R M$ and $s(-) = \mathrm{Hom}_R(-, M)$ for a fixed R-module M. We have already used these functors when we have defined homology and cohomology groups with coefficients.

6.2. Tensor product. The *tensor product* $A \otimes_R B$ of two R-modules A and B is the quotient of the free R-module over $A \times B$ and the ideal generated by the elements of the form

$$r(a, b) - (ra, b),\ r(a, b) - (a, rb),\ (a_1 + a_2, b) - (a_1, b) - (a_2, b),\ (a, b_1 + b_2) - (a, b_1) - (a, b_2)$$

where $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$, $r \in R$. The class of equivalence of the element (a, b) in $A \otimes_R B$ is denoted by $a \otimes b$. The map $\varphi : A \times B \to A \otimes_R B$, $\varphi(a, b) = a \otimes b$ is bilinear and has the following universal property:

Whenever an R-module C and a bilinear map $\psi : A \times B \to C$ are given, there is just one R-modul homomorphism $\Psi : A \otimes_R B \to C$ such that the diagram

$$\begin{tikzcd} A \times B \arrow[r, "\psi"] \arrow[d, "\varphi"] & C \\ A \otimes_R B \arrow[ur, "\Psi"'] & \end{tikzcd}$$

commutes. This property determines the tensor product uniquely up to isomorphism.

If $f : A \to C$ and $g : B \to D$ are homomorphisms of R-modules then $(a, b) \to f(a) \otimes g(b)$ is a bilinear map and the universal property above ensures the existence and uniqueness of an R-homomorphism $f \otimes g : A \otimes_R B \to C \otimes_R D$ with the property $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.

Homomorphisms between R-modules form an R-module denoted by $\mathrm{Hom}_R(A, B)$. If $R = \mathbb{Z}$, we will denote the tensor product of Abelian groups A and B without the subindex $\mathbb{Z}$, i.e. $A \otimes B$, and similarly, the group of homomorfisms from A to B will be denoted by $\mathrm{Hom}(A, B)$.

Exercise. Prove from the definition that

$$\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}, \quad \mathbb{Z} \otimes \mathbb{Z}_n = \mathbb{Z}_n, \quad \mathbb{Z}_n \otimes \mathbb{Z}_m = \mathbb{Z}_{d(n,m)}, \quad \mathbb{Z}_n \otimes \mathbb{Q} = 0$$

where $d(m, n)$ is the greatest common divisor of n and m. Further compute

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}), \quad \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_n), \quad \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}), \quad \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_m).$$

6.3. Additive functors and cofunctors. A functor (or a cofunctor) $t : \mathsf{Mod} \to \mathsf{Mod}$ is called *additive* if

$$t(\alpha + \beta) = t(\alpha) + t(\beta)$$

for all $\alpha, \beta \in \mathrm{Hom}_R(A, B)$. Additive functors and cofunctors have the following properties.

(1) $t(0) = 0$ for any zero homomorphism.
(2) $t(A \oplus B) = t(A) \oplus t(B)$
(3) Every additive functor (cofunctor) converts short exact sequences which split into short exact sequences which again split.
(4) Every additive functor (cofunctor) can be extended to a functor $\mathsf{Ch} \to \mathsf{Ch}$ (cofunctor $\mathsf{Ch} \to \mathsf{CoCh}$) which preserves chain homotopies (converts chain homotopies to cochain homotopies).

Proof of (2) and (3). Consider a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

which splits, i. e. there are homomorphisms $p : B \to A$, $q : C \to B$ such that $pi = \mathrm{id}_A$, $jq = \mathrm{id}_C$, $ip + qj = \mathrm{id}_B$. See 3.1. Applying an additive functor t we get a splitting short exact sequence described by homomorphisms $t(i)$, $t(j)$, $t(p)$, $t(q)$. $\square$

6.4. Exact functors and cofunctors. An additive functor (or an additive cofunctor) $t : \mathsf{Mod} \to \mathsf{Mod}$ is called *exact* if it preserves short exact sequences.

Example. The functor $t(-) = - \otimes \mathbb{Z}_2$ and the cofunctor $s(-) = \mathrm{Hom}(-, \mathbb{Z})$ are additive but not exact. To show it apply them on the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2\times} \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

On the other hand, the functor $t(-) = - \otimes \mathbb{Q}$ from Ab to Ab is exact.

Lemma. *Let (C, ∂) be a chain complex and let $t : \mathsf{Mod} \to \mathsf{Mod}$ be an exact functor. Then*

$$H_n(tC, t\partial) = tH_n(C, \partial).$$

Consequently, t converts all exact sequences into exact sequences.

Proof. Since t preserves short exact sequences, it preserves kernels, images and factors. So we get

$$H_n(tC) = \frac{\mathrm{Ker}\, t\partial_n}{\mathrm{Im}\, t\partial_{n+1}} = \frac{t(\mathrm{Ker}\, \partial_n)}{t(\mathrm{Im}\, \partial_{n+1})} = t\left(\frac{\mathrm{Ker}\, \partial_n}{\mathrm{Im}\, \partial_{n+1}}\right) = t(H_n(C)).$$

□

6.5. Right exact functors. An additive functor $t : \mathsf{Mod} \to \mathsf{Mod}$ is called *right exact* if it converts any exact sequence

$$A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

into an exact sequence

$$tA \xrightarrow{t(i)} tB \xrightarrow{t(j)} tC \to 0.$$

Theorem. *Consider an R-module M. The functor $t(-) = - \otimes_R M$ from Mod to Ab is right exact.*

Proof. The exact sequence $A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is converted into the sequence

$$A \otimes_R M \xrightarrow{i \otimes \mathrm{id}_M} B \otimes_R M \xrightarrow{j \otimes \mathrm{id}_M} C \otimes_R M \to 0.$$

It is clear that $j \otimes \mathrm{id}_M$ is an epimorphism. According to the lemma below $\mathrm{Ker}(j \otimes \mathrm{id}_M)$ is generated by elements $b \otimes m$ where $b \in \mathrm{Ker}\, j = \mathrm{Im}\, i$. Hence, $\mathrm{Ker}(j \otimes \mathrm{id}_M) = \mathrm{Im}(i \otimes \mathrm{id}_M)$. □

Lemma. *If $\alpha : A \to A'$ and $\beta : B \to B'$ are epimorphisms, then $\mathrm{Ker}(\alpha \otimes \beta)$ is generated by elements $a \otimes b$ where $a \in \mathrm{Ker}\, \alpha$ or $b \in \mathrm{Ker}\, \beta$.*

For the proof see [Spanier], Chapter 5, Lemma 1.5.

6.6. Left exact cofunctors. An additive cofunctor $t : \mathsf{Mod} \to \mathsf{Mod}$ is called *left exact* if it converts any exact sequence

$$A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

into an exact sequence

$$O \to tC \xrightarrow{t(j)} tB \xrightarrow{t(i)} tA.$$

Theorem. *Consider an R-module M. The cofunctor $t(-) = \mathrm{Hom}_R(-, M)$ from* Mod *to* Mod *is left exact.*

The proof is not difficult and is left as an exercise.

6.7. Projective modules. An R-modul is called *projective* if for any epimorphism $p : A \to B$ and any homomorphism $f : P \to B$ there is $F : P \to A$ such that the diagram

$$\begin{tikzcd} & A \arrow[d, "p"] \\ P \arrow[ur, "F"] \arrow[r, "f"'] & B \arrow[d] \\ & 0 \end{tikzcd}$$

commutes. Every free R-module is projective.

6.8. Projective resolution. A *projective resolution* of an R-module A is a chain complex (P, ε), $P_i = 0$ for $i < 0$ and a homomorphism $\alpha : P_0 \to A$ such that the sequence

$$\to P_i \xrightarrow{\varepsilon_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{\varepsilon_1} P_0 \xrightarrow{\alpha} A \to 0$$

is exact. It means that

$$H_i(P, \varepsilon) = \begin{cases} 0 & \text{for } i \neq 0, \\ P_0/\operatorname{Im}\varepsilon_1 = P_0/\operatorname{Ker}\alpha \cong A & \text{for } i = 0. \end{cases}$$

If all P_i are free modules, the resolution is called *free*.

Lemma A. *To every module there is a free resolution.*

Proof. For module A denote $F(A)$ a free module over A and $\pi : F(A) \to A$ a canonical projection. Then the free resolution of A is constructed in the following way

$$\begin{tikzcd} P_2 = F\operatorname{Ker}\varepsilon_1 \arrow[r] \arrow[rr, bend left, "\varepsilon_2"] & \operatorname{Ker}\varepsilon_1 \arrow[r] & P_1 = F(\operatorname{Ker}\pi) \arrow[r] \arrow[rr, bend left, "\varepsilon_1"] & \operatorname{Ker}\pi \arrow[r] & P_0 = F(A) \arrow[r, "\pi"'] & A \end{tikzcd}$$

□

Lemma B. *Every Abelian group A has the projective resolution*

$$0 \to \operatorname{Ker}\pi \to F(A) \xrightarrow{\pi} A \to 0.$$

Proof. $\operatorname{Ker}\pi$ as a subgroup of free Abelian group $F(A)$ is free. □

Theorem. *Consider a homomorphism of R-modules $\varphi : A \to A'$. Let (P_n, ε_n) and (P'_n, ε'_n) be projective resolutions of A and A', respectively. Then there is a chain homomorphism $\varphi_n : (P, \varepsilon) \to (P', \varepsilon')$ such that the diagram*

$$\begin{tikzcd}
\dots \arrow[r, "\varepsilon_3"] & P_2 \arrow[r, "\varepsilon_2"] \arrow[d, "\varphi_2"] & P_1 \arrow[r, "\varepsilon_1"] \arrow[d, "\varphi_1"] & P_0 \arrow[r, "\alpha"] \arrow[d, "\varphi_0"] & A \arrow[r] \arrow[d, "\varphi"] & 0 \\
\dots \arrow[r, "\varepsilon'_3"] & P'_2 \arrow[r, "\varepsilon'_2"] & P'_1 \arrow[r, "\varepsilon'_1"] & P'_0 \arrow[r, "\alpha'"] & A' \arrow[r] & 0
\end{tikzcd}$$

commutes. Moreover, any two such chain homomorphism $(P, \varepsilon) \to (P', \varepsilon')$ are chain homotopic.

Proof of the first part. α' is an epimorphism and P_0 is projective. Hence there is $\varphi_0 : P_0 \to P'_0$ such that the first square on the right side commutes.

Since $\alpha'(\varphi_0 \varepsilon_1) = \varphi(\alpha\varepsilon_1) = \varphi \circ 0 = 0$, we get that

$$\operatorname{Im}(\varphi_0 \varepsilon) \subseteq \operatorname{Ker} \alpha' = \operatorname{Im} \varepsilon'_1.$$

$\varepsilon'_1 : P'_1 \to \operatorname{Im} \varepsilon'_1$ is an epimorhism and P_1 is projective. Hence there is $\varphi_1 : P_1 \to P'_1$ such that the second square in the diagram commutes.

The proof of the rest of the first part proceeds in the same way by induction. □

Proof of the second part. Let φ_* and φ'_* be two chain homomorphisms making the diagram above commutative. Since $\alpha'(\varphi_0 - \varphi'_0) = \alpha(\varphi - \varphi) = 0$, we have

$$\operatorname{Im}(\varphi_0 - \varphi'_0) \subseteq \operatorname{Ker} \alpha' = \operatorname{Im} \varepsilon'_1.$$

Therefore there exists $s_0 : P_0 \to P'_1$ such that $\varepsilon'_1 s_0 = \varphi_0 - \varphi'_0$.

Next, $\varepsilon'_1(\varphi_1 - \varphi'_1 - s_0\varepsilon_1) = \varepsilon'_1(\varphi_1 - \varphi'_1) - \varepsilon'_1 s_0 \varepsilon_1 = (\varphi_0 - \varphi'_0)\varepsilon_1 - (\varphi_0 - \varphi'_0)\varepsilon_1 = 0$, hence

$$\operatorname{Im}(\varphi_0 - \varphi'_0 - s_0\varepsilon_1) \subseteq \operatorname{Ker} \varepsilon'_1 = \operatorname{Im} \varepsilon'_2,$$

and consequently, there is $s_1 : P_1 \to P'_2$ such that

$$\varepsilon'_2 s_1 = \varphi_1 - \varphi'_1 - s_0 \varepsilon_1.$$

The rest proceeds by induction in the same way. □

6.9. Derived functors. Consider a right exact functor $t : \mathsf{Mod} \to \mathsf{Mod}$ and a homomorphism of R-modules $\varphi : A \to A'$. Let (P, ε) and (P', ε') be projective resolutions of A and A', respectively, and let $\varphi_* : (P, \varepsilon) \to (P', \varepsilon')$ be a chain homomorphism induced by φ. The *derived functors* $t_i : \mathsf{Mod} \to \mathsf{Mod}$ of the functor t are defined

$$t_i A = H_i(tP, t\varepsilon)$$
$$t_i \varphi = H_i(t\varphi).$$

The functor t_0 is equal to t since

$$t_0 A = tP_0 / \operatorname{Im} t\varepsilon_1 = tP_0 / \operatorname{Ker} t\alpha \cong tA.$$

Using the previous theorem we can easily show that the definition does not depend on the choice of projective resolutions and a chain homomorphism φ_*.

Definition. The i-th derived functors of the functor $t(-) = - \otimes_R M$ is denoted

$$\mathrm{Tor}_i^R(-, M).$$

If $R = \mathbb{Z}$, the index $\mathbb{Z}$ in the notation will be omitted.

Example. Let $R = \mathbb{Z}$. Any Abelian group A has a free resolution with $P_i = 0$ for $i \geq 2$. Hence

$$\mathrm{Tor}_i(A, B) = 0 \quad \text{for } i \geq 2.$$

Hence we will omit the index 1 in $\mathrm{Tor}_1(A, B)$. We have

(1) $\mathrm{Tor}(A, B) = 0$ for any free Abelian group A.
(2) $\mathrm{Tor}(A, B) = 0$ for any free Abelian group B.
(3) $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{d(m,n)}$ where $d(m, n)$ is the greatest common divisor of m and n.
(4) $\mathrm{Tor}(-, B)$ is an additive functor.

The proof based on the definition is not difficult and is left to the reader as an exercise.

6.10. Derived cofunctors. Consider a left exact cofunctor $t : \mathsf{Mod} \to \mathsf{Mod}$ and a homomorphism of R-modules $\varphi : A \to A'$. Let (P, ε) and (P', ε') be projective resolutions of A and A', respectively, and let $\varphi_* : (P, \varepsilon) \to (P', \varepsilon')$ be a chain homomorphism induced by φ. The *derived cofunctors* $t^i : \mathsf{Mod} \to \mathsf{Mod}$ of the functor t are defined

$$t^i A = H^i(tP, t\varepsilon)$$
$$t^i \varphi = H^i(t\varphi).$$

The functor t^0 is equal to t since

$$t_0 A = \mathrm{Ker}\, t\varepsilon_1 = \mathrm{Im}\, t\alpha = tA.$$

Using Theorem 6.8 we can easily show that the definition does not depend on the choice of projective resolutions and a chain homomorphism φ_*.

Definition. The i-th derived functors of the functor $t(-) = \mathrm{Hom}_R(-, M)$ is denoted

$$\mathrm{Ext}_R^i(-, M).$$

If $R = \mathbb{Z}$, the index $\mathbb{Z}$ in the notation will be omitted.

Example. Let $R = \mathbb{Z}$. Since every Abelian group A has a free resolution with $P_i = 0$ for $i \geq 2$,

$$\mathrm{Ext}^i(A, B) = 0 \quad \text{for } i \geq 2.$$

Hence we will write $\mathrm{Ext}(A, B)$ for $\mathrm{Ext}^1(A, B)$. We have

(1) $\mathrm{Ext}(A, B) = 0$ for any free Abelian group A.
(2) $\mathrm{Ext}(\mathbb{Z}_n, \mathbb{Z}) = \mathbb{Z}_n$.
(3) $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{d(m,n)}$ where $d(m, n)$ is the greatest common divisor of m and n.
(4) $\mathrm{Ext}(-, B)$ is an additive cofunctor.

The proof is the application of the definition and it is again left to the reader.

6.11. Universal coefficient theorems. In this paragraph we first express the cohomology groups $H^n(X;G)$ with the aid of functors Hom and Ext using the homology groups $H_*(X)$.

Theorem A. *If a free chain complex C of Abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $C^n = \operatorname{Hom}(C_n, G)$ are determined by the following split short exact sequence*

$$0 \to \operatorname{Ext}(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \to 0$$

where $h[f]([c]) = f(c)$ for all cycles $c \in C_n$ and all cocycles $f \in \operatorname{Hom}(C_n; G)$.

Remark. The exact sequence is natural but the splitting not. In this case the naturality means that for every chain homomorphism $f : C \to D$ we have commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}(H_{n-1}(C), G) \arrow[r] \arrow[d, "\operatorname{Ext}(H_{n-1}f, \mathrm{id}_G)"] & H^n(C;G) \arrow[r, "h_C"] \arrow[d, "H^n f"] & \operatorname{Hom}(H_n(C), G) \arrow[r] \arrow[d, "\operatorname{Hom}(H_n f, \mathrm{id}_G)"] & 0 \\
0 \arrow[r] & \operatorname{Ext}(H_{n-1}(D), G) \arrow[r] & H^n(D;G) \arrow[r, "h_D"] & \operatorname{Hom}(H_n(D), G) \arrow[r] & 0
\end{tikzcd}$$

Proof. The free chain complex (C_n, ∂) determines two other chain complexes, the chain complex of cycles $(Z_n, 0)$ and the chain complex of boundaries $(B_n, 0)$. We have the short exact sequence of these chain complexes

$$0 \to Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \to 0.$$

Since B_{n-1} is a subgroup of the free Abelian group C_{n-1}, it is also free and the exact sequence splits. Since the functor $\operatorname{Hom}(-, G)$ is additive, it converts this sequence into the short exact sequence of cochain complexes

$$0 \to \operatorname{Hom}(B_{n-1}, G) \xrightarrow{\delta} \operatorname{Hom}(C_n, G) \xrightarrow{i^*} \operatorname{Hom}(Z_n, G) \to 0.$$

As in 5.6 we obtain the long exact sequence of cohomology groups of the given cochain complexes

$$\to \operatorname{Hom}(Z_{n-1}, G) \to \operatorname{Hom}(B_{n-1}, G) \to H^n(C;G) \xrightarrow{i^*} \operatorname{Hom}(Z_n, G) \xrightarrow{\delta^*} \operatorname{Hom}(B_n, G) \to$$

Next, one has to realize how the connecting homomorphism δ^* in this exact sequence looks like using its definition and the special form of the short exact sequence. The conclusion is that $\delta^* = j^*$ where $j : B_n \hookrightarrow Z_n$ is an iclusion. Now we can reduce the long exact sequence to the short one

$$0 \to \frac{\operatorname{Hom}(B_{n-1}, G)}{\operatorname{Im} j^*} \longrightarrow H^n(C;G) \xrightarrow{i^*} \operatorname{Ker} j^* \to 0.$$

We determine $\operatorname{Ker} j^*$ and $\operatorname{Hom}(B_{n-1}, G)/\operatorname{Im} j^*$. Consider the short exact sequence

$$0 \to B_n \xrightarrow{j} Z_n \to H_n(C) \to 0.$$

It is a free resolution of $H_n(C)$. Applying the cofunctor $\operatorname{Hom}(-, G)$ we get the cochain complex

$$0 \to \operatorname{Hom}(Z_n, G) \xrightarrow{j^*} \operatorname{Hom}(B_n, G) \to 0 \to 0 \to \dots$$

from which we can easily compute that

$$\mathrm{Hom}(H_n(C),G)=\mathrm{Ker}\, j^*,\quad \mathrm{Ext}(H_n(C),G)=\frac{\mathrm{Hom}(B_{n-1},G)}{\mathrm{Im}\, j^*}.$$

This completes the proof of exactness.

We will find a splitting $r:\mathrm{Hom}(H_n(C);G)\to H^n(C^*;G)$. Let $g\in \mathrm{Hom}(H_n(C),G)$. We can define $f\in\mathrm{Hom}(C_n,G)$ such that on cycles $c\in Z_n$

$$f(c)=g([c])$$

where $[c]\in H_n(C)$. f is a cocycle, hence $[f]\in H^n(C;G)$ and $h[f]([c])=f(c)=g([c])$. □

In a very similar way one can compute homology groups with coefficients in G using the tensor product $H_*(C)\otimes G$ and $\mathrm{Tor}(H_*(C),G)$.

Theorem B. *If a free chain complex C has homology groups $H_n(C)$, then the homology groups $H_n(C_*;G)$ of the chain complex $C_n\otimes G$ are determined by the split short exact sequence*

$$0\to H_n(C)\otimes G\xrightarrow{l} H_n(C;G)\to \mathrm{Tor}(H_{n-1}(C),G)\to 0$$

where $l([c]\otimes g)=[c\otimes g]$ for $c\in Z_n(C)$, $g\in G$.

6.12. Exercise. Compute cohomology of real projective spaces with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients using the universal coefficient theorem for cohomology.

Exercise. Using again the universal coefficient theorem for cohomology and Theorem 4.4 prove that that for a given CW-complex X the cohomology of the cochain complex

$$(H^n(X^n,X^{n-1};G),d^n)$$

where d^n is the composition

$$H^n(X^n,X^{n-1};G)\xrightarrow{j_n^*} H^n(X^n;G)\xrightarrow{\delta^*} H^{n+1}(X^{n+1},X^n;G)$$

is isomorphic to $H^*(X;G)$. See also Theorem 5.14.

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