INTRODUCTION TO ALGEBRAIC TOPOLOGY

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12. Homotopy and CW-complexes

This section demonstrates the importance of CW-complexes in homotopy theory. The main results derived here are Whitehead theorem and theorems on approximation of maps by cellular maps and spaces by CW-complexes.

12.1. n**-connectivity.** A space X is *n-connected* if $\pi_i(X, x_0) = 0$ for all $0 \leq i \leq n$ and some base point $x_0 \in X$ (and consequently, for all base points).

A pair (X, A) is called *n-connected* if each component of path connectivity of X contains a point from A and $\pi_i(X, A, x_0) = 0$ for all $x_0 \in A$ and all $1 \leq i \leq n$

We say that a map $f : X \to Y$ is an *n-equivalence* if $f_* : \pi_i(X, x_0) \to \pi_i(Y, f(x_0))$ is an isomorphism for all $x_0 \in X$ if $0 \leq i < n$ and an epimorphism for all x_0 if $i = n$.

Exercise. Prove that a pair (X, A) is n-connected if and only if the inclusion $i : A \hookrightarrow X$ is an n-equivalence.

12.2. Compression lemma is an important technical tool in what follows.

Lemma A (Compression lemma). *Let (X, A) be a pair of CW-complexes and (Y, B) a pair with $B \neq \emptyset$. Suppose that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$ whenever there is a cell in $X - A$ of dimension n. Then every $f : (X, A) \to (Y, B)$ is homotopic rel A with a map $g : X \to B$.*

$$\begin{array}{ccc} A & \xrightarrow{f/A} & B \\ \downarrow & {}^{g}\nearrow \; \sim & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

If $n = 0$, the condition $\pi_0(Y, B, y_0) = 0$ means that (Y, B) is 0-connected.

Proof. By induction we will define maps $f_n : X \to Y$ such that $f_n(X^n \cup A) \subseteq B$, and f_n is homotopic to f_{n-1} rel $A \cup X^{n-1}$. Put $f_{-1} = f$. Suppose that we have f_{n-1} and there is a cell e^n in $X - A$. Let $\varphi : D^n \to X$ be its characteristic map. Then $f_{n-1}\varphi : (D^n, \partial D^n) \to (Y, B)$ represents zero element in $\pi_n(Y, B)$. According to Proposition 10.2 it means that $f_{n-1}\varphi : (D^n, \partial D^n) \to (Y, B)$ is homotopic rel ∂D^n to a map $h_n : (D^n, \partial D^n) \to (B, B)$. Doing it for all cells of dimension n in $X - A$ we obtain a map $g_n : X^n \cup A \to B$ homotopic rel $A \cup X^{n-1}$ with f_{n-1} restricted to $X^n \cup A$. Using the homotopy extension property of the pair $(X, X^n \cup A)$ we can conclude that g_n can be extended to a map $f_n : X \to Y$ which is homotopic rel $A \cup X^{n-1}$ to f_{n-1}.

Now for $x \in X^n$ define $g(x) = f_n(x) = g_n(x)$. By the same trick as in the proof of Theorem 2.7 we can construct a homotopy rel A between f and g. □

The proof of the following extension lemma is similar but easier and hence left to the reader.

Lemma B (Extension lemma). *Consider a pair (X, A) of CW-complexes and a map $f : A \to Y$. If Y is path connected and $\pi_{n-1}(Y, y_0) = 0$ whenever there is a cell in $X - A$ of dimension n, then f can be extended to a map $X \to Y$.*

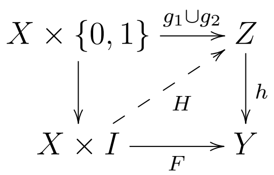
12.3. Whitehead Theorem. The compression lemma has two important consequences.

Corollary. *Let $h : Z \to Y$ be an n-equivalence and let X be a finite dimensional CW-complex. Then the induced map $h_* : [X, Z] \to [X, Y]$ is*

1. *a surjection if* $\dim X \leq n$,
2. *a bijection if* $\dim X \leq n - 1$.

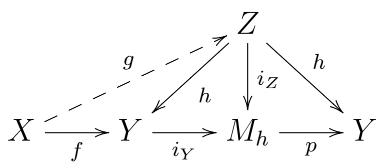
Proof. First, we will suppose that $h : Z \to Y$ is an inclusion and apply the compression lemma. Put $B = Z$, $A = \emptyset$ and consider a map $f : X \to Y$. If $\dim X \leq n$ then all the assumptions of the compression lemma are satisfied. Consequently, there is a map $g : X \to Z$ such that $hg \sim f$. Hence $h_* : [X, Z] \to [X, Y]$ is surjection.

Let $\dim X \leq n - 1$ and let $g_1, g_2 : X \to Z$ be two maps such that $hg_1 \sim hg_2$ via a homotopy $F : X \times I \to Y$. Then we can apply the compression lemma in the situation of the diagram



to get a homotopy $H : X \times I \to Z$ between g_1 and g_2.

If h is not an inclusion, we use the mapping cylinder M_h. (See 1.5 for the definition and basic properties.) Let $f : X \to Y$ be a map. Apply the result of the previous part of the proof to the inclusion $i_Z : Z \hookrightarrow M_h$ and to the map $i_Y f : X \to Y \hookrightarrow M_h$ to get $g : X \to Z$ such that $i_Z g \sim i_Y f$.



Since the right triangle in the diagram commutes and the middle one commutes up to homotopy and $pi_Y = \mathrm{id_Y}$, we get

$$hg = pi_Z g \sim pi_Y f = f.$$

The statement (2) can be proved in a similar way. □

A map $f : X \to Y$ is called a *weak homotopy equivalence* if $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for all n and all base points x_0.

Theorem (Whitehead Theorem). *If a map $h : Z \to Y$ between two CW-complexes is a weak homotopy equivalence, then h is a homotopy equivalence.*

Moreover, if Z is a subcomplex of Y and h is an inclusion, then Z is even deformation retract of Y.

Proof. Let h be an inclusion. We apply the compression lemma in the following situation:

$$\begin{array}{ccc} Z & \xrightarrow{\mathrm{id}_Z} & Z \\ {\scriptstyle h}\downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

Then $gh \sim \mathrm{id}_Y$ rel Z and consequently $hg = \mathrm{id}_Z$. So Z is a deformation retract of Y. The proof in a general case again uses mapping cylinder M_h. □

12.4. Simplicial approximation lemma. The following rather technical statement will play an important role in proofs of approximation theorems in this section and in the proof of homotopy excision theorem in the next section. Under convex polyhedron we mean an intersection of finite number of halfspaces in $\mathbb{R}^n$ with nonempty interior.

Lemma (Simplicial approximation lemma). *Consider a map $f : I^n \to Z$. Let Z be a space obtained from a space W by attaching a cell e^k. Then f is rel $f^{-1}(W)$ homotopic to f_1 for which there is a simplex $\Delta^k \subset e^k$ with $f_1^{-1}(\Delta^k)$ a union (possibly empty) of finitely many convex polyhedra such that f_1 is the restriction of a linear surjection $\mathbb{R}^n \to \mathbb{R}^k$ on each of them.*

The proof is elementary but rather technical and we omit it. See [Hatcher], Lemma 4.10, pages 350–351.

12.5. Cellular approximation. We recall that a map $g : X \to Y$ between two CW-complexes is called cellular, if $g(X^n) \subseteq Y^n$ for all n.

Theorem (Cellular approximation theorem). *If $f : X \to Y$ is a map between CW-complexes, then it is homotopic to a cellular map. If f is already cellular on a subcomplex A, then f is homotopic to a cellular map rel A.*

Corollary A. $\pi_k(S^n) = 0$ *for* $k < n$.

Corollary B. *Let (X, A) be a pair of CW-complexes such that $X - A$ contains only cells of dimension greater then n. Then (X, A) is n-connected.*

Proof of the cellular approximation theorem. By induction we will construct maps $f_n : X \to Y$ such that $f_{-1} = f$, f_n is cellular on X^n and $f_n \sim f_{n-1}$ rel $X^{n-1} \cup A$. Then we can define $g(x) = f_n(x)$ for $x \in X^n$ and by the same trick as in the proof of Theorem 2.7 we can construct homotopy rel A between f and g.

Suppose we have already f_{n-1} and there is a cell e^n such that $f_{n-1}(e^n)$ does not lie in Y^n. Then $f(e^n)$ meets a cell e^k in Y of dimension $k > n$. According to the simplicial approximation lemma f_{n-1} restricted to $\overline{e^n}$ is homotopic rel ∂e^n to $h : \overline{e^n} \to Y$ with the property that there is a simplex $\Delta^k \subset e^k$ and $h(e^n) \subset Y - \Delta^k$. (Since $n < k$, there is no linear surjection $\mathbb{R}^n \to \mathbb{R}^k$.) ∂e^k is a deformation retract of $\overline{e^k} - \Delta^k$ and that is why h is homotopic rel ∂e^n to a map $g : \overline{e^n} \to Y - e^k$. Since $f(e^n)$ meets only a finite number of cells, repeating the previous step we get a map f_n defined on $\overline{e^n}$ such that $f_n(e^n) \subseteq Y^n$ and homotopic rel ∂e^n to $f_{n-1}/\ \overline{e^n}$. In the same way we can define f_n on $A \cup X^n$ homotopic to $f_{n-1}/A \cup X^n$ rel $A \cup X^{n-1}$. Then using homotopy extension property for the pair $(X, A \cup X^n)$ we obtain $f_n : X \to Y$ homotopic to f_{n-1} rel $A \cup X^{n-1}$. □

12.6. Approximation by CW-complexes. Consider a pair (X, A) where A is a CW-complex. An *n-connected CW model* for (X, A) is an n-connected pair of CW-complexes (Z, A) together with a map $f : Z \to X$ such that $f/A = \mathrm{id}_A$ and $f_* : \pi_i(Z, z_0) \to \pi_i(X, f(z_0))$ is an isomorphism for $i > n$ and a monomorphism for $i = n$ and all base points $z_0 \in Z$.

If we take A a set containing one point from every path component of X, then 0-connected CW model gives a CW-complex Z and a map $Z \to X$ which is a weak homotopy equivalence.

Theorem A (CW approximation theorem). *For every $n \geq 0$ and for every pair (X, A) where A is a CW-complex there exists n-connected CW-model (Z, A) with the additional property that Z can be obtained from A by attaching cells of dimensions greater than n.*

Proof. We proceed by induction constructing $Z_n = A \subset Z_{n+1} \subset Z_{n+2} \subset \dots$ with Z_k obtained from Z_{k-1} by attaching cells of dimension k, and a map $f : Z_k \to X$ such that $f/A = \mathrm{id}_A$ and $f_* : \pi_i(Z_k) \to \pi_i(X)$ is a monomorhism for $n \leq i < k$ and an epimorphism for $n < i \leq k$. For simplicity we will consider X and A path connected with a fixed base point $x_0 \in A$.

Suppose we have already $f : Z_k \to X$. Let $\varphi_\alpha : S^k \to Z_k$ be maps representing generators in the kernel of $f_* : \pi_k(Z_k) \to \pi_k(X)$. Put

$$Y_{k+1} = Z_k \cup_{\varphi_\alpha} \bigcup_\alpha D_\alpha^{k+1}.$$

Since the map $f : Z_k \to X$ restricted to the boundaries of new cells is trivial, it can be extended to a map $f : Y_{k+1} \to X$.

By the cellular approximation theorem $\pi_i(Y_{k+1}) = \pi_i(Z_k)$ for all $i \leq k - 1$. Hence the new f_* has the same properties as the old f_* on homotopy groups π_i with $i \leq k - 1$. Since the composion $\pi_k(Z_k) \to \pi_k(Y_{k+1}) \to \pi_k(X)$ is surjective according to the induction assumptions, the homomorphism $f_* : \pi_k(Y_{k+1}) \to \pi_k(X)$ has to be surjective as well.

Now we prove that it is injective. Let $[\varphi] \in \pi_k(Y_{k+1})$ and let $f\varphi \sim 0$. By cellular approximation $\varphi : S^k \to Y_{k+1}$ is homotopic to $\widetilde{\varphi} : S^k \to Y_{k+1}^k = Z_k \subseteq Y_{k+1}$ and

$[f\widetilde{\varphi}] = 0$ in $\pi_k(X)$. Hence $[\widetilde{\varphi}] \in \operatorname{Ker} f_*$ is a sum of $[\varphi_\alpha]$, and consequenly, it is zero in $\pi_k(Y_{k+1})$.

Next, let maps $\psi_\alpha : S^{k+1}_\alpha \to X$ represent generators of $\pi_{k+1}(X)$. Put

$$Z_{k+1} = Y_{k+1} \vee \bigvee_\alpha S^{k+1}_\alpha$$

and define $f = \psi_\alpha$ on new $(k+1)$-cells. It is clear that $f_* : \pi_{k+1}(Z_{k+1}) \to \pi_{k+1}(X)$ is a surjection. Using cellular approximation it can be shown that $\pi_i(Z_{k+1}, Y_{k+1}) = 0$ for $i \leq k$. From the long exact sequence of the pair (Z_{k+1}, Y_{k+1}) we get that $\pi_i(Y_{k+1}) = \pi_i(Z_{k+1})$ for $i \leq k-1$. Consequently, $f_* : \pi_i(Z_{k+1}) \to \pi_i(X)$ is an isomorphism for $n < i \leq k-1$ and a monomorphism for $i = n$. The same long exact sequence implies that $\pi_k(Y_{k+1}) \to \pi_k(Z_{k+1})$ is surjective. We have already proved that $f_* : \pi_k(Y_{k+1}) \to \pi_k(X)$ is an isomorphism. From the diagram

$$\begin{array}{ccc} \pi_k(Y_{k+1}) & \xrightarrow{\text{epi}} & \pi_k(Z_{k+1}) \\ & \searrow^{\text{iso}} & \downarrow f_* \\ & & \pi_k(X) \end{array}$$

we can see that $f_* : \pi_k(Z_{k+1}) \to \pi_k(X)$ is also an isomorphism. □

Corollary. *If (X, A) is an n-connected pair of CW-complexes, then there is a pair (Z, A) homotopy equivalent to (X, A) rel A such that the cells in $Z - A$ have dimension greater than n.*

Proof. Let $f : (Z, A) \to (X, A)$ be an n-connected model for (X, A) obtained by attaching cells of dimension $> n$ to A. Then $f_* : \pi_j(Z) \to \pi_j(X)$ is a monomorphism for $j = n$ and an isomorphism for $j > n$. We will show that f_* is an isomorphism also for $j \leq n$. Consider the diagram:

$$\begin{array}{ccc} A & & \\ {\scriptstyle i_Z}\downarrow & \searrow^{i_X} & \\ Z & \xrightarrow[f]{} & X \end{array}$$

The inclusions i_X and i_Z are n-equivalences. Consequently, $f_* i_{Z*} = i_{X*} : \pi_j(A) \to \pi_j(X)$ is an epimorphism for $j = n$. Hence so is f_*. Next, i_{X*} and i_{Z*} are isomorphisms for $j < n$, hence so is f_*.

Finally, according to Whitehead Theorem, the weak homotopy equivalence f between two CW-complexes is a homotopy equivalence. □

Theorem B. *Let $f : (Z, A) \to (X, A)$ and $f' : (Z', A') \to (X', Z')$ be two n-connected CW-models. Given a map $g : (X, A) \to (X', A')$ there is a map $h : (Z, A) \to (Z', A')$ such that the following diagram commutes up to homotopy rel A:*

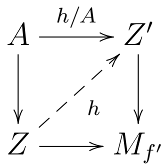
$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow[f']{} & X' \end{array}$$

The map h is unique up to homotopy rel A.

Proof. By the previous corollary we can suppose that $Z-A$ has only cells of dimension $\geq n+1$. We can define h/A as g/A.

$$\begin{CD} A @>{h/A}>> Z' \\ @VVV @VV{f'}V \\ Z @>>{gf}> X' \end{CD}$$

Replace X' by the mapping cylinder $M_{f'}$ which is homotopy equivalent to X'. Since $f' : Z' \to X'$ is an n-connected model, from the long exact sequence of the pair $(M_{f'}, Z')$ we get that $\pi_i(M_{f'}, Z') = 0$ for $i \geq n+1$. According to compression lemma 12.2 there exists $h : Z \to Z'$ such that the diagram



commutes up to homotopy rel A. This h has required properties. The proof that it is unique up to homotopy follows the same lines. $\square$

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