COMPUTING HOMOLOGY GROUPS

We are given the following (model of) topological space X via its Δ–complex We

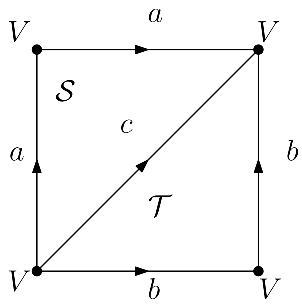

FIGURE 1. Model of the space X

decompose X into the sequence of cells:

$$X_0 = \{V\}, \quad X_1 = \{a, b, c\}, \quad X_2 = \{\mathcal{T}, \mathcal{S}\}, \quad X_n = \emptyset, n \geq 3.$$

The chain complex $C_*(X)$ is given by

$$C_i =< X_i >, i \geq 0$$

as every group C_i is an abelian group ($=\mathbb{Z}$ module) finitely genrated by the i–cells.

As such, we can see C_i as a kind of "vector space" and we can describe the chain homomorphisms ∂_i in terms of matrices. Let us describe ∂_2: We have $\partial_2(\mathcal{T}) = 2b - c$, $\partial_2(\mathcal{S}) = 2a - c$, hence in our ordering

$$\partial_2 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \end{pmatrix}$$

We can see that the kernel of ∂_2 is 0. Then it is rather easy to see that $0 = Z_2(X) = H_2(X)$.

In order to compute $H_1(X)$, we first see that $\ker \partial_1 = C_1 =< a, b, c >$. (because the matrix is the zero matrix).

The group H_1 is a factor group of C_1 where the relations that definie factoring are $2b - c = 0$ and $2a - c = 0$.

I am not aware of any very easy algorithmic way how to compute the factoring now (there is in fact a way using the Smith normal form, but I deem it unnecessary). We have to use our wit and some dirty tricks:

(1) we observe that c depends on both a and b and so we have

$$< a, b, c | 2a = -c, 2b = -c >=< a, b | 2a = 2b >$$

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(2) From our "vector space" idea we can change our basis from (a, b) to $(a-b, a)$. Let us denote $d = a - b$. Then
$$H_1 =< a, d|2d = 0 >\cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

This concludes the calculation. Good luck with the homework. And by the way what is really our space X geometrically? (HINT: First think just about one of the triangles).