Exercise session 4

Course: Mathematical methods in Economics
Lecturer: Dmytro Vikhrov
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Problem 1 *(wealth transfer over time)*

Suppose that a consumer lives for two periods with instantaneous utility in each period equal to $U(c_t)$. The consumer can transfer her wealth across periods by buying bonds b in $t = 1$ and selling them at $t = 2$. The interest rate paid on bonds is R and the discount factor is β. Consumer's flow of income in respective periods is y_1 and y_2.

1. Explain the meaning of discount factor β and bonds b.
2. Carefully setup the maximization problem.
3. Derive the Euler equation for consumption. Explain how the consumer decides to shift consumption over time.
4. For $U(c_t) = log(c_t)$ derive the optimal consumption sequence, $\{c_1^*, c_2^*\}$ and bond holdings b^*. Interpret your findings.
5. Define the competitive equilibrium and find the market clearing R^*. Compute $\frac{\partial R^*}{\partial \beta}$, $\frac{\partial R^*}{\partial y_1}$, $\frac{\partial R^*}{\partial y_2}$.

Problem 2 *(pay-as-you-go social security system)*

Consider the two-period model from Problem 1. Assume that workers get income y while young and no income while old. The government taxes the young generation at per-unit tax τ and distributes the proceeds amongst the old generation. Each population cohort grows in size, $L_{t+1} = (1 + g)L_t$.

1. Draw a graph to depict the intuition of the overlapping generations.
2. Setup the government budget constraint. In what cases can the government run a deficit of the PAY-GO pension system. What tools does the government have to reduce the deficit?
3. Setup the consumer maximization problem.
4. Derive the optimal consumption sequence $\{c_1^*, c_2^*\}$.
5. Derive $\frac{\partial c^*}{\partial g}$ and $\frac{\partial c^*}{\partial \tau}$ and interpret your findings.
6. In what cases does the consumer wish to opt out of the mandatory PAY-GO system?

Problem 3 *(funded social security system)*

Assume the two period setup as in the previous problem. At $t = 1$, The consume has a choice to invest in bonds b that yield an interest rate R at $t = 2$ or participate in the government funded scheme. In this scheme the consumer is taxed at the per-unit rate τ and the proceeds are invested at the rate r. Setup the consumer maximization problem and derive conditions for the consumer to participate in the government funded scheme.

Problem 4 *(infinite period setup)*

Recall the setup of problem 1, however now assume that the consumer is infinitely lived. Setup the maximization problem and derive the Euler equation for consumption.

Problem 5 *(dynamic programming)*

Setup the maximization problem of an infinitely lived agent in the recursive form.

1. Carefully define the state and control variables.
2. Write down conditions under which the policy function $y = \pi(x)$ maximizes the value function.
3. Find the Euler equation for consumption.
4. Using the Envelope theorem, establish that $V'(x) = \frac{\partial U(x,y^*)}{\partial x}$. State the transversality conditions.

Problem 6 *(DP example 1)*

Consumer's maximization problem is given by:

$$\max_{\{k_t,\,c_t\}} \sum_{t=0}^{\infty} \beta^t log(c_t)$$
$$\text{s.t.: } k_{t+1} = k_t^\alpha - c_t,\ k(0) > 0.$$

1. Setup the Bellman equation and take the first - order conditions.
2. Somebody tells you that the policy function is of the form $\pi(x) = Ax^\alpha$. Find the parameter A.

Problem 7 *(DP example 2)*

Consider the following DP problem:

$$\max_{\{k_t,\,c_t\}} \sum_{t=0}^{\infty} \beta^t log(c_t)$$
$$\text{s.t.: } c_t = (1+r)a_t - a_{t+1} + w,\ a(0) > 0.$$

1. Setup the Bellman equation and take the first - order conditions.
2. Guess and verify that the solution is a policy function $\pi(x) = A + Bln(x)$.