(10.5) generated by $[\hat{s}(t), \hat{h}(t)]_{t=0}^{T}$ be denoted by $\hat{Y}$. Since $[\hat{s}(t), \hat{h}(t)]_{t=0}^{T}$ does not maximize (10.5), there exists $[\tilde{s}(t), \tilde{h}(t)]_{t=0}^{T}$ reaching a value of (10.5) $\tilde{Y} > \hat{Y}$. By the hypothesis that $[\hat{c}(t), \hat{s}(t), \hat{h}(t)]_{t=0}^{T}$ is a solution to (10.1), the budget constraint (10.4) implies

$$\int_0^T \exp(-rt)\hat{c}(t)dt \leq \hat{Y}.$$

Let $\varepsilon > 0$ and consider $[c(t), s(t), h(t)]_{t=0}^{T}$ such that $c(t) = \hat{c}(t) + \varepsilon$, $s(t) = \tilde{s}(t)$, and $h(t) = \tilde{h}(t)$ for all t. We have that

$$\int_0^T \exp(-rt)c(t)dt = \int_0^T \exp(-rt)\hat{c}(t)dt + \frac{[1 - \exp(-rT)]}{r}\varepsilon.$$
$$\leq \hat{Y} + \frac{[1 - \exp(-rT)]}{r}\varepsilon.$$

Since $\tilde{Y} > \hat{Y}$, for ε sufficiently small, $\int_0^T \exp(-rt)c(t)\,dt \leq \tilde{Y}$, and thus $[c(t), s(t), h(t)]_{t=0}^{T}$ is feasible. Since $u(\cdot)$ is strictly increasing, $[c(t), s(t), h(t)]_{t=0}^{T}$ is strictly preferred to $[\hat{c}(t), \hat{s}(t), \hat{h}(t)]_{t=0}^{T}$, leading to a contradiction and proving the "only if" part.

The proof of the "if" part is similar. Suppose that $[\hat{s}(t), \hat{h}(t)]_{t=0}^{T}$ maximizes (10.5). Let the maximum value be denoted by $\hat{Y}$. Consider the maximization of (10.1) subject to the constraint that $\int_0^T \exp(-rt)c(t)dt \leq \hat{Y}$. Let $[\hat{c}(t)]_{t=0}^{T}$ be a solution. Thus if $[\tilde{c}'(t)]_{t=0}^{T}$ is strictly preferred to $[\hat{c}(t)]_{t=0}^{T}$, then $\int_0^T \exp(-rt)\tilde{c}'(t)dt > \hat{Y}$. Then $[\hat{c}(t), \hat{s}(t), \hat{h}(t)]_{t=0}^{T}$ must be a solution to the original problem, because any other $[s(t), h(t)]_{t=0}^{T}$ leads to a value of (10.5) equal to $Y \leq \hat{Y}$, and if $[\tilde{c}'(t)]_{t=0}^{T}$ is strictly preferred to $[\hat{c}(t)]_{t=0}^{T}$, then $\int_0^T \exp(-rt)c'(t)dt > \hat{Y} \geq Y$ for any Y associated with any feasible $[s(t), h(t)]_{t=0}^{T}$. ■

The intuition for this theorem is straightforward: in the presence of perfect capital markets, the best human capital accumulation decisions are those that maximize the lifetime budget set of the individual. Exercise 10.2 shows that this theorem does not hold when there are imperfect capital markets and also does not generalize to the case where leisure is also an argument of the utility function.

10.2 Schooling Investments and Returns to Education

Let us next turn to the simplest model of schooling decisions in partial equilibrium, which illustrate the main trade-offs in human capital investments. The model presented here is a version of Mincer's (1974) seminal contribution. This model also enables a simple mapping from the theory of human capital investments to the large empirical literature on returns to schooling.

Let us first assume that $T = \infty$, which simplifies the expressions. The flow rate of death, ν, is positive, so that individuals have finite expected lives. Suppose that (10.2) is such that the individual has to spend an interval S with $s(t) = 1$—that is, in full-time schooling—and $s(t) = 0$ thereafter. At the end of the schooling interval, the individual has a schooling level of

$$h(S) = \eta(S),$$

where $\eta(\cdot)$ is a continuous, increasing, differentiable, and concave function. For $t \in [S, \infty)$, human capital accumulates over time (as the individual works) according to the differential equation

$$\dot{h}(t) = g_h h(t), \tag{10.6}$$

for some $g_h \geq 0$. Suppose also that wages grow exponentially,

$$\dot{w}(t) = g_w w(t), \tag{10.7}$$

with initial value $w(0) > 0$. Suppose that

$$g_w + g_h < r + \nu,$$

so that the net present discounted value of the individual is finite. Now using Theorem 10.1, the optimal schooling decision must be a solution to the following maximization problem:

$$\max_S \int_S^\infty \exp(-(r+\nu)t)w(t)h(t)dt. \tag{10.8}$$

Using (10.6) and (10.7), this equation is equivalent to (see Exercise 10.3):

$$\max_S \frac{\eta(S)w(0)\exp(-(r+\nu-g_w)S)}{r+\nu-g_h-g_w}. \tag{10.9}$$

Since $\eta(S)$ is concave, the objective function in (10.9) is strictly concave. Therefore the unique solution to this problem is characterized by the first-order condition

$$\frac{\eta'(S^*)}{\eta(S^*)} = r + \nu - g_w. \tag{10.10}$$

Equation (10.10) shows that higher interest rates and higher values of ν (corresponding to shorter planning horizons) reduce human capital investments, while higher values of g_w increase the value of human capital and thus encourage further investments.

Integrating both sides of this equation with respect to S (or taking the antiderivative), we obtain

$$\log \eta(S^*) = \text{constant} + (r + \nu - g_w)S^*. \tag{10.11}$$

Next, the wage earnings of the worker of age $\tau \geq S^*$ in the labor market at time t are

$$W(S^*, t) = \exp(g_w t)\exp(g_h(t - S^*))\eta(S^*).$$

Taking logs and using (10.11) implies that

$$\log W(S^*, t) = \text{constant} + (r + \nu - g_w)S^* + g_w t + g_h(t - S^*),$$

where $t - S^*$ can be thought of as worker experience (time after schooling). If we make a cross-sectional comparison across workers, the time trend term $g_w t$ is also included in the constant, so that we obtain the canonical Mincer equation, where, in the cross section, log wage earnings are proportional to schooling and experience. Written differently, we have the following cross-sectional equation:

$$\log W_j = \text{constant} + \gamma_s S_j + \gamma_e \text{ experience}, \tag{10.12}$$

where j refers to individual j. However, there is so far no source of heterogeneity that can generate different levels of schooling across individuals. Nevertheless (10.12) is important, since it is the typical empirical model for the relationship between wages and schooling estimated in labor economics.

The economic insight provided by this equation is quite important. It suggests that the functional form of the Mincerian wage equation is not a mere coincidence, but has economic content: the opportunity cost of one more year of schooling is foregone earnings. So the benefit has to be commensurate with these foregone earnings, and thus should lead to a proportional increase in earnings in the future. In particular, this proportional increase should be at the rate $(r + \nu - g_w)$.

As already discussed in Chapter 3, empirical work using equations of the form (10.12) leads to estimates for the returns to schooling coefficient, γ_s, in the range of 0.06–0.10. Equation (10.12) suggests that these returns to schooling are not unreasonable. For example, we can think of the annual interest rate r as approximately 0.10, ν as corresponding to 0.02 (which gives an expected life of 50 years), and g_w corresponding to the rate of wage growth holding the human capital level of the individual constant (which should be approximately 0.02). Thus we should expect an estimate of γ about 0.10, which is consistent with the upper range of the empirical estimates.

.3 The Ben-Porath Model

The baseline Ben-Porath model enriches the model studied in the previous section by allowing human capital investments and nontrivial labor supply decisions throughout the lifetime of the individual. In particular, we now let $s(t) \in [0, 1]$ for all $t \geq 0$. Together with the Mincer equation (10.12) (and the model underlying this equation presented in the previous section), the Ben-Porath model is the basis of much of labor economics. Here it is sufficient to consider a simple version of this model in which the human capital accumulation equation (10.2) takes the form

$$\dot{h}(t) = \phi(s(t)h(t)) - \delta_h h(t), \tag{10.13}$$

where $\delta_h > 0$ captures the depreciation of human capital, which comes about, for example, because new machines and techniques are introduced that erode the existing human capital of the worker. The individual starts with an initial value of human capital $h(0) > 0$. The function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, twice differentiable, and strictly concave. Furthermore, let us simplify the analysis by assuming that this function satisfies the Inada-type conditions

$$\lim_{x \to 0} \phi'(x) = \infty, \quad \text{and} \quad \lim_{x \to h(0)} \phi'(x) = 0.$$

The latter condition removes the need to impose additional constraints to ensure $s(t) \in (0, 1)$ (see Exercise 10.6).

Let us also suppose that there is no non–human capital component of labor, so that $\omega(t) = 0$ for all t, $T = \infty$, and there is a flow rate of death $\nu > 0$. Finally let us assume that the wage per unit of human capital is constant at w and the interest rate is constant and equal to r. We can also normalize $w = 1$ without loss of generality.

Again using Theorem 10.1, human capital investments can be determined as a solution to the following problem:

$$\max \int_0^\infty \exp(-(r + \nu)t)(1 - s(t))h(t)\, dt,$$

subject to (10.13).