Chapter 1: Sobolev Spaces

Introduction

In many problems of mathematical physics and variational calculus it is not sufficient to deal with the classical solutions of differential equations. It is necessary to introduce the notion of weak derivatives and to work in the so called Sobolev spaces.
Let us consider the simplest example — the Dirichlet problem for the Laplace equation in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\left.\begin{array}{ll} \triangle u = 0, & x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega, \end{array}\right\} \qquad (*)$$

where $\varphi(x)$ is a given function on the boundary $\partial\Omega$. It is known that the Laplace equation is the Euler equation for the functional

$$l(u) = \int_\Omega \sum_{j=1}^n \left| \frac{\partial u}{\partial x_j} \right|^2 dx.$$

We can consider $(*)$ as a variational problem: to find the minimum of $l(u)$ on the set of functions satisfying condition $u|_{\partial\Omega} = \varphi$. It is much easier to minimize this functional not in $C^1(\overline{\Omega})$, but in a larger class.
Namely, in the Sobolev class $W_2^1(\Omega)$.
$W_2^1(\Omega)$ consists of all functions $u \in L_2(\Omega)$, having the *weak derivatives* $\partial_j u \ \in L_2(\Omega), \ \ j = 1, \dots, n$. If the boundary $\partial\Omega$ is smooth, then the trace of $u(x)$ on $\partial\Omega$ is well defined and relation $u|_{\partial\Omega} = \varphi$ makes sense. (This follows from the so called „boundary trace theorem“ for Sobolev spaces.)
If we consider $l(u)$ on $W_2^1(\Omega)$, it is easy to prove the existence and uniqueness of solution of our variational problem.
The function $u \in W_2^1(\Omega)$, that gives minimum to $l(u)$ under the condition $u|_{\partial\Omega} = \varphi$, is called the *weak solution* of the Dirichlet problem $(*)$.

We'll study the Sobolev spaces, the extension theorems, the boundary trace theorems and the embedding theorems.
Next, we'll apply this theory to elliptic boundary value problems.

§1: Preliminaries

Let us recall some definitions and notation.

Definition

An *open connected set* $\Omega \subset \mathbb{R}^n$ is called a *domain.*
By $\overline{\Omega}$ we denote the closure of Ω; $\partial\Omega$ is the boundary.

Definition

We say that a domain $\Omega' \subset \Omega \subset \mathbb{R}^n$ is a *strictly interior* subdomain of Ω and write $\Omega' \subset\subset \Omega$, if $\overline{\Omega'} \subset \Omega$.

If Ω' is *bounded* and $\Omega' \subset\subset \Omega$, then *dist* $\{\Omega', \partial\Omega\} > 0$. We use the following notation:

$$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \quad \partial_j u = \frac{\partial u}{\partial x_j},$$

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n_+ \quad \text{is a multi–index}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n, \quad \partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}$$

$$\text{Next,} \ \ \nabla u = (\partial_1 u, \dots, \partial_n u), \quad |\nabla u| = \left(\sum_{j=1}^n |\partial_j u|^2 \right)^{1/2}$$

Definition

$L_q(\Omega), \quad 1 \le q < \infty$, is the set of all measurable functions $u(x)$ in Ω such that the norm

$$\|u\|_{q,\Omega} = \left(\int_\Omega |u(x)|^q dx \right)^{1/q}$$

is finite.

$L_q(\Omega)$ is a *Banach space*. We'll use the following *property*:
Let $u \in L_q(\Omega), 1 \le q < \infty$. We denote

$$J_\rho(u; L_q) = \sup_{|z| \le \rho} \left(\int_{\mathbb{R}^n} |u(x+z) - u(x)|^q \, dx \right)^{1/q}.$$

Here $u(x)$ is extended by zero on $\mathbb{R}^n \backslash \Omega$. $J_\rho(u; L_q)$ is called the modulus of continuity of a function u in $L_q(\Omega)$. Then

$$J_\rho(u; L_q) \to 0 \quad \text{as} \ \ \rho \to 0.$$

Definition

$L_{q,loc}(\Omega), 1 \le q < \infty$, is the set of all measurable functions $u(x)$ in Ω such that $\int_{\Omega'} |u(x)|^p \, dx < \infty$ for any *bounded strictly interior* subdomain $\Omega' \subset\subset \Omega$.

$L_{q,loc}(\Omega)$ is a *topological space* (but not a Banach space).
We say that $u_k \stackrel{k\to\infty}{\longrightarrow} u$ in $L_{q,loc}(\Omega)$, if $\|u_k - u\|_{q,\Omega'} \stackrel{k\to\infty}{\longrightarrow} 0$ for any bounded $\Omega' \subset\subset \Omega$

Definition

$L_\infty(\Omega)$ is the set of all bounded measurable functions in Ω; the norm is defined by

$$\|u\|_{\infty,\Omega} = ess \sup_{x\in\Omega} |u(x)|$$

Definition

$C^l(\overline{\Omega})$ is the *Banach space* of all functions in $\overline{\Omega}$ such that $u(x)$ and $\partial^\alpha u(x)$ with $|\alpha| \le l$ are *uniformly continuous in* $\overline{\Omega}$ and the norm

$$\|u\|_{C^l(\overline{\Omega})} = \sum_{|\alpha|\le l} \sup_{x\in\Omega} |\partial^\alpha u(x)|$$

is finite. If $l = 0$, we denote $C^0(\overline{\Omega}) = C(\overline{\Omega})$.

Remark

If Ω is *bounded*, then $\|u\|_{C^l(\overline{\Omega})} < \infty$ follows from the uniform continuity of $u, \ \partial^\alpha u, \ |\alpha| \le l$

Definition

$C^l(\Omega)$ is the class of functions in Ω such that $u(x)$ and $\partial^\alpha u$, $|\alpha| \le l$, are continuous in Ω.

Remark

Even if Ω is bounded, a function $u \in C^l(\Omega)$ may be not bounded; it may grow near the boundary.

Definition

$C_0^\infty(\Omega)$ is the class of the functions $u(x)$ in Ω such that

a) $u(x)$ is infinitely smooth, which means that $\partial^\alpha u$ is uniformly continuous in $\overline{\Omega}, \quad \forall\alpha$;

b) $u(x)$ is compactly supported: *supp* u is a compact subset of Ω.

§2: Mollification of functions

1. Definition of mollification

The procedure of mollification allows us to approximate function $u \in L_q(\Omega)$ by smooth functions.
Let $\omega(x), x \in \mathbb{R}^n$, be a function such that

$$\omega \in C_0^\infty(\mathbb{R}^n), \quad \omega(x) \geq 0, \quad \omega(x) = 0 \text{ if } |x| \geq 1, \text{ and}$$

$$\int_{\mathbb{R}} \omega(x)dx = 1. \tag{1}$$

For example, we may take

$$\omega(x) = \begin{cases} c \ \exp\left\{-\frac{1}{1-|x|^2}\right\} & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| \geq 1 \end{cases}$$

where constant c is chosen so that condition (1) is satisfied.
For $\rho > 0$ we put

$$\omega_\rho(x) \quad = \quad \rho^{-n}\omega\left(\frac{x}{\rho}\right), \quad x \in \mathbb{R}^n. \tag{2}$$

Then $\omega_\rho \in C_0^\infty(\mathbb{R}^n), \quad \omega_\rho(x) \geq 0,$

$$\omega_\rho(x) = 0 \quad \text{if} \quad |x| \geq \rho, \tag{3}$$

$$\int_{\mathbb{R}^n} \omega_\rho(x)dx = 1. \tag{4}$$

Definition

w_ρ is called a *mollifier.*

Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $u \in L_q(\Omega)$ with some $1 \leq q \leq \infty$. We *extend* $u(x)$ *by zero* on $\mathbb{R}^n \backslash \Omega$ and consider the convolution $\omega_\rho * u =: u_\rho$

$$u_\rho(x) = \int_{\mathbb{R}^n} \omega_\rho(x-y)u(y)dy. \tag{5}$$

In fact, the integral is over $\Omega \cap \{y : |x-y| < \rho\}$.

Definition

$u_\rho(x)$ is called a *mollification* or *regularization* of u(x).

2. Properties of mollification

1) $u_\rho \in C^\infty(\mathbb{R}^n)$, and
$\partial^\alpha u_\rho(x) = \int_{\mathbb{R}^n} \partial_x^\alpha \omega_\rho(x-y)u(y)dy$.
This follows from $\omega_\rho \in C^\infty$.

2) $u_\rho(x) = 0$ if $dist\,\{x;\Omega\} \geq \rho$, since $\omega_\rho(x-y) = 0$, $y \in \Omega$.

3) Let $u \in L_q(\Omega)$ with some $q \in [1,\infty]$. Then

$$\|u_\rho\|_{q,\mathbb{R}^n} \leq \|u\|_{q,\Omega}\,. \tag{6}$$

In other words, the operator $\mathcal{Y}_\rho : u \mapsto u_\rho$ is a linear continuous operator from $L_q(\Omega)$ to $L_q(\mathbb{R}^n)$ and
$\|\mathcal{Y}_\rho\|_{L_q(\Omega)\to L_q(\mathbb{R}^n)} \leq 1$.
Proof:
Case 1 : $1 < q < \infty$.
Let $\frac{1}{q} + \frac{1}{q'} = 1$. By the Hölder inequality and (4), we have

$$\begin{aligned} |u_\rho(x)| &= \left| \int_{\mathbb{R}^n} \omega_\rho(x-y)^{1/q} \omega_\rho(x-y)^{1/q'} u(y)dy \right| \\ &\leq \underbrace{\left(\int_{\mathbb{R}^n} \omega_\rho(x-y) \right)^{1/q'}}_{=1} \left(\int_{\mathbb{R}^n} \omega_\rho(x-y)|u(y)|^q dy \right)^{1/q} \end{aligned}$$

$$\Rightarrow \quad |u_\rho(x)|^q \leq \int_{\mathbb{R}^n} \omega_\rho(x-y)|u(y)|^q dy$$

By (4), we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |u_\rho(x)|^q\,dx &\leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \omega_\rho(x-y)\,|u(y)|^q\,dy \\ &= \int_{\mathbb{R}^n} dy\,|u(y)|^q \underbrace{\left(\int_{\mathbb{R}^n} \omega_\rho(x-y)dx \right)}_{=1} \\ &= \int_{\mathbb{R}^n} |u(y)|^q\,dy \end{aligned}$$

Case 2 : $q = \infty$. We have

$$\begin{aligned} |u_\rho(x)| &\leq \int_{\mathbb{R}^n} \omega_\rho(x-y)|u(y)|dy \\ &\leq \|u\|_\infty \underbrace{\int_{\mathbb{R}^n} \omega_\rho(x-y)dy}_{=1} \end{aligned}$$

$\Rightarrow \|u_\rho\|_\infty \leq \|u\|_\infty$

<u>Case 3</u> : $q = 1$.

We integrate the inequality

$$|u_\rho(x)| \leq \int_{\mathbb{R}^n} \omega_\rho(x-y)|u(y)|dy$$

and obtain:

$$\int_{\mathbb{R}^n} |u_\rho(x)|\, dx \leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \omega_\rho(x-y)|u(y)|dy = \int_{\mathbb{R}^n} |u(y)|dy$$

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4) Let $u \in L_q(\Omega)$, $\underline{1 \leq q < \infty}$. Then

$$\|u_\rho - u\|_{q,\mathbb{R}^n} \to 0 \quad \text{as} \quad \rho \to 0. \tag{7}$$

Consequently,

$$\|u_\rho - u\|_{q,\Omega} \to 0 \quad \text{as} \quad \rho \to 0.$$

<u>Proof</u>

The proof is based on the following property: if $u \in L_q(\Omega)$ (and $u(x)$ is extended by 0), then

$$\sup_{|z|\leq\rho} \left(\int_{\mathbb{R}^n} |u(x+z) - u(x)|^q \, dx \right)^{1/q} =: J_\rho(u; L_q) \to 0 \quad \text{as} \quad \rho \to 0.$$

($J_\rho(u; L_q)$ is called the modulus of continuity of u in L_q.)

<u>Case 1</u> : $1 < q < \infty$. By (4) and (5) we have

$$\begin{aligned} |u_\rho(x) - u(x)| &\leq \int_{\mathbb{R}^n} \omega_\rho(x-y)\,(u(y) - u(x))\, dy \\ &= \int_{\mathbb{R}^n} \omega_\rho(x-y)^{1/q'} \omega_\rho(x-y)^{1/q}\,(u(y) - u(x))\, dy \end{aligned}$$

Then, by the Hölder inequality, it follows that

$$|u_\rho(x) - u(x)| \leq \underbrace{\left(\int_{\mathbb{R}^n} \omega_\rho(x-y) dy \right)^{1/q'}}_{=1} \left(\int_{\mathbb{R}^n} \omega_\rho(x-y)\,|u(y) - u(x)|^q \, dy \right)^{1/q}$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}^n} |u_\rho(x) - u(x)|^q \, dx &\leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \omega_\rho(x-y) \, |u(y) - u(x)|^q \, dy \\
&\overset{x-y=z}{=} \int_{|z|<\rho} dz \;\; \omega_\rho(z) \int_{\mathbb{R}^n} |u(y+z) - u(y)|^q \, dy \\
&\leq \sup_{|z|\leq\rho} \int_{\mathbb{R}^n} |u(y+z) - u(y)|^q \, dy \underbrace{\int_{|z|<\rho} dz \quad \omega_\rho(z)}_{=1} \\
&= (J_\rho(u; L_q))^q \, .
\end{aligned}$$

$$\Rightarrow \|u_\rho - u\|_{q,\mathbb{R}^n} \leq J_\rho(u; L_q) \to 0 \quad \text{as} \quad \rho \to 0.$$

Case 2 : $q = 1$. We have

$$|u_\rho(x) - u(x)| \leq \int_{\mathbb{R}^n} \omega_\rho(x-y) \, |u(y) - u(x)| \, dy$$

$$\begin{aligned}
\Rightarrow \int_{\mathbb{R}^n} |u_\rho(x) - u(x)| \, dx &\leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \omega_\rho(x-y) \, |u(y) - u(x)| \, dy \\
&\overset{x-y=z}{=} \int_{|z|<\rho} dz \;\; \omega_\rho(z) \int_{\mathbb{R}^n} |u(y+z) - u(y)| \, dy \\
&\leq J_\rho(u; L_1) \\
&\to 0 \quad \text{as} \quad \rho \to 0.
\end{aligned}$$

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Remark

If $q = \infty$, there is **NO** such property, since L_∞–limit of smooth functions $u_\rho(x)$ must be a continuous function.
If $u \in C(\overline{\Omega})$ and we extend $u(x)$ by zero, then we may loose continuity.
In general, $\|u_\rho - u\|_{C(\overline{\Omega})} \nrightarrow 0 \quad \text{as} \quad \rho \to 0.$

However, we have the following property:

5) If $u \in C(\overline{\Omega})$, $\Omega' \subset\subset \Omega$ and Ω' is bounded, then

$$\|u_\rho - u\|_{C(\overline{\Omega'})} \to 0 \quad \text{as} \quad \rho \to 0$$

Proof

Let $\rho < dist\,\{\Omega'; \partial\Omega\}$. Then

$$\begin{aligned} u_\rho(x) - u(x) &= \int_{\mathbb{R}^n} \omega_\rho(x-y)\,(u(y)-u(x))\,dy \\ &\overset{x-y=z}{=} \int_{\mathbb{R}^n} \omega_\rho(z)\,(u(x-z)-u(x))\,dz \\ \Rightarrow \sup_{x\in\overline{\Omega'}} |u_\rho(x)-u(x)| &\leq \sup_{x\in\overline{\Omega'}} \sup_{|z|\leq\rho} |u(x-z)-u(x)| \\ &\to 0 \quad \text{as} \quad \rho \to 0 \end{aligned}$$

(since $u(x)$ is continuous in $\overline{\Omega}$).

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§3: Class $C_0^\infty(\Omega)$

By $C_0^\infty(\Omega)$ we denote the class of infinitely smooth functions in Ω with compact support:

$$u \in C_0^\infty(\Omega) \quad \Leftrightarrow \quad u \in C^\infty(\overline{\Omega}) \text{ and } \operatorname{supp} u \subset \Omega.$$

Theorem 1

$C_0^\infty(\Omega)$ is dense in $L_q(\Omega), 1 \leq q < \infty$

Proof

Let $u \in L_q(\Omega)$ and $\varepsilon > 0$. Let Ω' be a bounded domain, $\Omega' \subset\subset \Omega$, and

$$\|u\|_{q,\Omega\setminus\Omega'} \leq \frac{\varepsilon}{2}.$$

We put

$$u^{(\varepsilon)}(x) = \begin{cases} u(x) & \text{if } x \in \Omega' \\ 0 & \text{if } x \in \Omega\setminus\Omega' \end{cases}$$

Then $\left\|u - u^{(\varepsilon)}\right\|_{q,\Omega} \leq \frac{\varepsilon}{2}$. Let $u_\rho^{(\varepsilon)}(x)$ be the mollification of $u^{(\varepsilon)}(x)$. By property 4) of mollification, $\left\|u_\rho^{(\varepsilon)} - u^{(\varepsilon)}\right\|_{q,\Omega} \leq \frac{\varepsilon}{2}$ for sufficiently small ρ.
Hence, $\left\|u_\rho^{(\varepsilon)} - u\right\|_{q,\Omega} \leq \varepsilon$ for sufficiently small ρ.
Note that $u_\rho^{(\varepsilon)} \in C_0^\infty(\Omega)$ if $\rho < dist\,\{\Omega', \partial\Omega\}$.

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Theorem 2

Let $u \in L_{1,loc}(\Omega)$, and suppose that

$$\int_\Omega u(x)\eta(x)dx = 0, \quad \forall \eta \in C_0^\infty(\Omega). \tag{8}$$

Then $u(x) = 0, \quad$ a. e. $x \in \Omega$.

Theorem 2 is an analog of the Main Lemma of variational calculus.

Proof

1) First, let us prove that

$$\int_\Omega u(x)\eta(x)dx = 0$$

for any $\eta \in L_\infty(\Omega)$ with compact support supp $\eta \subset \Omega$. Suppose that supp $\eta \subset \overline{\Omega'}$, where Ω' is a bounded domain and $\Omega' \subset\subset \Omega$. Then

$$\eta_\rho \in C_0^\infty(\Omega) \text{ if } \rho < dist\left\{\Omega'; \partial\Omega\right\} =: 2\rho_0.$$

Let $\Omega'_{\rho_0} = \{x : dist\{x; \Omega'\} < \rho_0\}$, and let

$$\chi_{\rho_0}(x) = \begin{cases} 1 & \text{if } x \in \Omega'_{\rho_0} \\ 0 & \text{otherwise} \end{cases}$$

By (8),

$$\int_\Omega u(x)\eta_\rho(x)dx = 0, \quad \rho < \rho_0. \tag{9}$$

Since $\eta \in L_1(\Omega)$, by property 4) of mollification,
$\|\eta_\rho - \eta\|_{1,\Omega} \to 0$ as $\rho \to 0$.
Then there exists a sequence $\{\rho_k\}_{k\in\mathbb{N}}$, $\rho_k \to 0, \rho_k < \rho_0$, such that

$$\eta_{\rho_k}(x) \overset{k\to\infty}{\longrightarrow} \eta(x) \quad \textit{for almost every } x \in \Omega.$$

Then also $\eta_{\rho_k}(x)u(x) \overset{k\to\infty}{\longrightarrow} \eta(x)u(x)$ for a. e. $x \in \Omega$.
Using property 3) (that $\|\eta_\rho\|_\infty \le \|\eta\|_\infty$), we have

$$|u(x)\eta_{\rho_k}(x)| \le \chi_{\rho_0}(x)|u(x)| \, \|\eta\|_\infty \,, \tag{10}$$

and the right-hand side in (10) belongs to $L_1(\Omega)$.
Then, by the Lebesgue Theorem,

$$\int_\Omega u(x)\eta_{\rho_k}(x)dx \overset{k\to\infty}{\longrightarrow} \int_\Omega u(x)\eta(x)dx.$$

By (9), the left-hand side is equal to zero.
Hence, $\int_\Omega u(x)\eta(x)dx = 0$.

2) Now, let Ω' be a bounded domain such that $\Omega' \subset\subset \Omega$. We put

$$\eta(x) = \begin{cases} \frac{\overline{u(x)}}{|u(x)|} & , \text{if } u(x) \neq 0, x \in \Omega' \\ 0 & , \text{otherwise} \end{cases}$$

Then

$$u(x)\eta(x) = \begin{cases} |u(x)| & , x \in \Omega' \\ 0 & , x \in \Omega\backslash\Omega' \end{cases}$$

Since $\eta(x)$ is L_∞–function with compact support *supp* $\eta \subset \overline{\Omega'} \subset \Omega$, then, by part 1),

$$0 = \int_\Omega u(x)\eta(x)dx = \int_{\Omega'} |u(x)|dx.$$

It follows that $u(x) = 0$ for a. e. $x \in \Omega'$. Since Ω' is an *arbitrary* bounded domain such that $\Omega' \subset\subset \Omega$, then

$$u(x) = 0, \quad \text{a.e. } x \in \Omega$$

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§4: Weak derivatives

1. Definition and properties of weak derivatives

Definition 1

Let α be a multi–index. Suppose that $u, v \in L_{1,loc}(\Omega)$, and

$$\int_\Omega u(x)\partial^\alpha \eta(x)dx = (-1)^{|\alpha|} \int_\Omega v(x)\eta(x)dx, \quad \forall \eta \in C_0^\infty(\Omega). \tag{11}$$

Then v is called the *weak* (or *distributional*) partial derivative of u in Ω, and is denoted by $\partial^\alpha u$.

If $u(x)$ is sufficiently smooth to have continuous derivative $\partial^\alpha u$, we can integrate by parts:

$$\int_\Omega u(x)\partial^\alpha \eta(x)dx = \int_\Omega (-1)^{|\alpha|} \partial^\alpha u(x)\eta(x)dx.$$

Hence, the classical derivative $\partial^\alpha u$ is also the weak derivative. Of course, $\partial^\alpha u$ may exist in the weak sense wihout existing in the classical sense.

Remark

1) To define the weak derivative $\partial^\alpha u$, we don't need the existence of derivatives of the smaller order (like in the classical definition).
2) The weak derivative is defined as an element of $L_{1,loc}(\Omega)$, so we can change it on some set of measure zero.

Properties of $\partial^\alpha u$

1) Uniqueness

Proof

Uniqueness of the weak derivative follows from Theorem 2. Suppose that $u \in L_{1,loc}(\Omega)$ and $v, w \in L_{1,loc}(\Omega)$ are both weak derivatives of u. Then, by (11),

$$\int_\Omega (v(x) - w(x))\, \eta(x)dx = 0, \quad \forall \eta \in C_0^\infty(\Omega).$$

By Theorem 2, $\quad v(x) = w(x), \quad a.e.\ x \in \Omega.$

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2) Linearity
If $u_1, u_2 \in L_{1,loc}(\Omega)$ and there exist weak derivatives $v_1 = \partial^\alpha u_1$, $v_2 = \partial^\alpha u_2 \in L_{1,loc}(\Omega)$, then there exists $\partial^\alpha (c_1 u_1 + c_2 u_2)$ and

$$\partial^\alpha (c_1 u_1 + c_2 u_2) = c_1 \partial^\alpha u_1 + c_2 \partial^\alpha u_2, \quad c_1, c_2 \in \mathbb{C}.$$

Proof
Obviously,

$$\begin{aligned}
\int_\Omega (c_1 u_1 + c_2 u_2)\, \partial^\alpha \eta dx &= c_1 \int_\Omega u_1 \partial^\alpha \eta dx + c_2 \int_\Omega u_2 \partial^\alpha \eta dx \\
&= (-1)^{|\alpha|} c_1 \int_\Omega v_1 \eta dx + (-1)^{|\alpha|} c_2 \int_\Omega v_2 \eta dx \\
&= (-1)^{|\alpha|} \int_\Omega \underbrace{(c_1 v_1 + c_2 v_2)}_{=\partial^\alpha (c_1 u_1 + c_2 u_2)} \eta dx.
\end{aligned}$$

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3) If $v = \partial^\alpha u$ in Ω , then $v = \partial^\alpha u$ in Ω' for any $\Omega' \subset \Omega$.
Obvious

4) Mollification of the weak derivative
„Derivative of mollification is equal to mollification of derivative“. This is true in any *bounded strictly interior* domain $\Omega' \subset\subset \Omega$.
Suppose that $u, v \in L_{1,loc}(\Omega)$ and $v = \partial^\alpha u$. Then

$$v_\rho(x) = \partial^\alpha u_\rho(x) \quad \text{if} \quad \rho < dist\,\{x, \partial\Omega\}\,. \tag{12}$$

The functions u_ρ and v_ρ are smooth; the derivative $\partial^\alpha u_\rho$ in (12) is understood in the classical sense.
Proof
Let $\rho < dist\,\{x, \partial\Omega\}$. We have

$$u_\rho(x) = \int_\Omega \omega_\rho(x-y) u(y) dy$$

Then $\quad \partial^\alpha u_\rho(x) = \int_\Omega \partial_x^\alpha \omega_\rho(x-y) u(y) dy$.
Note that $\partial_x^\alpha \omega_\rho(x-y) = (-1)^{|\alpha|} \partial_y^\alpha \omega_\rho(x-y)$.
Hence,

$$\partial^\alpha u_\rho(x) = (-1)^{|\alpha|} \int_\Omega \partial_y^\alpha \omega_\rho(x-y) u(y) dy.$$

Since $\rho < dist\,\{x, \partial\Omega\}$, then for $\eta(y) := \omega_\rho(x-y)$ we have $\eta \in C_0^\infty(\Omega)$.
By definition of the weak derivative $\partial^\alpha u = v$, we obtain

$$\partial^\alpha u_\rho(x) = \int_\Omega \omega_\rho(x-y) v(y) dy = v_\rho(x).$$

■

5) Suppose that $u \in L_{1,loc}(\Omega)$ and there exists the weak derivative $\partial^\alpha u$ such that
$\partial^\alpha u \in L_q(\Omega), 1 \le q < \infty$.
Then $\|\partial^\alpha u_\rho - \partial^\alpha u\|_{q,\Omega'} \to 0$ as $\rho \to 0$, for any bounded strictly interior domain $\Omega' \subset\subset \Omega$.

Proof
This follows from property 4) of mollification and property 4) of weak derivatives:

$$\partial^\alpha u = v \in L_q(\Omega); \quad \partial^\alpha u_\rho = v_\rho \text{ in } \Omega' \quad (\text{ for sufficiently small } \rho)\,;$$

$$\|v_\rho - v\|_{q,\Omega'} \to 0 \quad \text{as} \quad \rho \to 0.$$

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Remark
If we extend $u(x)$ by zero on $\mathbb{R}^n \backslash \Omega$, then, in general, the weak derivative $\partial^\alpha u$ in $\mathbb{R}^n$ *does not exist.* Hence, we have convergence $\partial^\alpha u_\rho \overset{\rho \to 0}{\longrightarrow} \partial^\alpha u$ in $L_q(\Omega')$ *only for bounded strictly interior domain* Ω'.

Exclusion:
if $u(x) = 0$, if $dist\,\{x; \partial\Omega\} < \rho_o$, and $\partial^\alpha u \in L_q(\Omega)$,
then $\|\partial^\alpha u_\rho - \partial^\alpha u\|_{q,\Omega} \overset{\rho \to 0}{\longrightarrow} 0$.

2. Another definition of the weak derivative

Definition 2

Suppose that $u, v \in L_{1,loc}(\Omega)$ and there exists a sequence $u_m \in C^l(\Omega)$, $m \in \mathbb{N}$, such that $u_m \overset{m \to \infty}{\longrightarrow} u$ and $\partial^\alpha u_m \overset{m \to \infty}{\longrightarrow} v$ in $L_{1,loc}(\Omega)$.
Here α is a multi–index and $|\alpha| = l$. Then v is called the weak derivative of u in $\Omega : \partial^\alpha u = v$.

Definition 1 $\Leftrightarrow$ Definition 2
Proof

1) Definition 1 $\Leftarrow$ Definition 2.
Since $u_m \in C^l(\Omega)$, then

$$\int_\Omega u_m \partial^\alpha \eta dx = (-1)^{|\alpha|} \int_\Omega \partial^\alpha u_m \eta dx, \quad \forall \eta \in C_0^\infty(\Omega). \tag{13}$$

For η fixed, the left–hand side of (13) tends to $\int_\Omega u\partial^\alpha\eta dx$ as $m \to \infty$:

$$\left|\int_\Omega (u_m - u)\,\partial^\alpha\eta dx\right| \leq \max|\partial^\alpha\eta| \int_{supp\,\eta} |u_m - u|\,dx \overset{m\to\infty}{\longrightarrow} 0.$$

Similarly, the right–hand side of (13) tends to $(-1)^{|\alpha|}\int_\Omega v\eta dx$. Consequently,

$$\int_\Omega u\partial^\alpha\eta dx = (-1)^{|\alpha|}\int_\Omega v\eta dx, \quad \forall\eta \in C_0^\infty(\Omega).$$

It means that $v = \partial^\alpha u$ in the sense of Definition 1.

2) Definition 1 $\Rightarrow$ Definition 2.
Let $u, v \in L_{1,loc}(\Omega)$, and let $v = \partial^\alpha u$ in the sense of Definition 1. We want to find a sequence $u_m \in C^\infty(\Omega)$ such that $u_m \overset{m\to\infty}{\longrightarrow} u$ and $\partial^\alpha u_m \overset{m\to\infty}{\longrightarrow} v$ in $L_{1,loc}(\Omega)$.
Let $\{\Omega'_m\}, m \in \mathbb{N}$, be a sequence of bounded domains such that

$$\Omega'_m \subset\subset \Omega, \quad \Omega'_m \subset \Omega'_{m+1} \text{ and } \bigcup_{m\in\mathbb{N}} \Omega'_m = \Omega.$$

We put

$$u^{(m)}(x) = \begin{cases} u(x) & \text{if } x \in \Omega'_m \\ 0 & \text{otherwise} \end{cases}$$

Then $u^{(m)} \in L_1(\Omega)$. Consider the mollification of $u^{(m)} : u_\rho^{(m)} \in C^\infty(\Omega)$.
Let $\{\rho_m\}_{m\in\mathbb{N}}$ be a sequence of positive numbers such that $\rho_m \to 0$ as $m \to \infty$.
We put

$$u_m(x) = u_{\rho_m}^{(m)}(x), \quad x \in \Omega.$$

Then $u_m \in C^\infty(\Omega)$ and $u_m \overset{m\to\infty}{\longrightarrow} u$ in $L_{1,loc}(\Omega)$. Prove this yourself, using property 4) of mollification.
Next, by property 5) of $\partial^\alpha u$, prove that $\partial^\alpha u_m \overset{m\to\infty}{\longrightarrow} v$ in $L_{1,loc}(\Omega)$.
Thus, $v = \partial^\alpha u$ in the sense of Definition 2.

■

Theorem 3

Let $u_m \in L_{1,loc}(\Omega)$ and $u_m \overset{m\to\infty}{\longrightarrow} u$ in $L_{1,loc}(\Omega)$. Suppose that there exist weak derivatives $\partial^\alpha u_m \in L_{1,loc}(\Omega)$ and $\partial^\alpha u_m \overset{m\to\infty}{\longrightarrow} v$ in $L_{1,loc}(\Omega)$. Then $v = \partial^\alpha u$.
In other words, the operator ∂^α is closed.

Proof
By Definition 1, for $\partial^\alpha u_m$ we have

$$\begin{aligned} \int_\Omega u_m \partial^\alpha \eta dx &= (-1)^{|\alpha|} \int_\Omega \partial^\alpha u_m \eta dx, \quad \forall \eta \in C_0^\infty(\Omega) \\ \downarrow m \to \infty & \qquad\qquad \downarrow m \to \infty \\ \int_\Omega u \partial^\alpha \eta dx &= (-1)^{|\alpha|} \int_\Omega v \eta dx, \quad \forall \eta \in C_0^\infty(\Omega) \end{aligned}$$

$\Rightarrow v = \partial^\alpha u$ in the sense of Definition 1.

■

Remark

The conclusion of Theorem 3 remains true under weaker assumptions that

$$\int_\Omega u_m \eta dx \overset{m\to\infty}{\longrightarrow} \int_\Omega u \eta dx \quad \text{and} \quad \int_\Omega \partial^\alpha u_m \eta dx \overset{m\to\infty}{\longrightarrow} \int_\Omega v \eta dx, \quad \forall \eta \in C_0^\infty(\Omega).$$

(It means that $u_m \to u$ and $\partial^\alpha u_m \to v$ in $\mathcal{D}'(\Omega)$.)

3. Weak derivatives of the product of functions

Proposition

If $u, \partial_j u \in L_{q,loc}(\Omega)$, and $v, \partial_j v \in L_{q',loc}(\Omega)$ with some $1 < q < \infty, \frac{1}{q} + \frac{1}{q'} = 1$ **or** if $u, \partial_j u \in L_{1,loc}(\Omega)$ and $v, \partial_j v \in C(\Omega)$, then

$$\partial_j (uv) = (\partial_j u) v + u (\partial_j v).$$

Proof

1) Case 1: $\quad 1 < q < \infty$
Let us fix $\eta \in C_0^\infty(\Omega)$. Let Ω' be a bounded domain such that $supp\, \eta \subset \Omega' \subset\subset \Omega$. We put

$$\tilde{u}(x) = \begin{cases} u(x) & , x \in \Omega' \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{v}(x) = \begin{cases} v(x) & , x \in \Omega' \\ 0 & \text{otherwise} \end{cases}$$

Then $\tilde{u} \in L_q(\Omega')$, $\tilde{v} \in L_{q'}(\Omega')$. By property 4) of mollifications,

$$\|\tilde{u_\rho} - \tilde{u}\|_{q,\Omega'} \to 0, \quad \|\tilde{v}_\rho - \tilde{v}\|_{q',\Omega'} \to 0 \quad \text{as } \rho \to 0$$

Next, $\partial_j \tilde{u} = \partial_j u$ in Ω', $\partial_j \tilde{v} = \partial_j v$ in Ω' (it is clear from Definition 1).
So, $\partial_j \tilde{u} \in L_q(\Omega')$, $\partial_j \tilde{v} \in L_{q'}(\Omega')$.
By property 5) of weak derivatives,

$$\|\partial_j \tilde{u_\rho} - \partial_j \tilde{u}\|_{q,supp\,\eta} \to 0, \quad \text{as } \rho \to 0,$$
$$\|\partial_j \tilde{v_\rho} - \partial_j \tilde{v}\|_{q',supp\,\eta} \to 0, \quad \text{as } \rho \to 0.$$

Since $\tilde{u}_\rho, \tilde{v}_\rho$ are smooth functions, we have

$$\begin{aligned} \int_{\Omega'} \tilde{u}_\rho \tilde{v}_\rho \partial_j \eta dx &= -\int_{\Omega'} \partial_j \left(\tilde{u}_\rho \tilde{v}_\rho\right) \eta dx \\ &= -\int_{\Omega'} \left(\partial_j \tilde{u}_\rho\right) \tilde{v}_\rho \eta dx - \int_{\Omega'} \tilde{u}_\rho \left(\partial_j \tilde{v}_\rho\right) \eta dx. \end{aligned} \quad (14)$$

Let us show that

$$\int_{\Omega'} \tilde{u}_\rho \tilde{v}_\rho \partial_j \eta dx \stackrel{\rho \to 0}{\longrightarrow} \int_{\Omega'} \tilde{u} \tilde{v} \partial_j \eta dx = \int_{\Omega} uv \partial_j \eta dx \quad (15)$$

We have

$$\begin{aligned} \left|\int_{\Omega'} \left(\tilde{u}_\rho \tilde{v}_\rho - \tilde{u}\tilde{v}\right) \partial_j \eta dx\right| &\le \left|\int_{\Omega'} \left(\tilde{u}_\rho - \tilde{u}\right) \tilde{v}_\rho \partial_j \eta dx\right| + \left|\int_{\Omega'} \tilde{u} \left(\tilde{v}_\rho - \tilde{v}\right) \partial_j \eta dx\right| \\ &\le \underbrace{\|\tilde{u}_\rho - \tilde{u}\|_{q,\Omega'}}_{\to 0} \underbrace{\|\tilde{v}_\rho\|_{q',\Omega'}}_{bounded} \max|\partial_j \eta| + \\ &+ \|\tilde{u}\|_{q,\Omega'} \underbrace{\|\tilde{v}_\rho - \tilde{v}\|_{q',\Omega'}}_{\to 0} \max|\partial_j \eta| \\ &\to 0 \quad \text{as} \quad \rho \to 0 \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} \int_{\Omega'} \left(\left(\partial_j \tilde{u}_\rho\right) \tilde{v}_\rho + \tilde{u}_\rho \left(\partial_j \tilde{v}_\rho\right)\right) \eta dx \quad &\stackrel{\rho \to 0}{\to} \quad \int_{\Omega'} \left(\left(\partial_j \tilde{u}\right) \tilde{v} + \tilde{u} \left(\partial_j \tilde{v}\right)\right) \eta dx \\ &= \quad \int_{\Omega} \left(\left(\partial_j u\right) v + u \left(\partial_j v\right)\right) \eta dx \end{aligned} \quad (16)$$

From (14) - (16) it follows that

$$\int_{\Omega} uv \partial_j \eta dx = -\int_{\Omega} \left(\left(\partial_j u\right) v + u \left(\partial_j v\right)\right) \eta dx$$

This identity is proved for any $\eta \in C_0^\infty(\Omega)$. It means (by Definition 1) that there exists the weak derivative $\partial_j(uv)$ and

$$\partial_j(uv) = \left(\partial_j u\right) v + u \left(\partial_j v\right)$$

2) Case q=1.
<u>Prove yourself</u>

■

4. Change of variables

Suppose that $u \in L_{1,loc}(\Omega)$ and there exist weak derivatives $\partial_j u \in L_{1,loc}(\Omega), j = 1, \ldots, n$.
Let $y = f(x)$ be a diffeomorphism of class C^1 and $f(\Omega) = \tilde{\Omega}$.
We put $\tilde{u}(y) = u(f^{-1}(y))$. Then $\tilde{u} \in L_{1,loc}(\tilde{\Omega})$. Let us show that there exist weak derivatives $\frac{\partial \tilde{u}}{\partial y_\kappa}, \kappa = 1, \ldots, n$, and

$$\frac{\partial \tilde{u}}{\partial y_k} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial y_\kappa}$$

Proof
Since there exist weak derivatives $\frac{\partial u}{\partial x_j} \in L_{1,loc}(\Omega), j = 1, \ldots, n$, then there exists a sequence $u_m \in C^1(\Omega)$ such that $u_m \overset{m\to\infty}{\longrightarrow} u$ and $\frac{\partial u_m}{\partial x_j} \overset{m\to\infty}{\longrightarrow} \frac{\partial u}{\partial x_j}$ in $L_{1,loc}(\Omega)$ for all $j = 1, \ldots, n$.
(We can construct this sequence like in the proof, that Definition 1 and Definition 2 are equivalent).
We denote $\tilde{u}_m(y) = u_m(f^{-1}(y))$. Then $\tilde{u}_m \in C^1(\tilde{\Omega})$, and, by usual rule (for classical derivatives),

$$\frac{\partial \tilde{u}_m}{\partial y_k} = \sum_{j=1}^{n} \frac{\partial u_m}{\partial x_j} \frac{\partial x_j}{\partial y_\kappa}$$

Let us check that $\tilde{u}_m \overset{m\to\infty}{\longrightarrow} \tilde{u}$ in $L_{1,loc}(\tilde{\Omega})$. Indeed, for every bounded domain $\tilde{\Omega}' \subset\subset \tilde{\Omega}$ we have

$$\begin{aligned}\int_{\tilde{\Omega}'} |\tilde{u}_m(y) - \tilde{u}(y)|\, dy &= \int_{\tilde{\Omega}'} \left|u_m(f^{-1}(y)) - u(f^{-1}(y))\right| dy \\ &= \int_{\Omega'} |u_m(x) - u(x)|\, |J(x)|\, dx \\ &\to 0 \quad \text{as} \quad m \to \infty\end{aligned}$$

Here $\Omega' = f^{-1'}(\tilde{\Omega}')$ and $J(x)$ is the Jacobian of the transformation $f(x)$ $\left(J(x) = det\left\{\frac{\partial y}{\partial x}\right\}\right)$.
Here the right-hand side tends to zero, since $|J(x)|$ is bounded in Ω';
Ω' is a bounded domain such that $\Omega' \subset\subset \Omega$; and $u_m \overset{m\to\infty}{\longrightarrow} u$ in $L_{1,loc}(\Omega)$.
Similarly, using that $\frac{\partial u_m}{\partial x_j} \overset{m\to\infty}{\longrightarrow} \frac{\partial u}{\partial x_j}$ in $L_{1,loc}(\Omega)$, one can show that

$$\frac{\partial \tilde{u}_m}{\partial y_k} = \sum_{j=1}^{n} \frac{\partial u_m}{\partial x_j} \frac{\partial x_j}{\partial y_\kappa} \overset{m\to\infty}{\longrightarrow} \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial y_\kappa} \quad \text{in} \quad L_{1,loc}(\tilde{\Omega})$$

Then, by Definition 2, there exist weak derivatives

$$\frac{\partial \tilde{u}}{\partial y_k} \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial y_k} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial y_k}$$

■

Thus, for weak derivatives we have the usual rule of change of variables. The same is true for derivatives of higher order.

5.

For ordinary derivatives we have the following property:
if $\frac{\partial u}{\partial x_j} = 0$ in $\Omega, j = 1, \dots, n$, then $u = const$. The same is true for weak derivatives.

Theorem 4

Suppose that $u \in L_{1,loc}(\Omega)$ and there exist weak derivatives $\partial^\alpha u$ for any multi-index α such that $|\alpha| = l \quad (l \in \mathbb{N})$ and $\partial^\alpha u = 0$ in Ω, $|\alpha| = l$. Then $u(x)$ is a polynomial of order $\leq l - 1$ in Ω.

Proof

1) Let Ω' be a bounded domain such that $\Omega' \subset\subset \Omega$. Let Ω'' be another bounded domain such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. We put

$$\tilde{u}(x) = \begin{cases} u(x) & , x \in \Omega'' \\ 0 & \text{otherwise} \end{cases}$$

Then $\tilde{u} \in L_1(\Omega'')$, and $\quad \partial^\alpha \tilde{u} = \partial^\alpha u = 0, \quad |\alpha| = l,$ in Ω''. Consider the mollification $\tilde{u}_\rho(x)$. If $\rho < dist\,\{\Omega', \partial\Omega''\}$, then, by property 4) of $\partial^\alpha u$,

$$\partial^\alpha \tilde{u}_\rho(x) = \left(\partial^\alpha \tilde{u}\right)_\rho(x), \quad x \in \Omega', \quad |\alpha| = l.$$

Hence, $\partial^\alpha \tilde{u}_\rho = 0$ in Ω'. Thus, $\tilde{u}_\rho(x)$ is a smooth function in Ω' and all its derivatives of order l are equal to zero. It follows that
$\tilde{u}_\rho(x) = P^{(\rho)}_{l-1}(x), \quad x \in \Omega'$, where $P^{(\rho)}_{l-1}$ is a polynomial of order $\leq l-1$.
By property 4) of mollification,

$$\|\tilde{u}_\rho - u\|_{1,\Omega'} \to 0 \quad \text{as} \quad \rho \to 0, \text{ i. e. }, \quad P^{(\rho)}_{l-1} \overset{\rho\to 0}{\longrightarrow} u \quad \text{in} \quad L_1(\Omega').$$

The set of all polynomials in Ω' of order $\leq l-1$ is a finite-dimensional (and, so, *closed*!) subspace in $L_1(\Omega')$. Therefore, the limit $u(x)$ must be also a polynomial of order $\leq l - 1$:

$$u(x) = P_{l-1}(x), \quad x \in \Omega'.$$

2) Now it is easy to complete the proof by the standard procedure. Let $\{\Omega'_k\}_{k\in\mathbb{N}}$ be a sequence of bounded domains such that

$$\Omega'_k \subset\subset \Omega, \quad \Omega'_k \subset \Omega'_{k+1}, \quad \text{and} \quad \bigcup_{k\in\mathbb{N}} \Omega'_k = \Omega.$$

We have poved that for each domain Ω'_k

$$u(x) = P^{(k)}_{l-1}(x), \quad x \in \Omega'_k$$

Then $P^{(k+1)}_{l-1}(x)$ is continuation of $P^{(k)}_{l-1}(x)$, but *continuation of a polynomial is unique.*
$\Rightarrow$ There exists a polynomial $P_{l-1}(x)$ such that

$$u(x) = P_{l-1}(x), \quad x \in \Omega.$$

■

6. Absolute continuity property

The existence of the weak derivative is related to the absolute continuity property. Recall the definition of absolute continuity for function of one variable.

Definition

Function $u : [a, b] \to \mathbb{R}$ is called *absolutely continuous*, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite set of disjoint intervals

$$(x_1, x'_1), \ (x_2, x'_2), \ \ldots, (x_m, x'_m) \quad (\subset [\, a, b\,])$$

with $\sum_{j=1}^{m} \left| x'_j - x_j \right| < \delta$, one has

$$\sum_{j=1}^{m} \left| u(x'_j) - u(x_j) \right| < \varepsilon.$$

We'll use the *following facts*:

1) $u : [a, b] \to \mathbb{R}$ is absolutely continuous if and only if there exists a function $v \in L_1(a, b)$ such that

$$u(x) = u(a) + \int_a^x v(t)dt, \quad x \in [a, b]\,.$$

2) If $u : [a, b] \to \mathbb{R}$ is absolutely continuous, then there *exists derivative* $\frac{du}{dx}$ *for almost every* $x \in (a, b)$ and $\frac{du}{dx} = v \ \ (\in L_1(a, b))$.

Theorem 5

Let $n = 1$. A measurable function $u(x)$ is absolutely continuous on $[a, b]$ *if and only if* there exists the weak derivative $\frac{du}{dx} \in L_1(a, b)$. The weak derivative coincides with the classical derivative almost everywhere.

Remark

When we speak about measurable functions, we mean not just one function but *a class of functions*, that are equal to each other almost everywhere. So, when we say that a measurable function $u(x)$ is absolutely continuous, it means that in the class of functions equivalent to u, there exists an absolutely continuous representative.

Proof

1) $u(x)$ is absolutely continuous. $\Rightarrow \exists$ weak derivative $\frac{du}{dx} \in L_1$. If $u(x)$ is a. c., then there exists the *classical derivative* $\frac{du}{dx} = v$ almost everywhere and $v \in L_1(a,b)$. Next, let $\eta \in C_0^\infty(a,b)$. Then the product ηu is also absolutely continuous. There exists the classical derivative $\frac{d(\eta u)}{dx}$ for almost every $x \in (a,b)$. We have the usual rule:

$$\frac{d\,(\eta u)}{dx} = v\eta + u\frac{d\eta}{dx}.$$

Integrate this identity over (a,b). Then $\int_a^b \frac{d(\eta u)}{dx} dx = 0$.
(Since $\eta(x) = 0$ near a and b). Hence,

$$\int_a^b \left(v\eta + u\frac{d\eta}{dx} \right) dx = 0.$$

The obtained identity

$$\int_a^b u\frac{d\eta}{dx} = -\int_a^b v\eta dx, \quad \forall \eta \in C_0^\infty(a,b),$$

by Definition 1, means that v is the *weak derivative* $\frac{du}{dx}$

2) $\exists$ weak derivative $v = \frac{du}{dx} \in L_1(a,b) \Rightarrow u$ is a. c.
Consider $w(x) = \int_a^x v(t)dt$.
Then $w(x)$ is *absolutely continuous*. There exists *classical derivative* $\frac{dw}{dx} = v$, a.e. $x \in (a,b)$.
By statement 1) (already proved), there exists the *weak derivative* $\frac{dw}{dx}$ which coincides with the *classical* one and with v.
Thus,

$$\frac{du}{dx} = \frac{dw}{dx}, \text{ i. e. } \frac{d\,(u-w)}{dx} = 0.$$

(the *weak* derivative is equal to zero.)
By Theorem 4, $u - w = const$. Since $w(x)$ is absolutely consinuous, then $u = c + w$ *is* also *absolutely continuous*.
If $x = a$, we have $u(a) = c + \underbrace{w(a)}_{=0} = c$. Thus,

$$u(x) = u(a) + \int_a^x v(t)dt.$$

$u(x)$ is *absolutely continuous*; it has classical derivative for a. e. $x \in (a,b)$;
classical derivative = weak derivative = $v \in L_1(a,b)$.

■

Theorem 6

Let $\Omega \subset \mathbb{R}^n, n > 1$. We denote $x' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ and write $x = \{x', x_j\}$. Suppose that $[a(x'), b(x')]$ are some intervals such that $\{x'\} \times [a(x'), b(x')] \subset \Omega$.
Let $u \in L_{1,loc}(\Omega)$ and there exists the weak derivative $\frac{\partial u}{\partial x_j} \in L_{1,loc}(\Omega)$.
Then for almost every x' the function $u(x', x_j)$ is absolutely continuous on interval $[a(x'), b(x')]$ (as a function of one variable x_j).

Exercise: Prove Theorem 6.

7. Examples

1) Let $\Omega = (0,1)^2$ and $u(x_1, x_2) = \varphi(x_1) + \psi(x_2)$, where φ and ψ are *not* absolutely continuous on $[0,1]$, but $\varphi, \psi \in L_1(0,1)$.
Then, by Theorem 6, $u(x_1, x_2)$ *does not* have weak derivatives $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}$ in Ω. (Since, if they exist, then $u(x)$ must be also absolutely continuous in x_1 for x_2 fixed, and in x_2 for x_1 fixed.)
However, there exists the weak derivative $\frac{\partial^2 u}{\partial x_1 \partial x_2} = 0$.
Indeed, for $\forall\ \eta \in C_0^\infty(\Omega)$,

$$\begin{aligned}
\int_\Omega u \frac{\partial^2 \eta}{\partial x_1 \partial x_2} dx &= \int_0^1 \int_0^1 \varphi(x_1) \frac{\partial^2 \eta}{\partial x_1 \partial x_2} dx_1 dx_2 + \int_0^1 \int_0^1 \psi(x_2) \frac{\partial^2 \eta}{\partial x_1 \partial x_2} dx_1 dx_2 \\
&= \int_0^1 dx_1 \varphi(x_1) \underbrace{\left(\int_0^1 \frac{\partial^2 \eta}{\partial x_1 \partial x_2} \right) dx_2}_{=0} + \int_0^1 dx_2 \psi(x_2) \underbrace{\left(\int_0^1 \frac{\partial^2 \eta}{\partial x_1 \partial x_2} dx_1 \right)}_{=0} \\
&= 0.
\end{aligned}$$

By Definition 1 of weak derivative, it means that there exists weak derivative $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ and $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ =0. This example shows that functions may have derivative of higher order, not having derivatives of lower order.

2) Suppose that the domain $\Omega \subset \mathbb{R}^n$ is devided by a smooth (n-1)–dimensional surface Γ into two parts Ω_1 and Ω_2. So, $\Omega = \Omega_1 \cup$

$\Omega_2 \cup \Gamma$.
Let $u_1 \in C^1(\overline{\Omega_1}), u_2 \in C^1(\overline{\Omega_2})$,

$$u(x) = \begin{cases} u_1(x) & , x \in \Omega_1 \\ u_2(x) & , x \in \Omega_2 \end{cases}$$

If $u_1|_\Gamma \neq u_2|_\Gamma$, then, *in general*, weak derivatives *do not exist.*
Let $\vec{n}(x)$ be the unit normal vector to Γ exterior with respect to Ω_1. Since $u_k(x), \quad k = 1, 2$, is a C^1–function in $\overline{\Omega}_k$, we can integrate by parts in Ω_k: for $\eta \in C_0^\infty(\Omega)$ we have:

$$\begin{aligned} \int_\Omega u \frac{\partial \eta}{\partial x_j} dx &= \int_{\Omega_1} u_1 \frac{\partial \eta}{\partial x_j} dx + \int_{\Omega_2} u_2 \frac{\partial \eta}{\partial x_j} dx \\ &= -\int_{\Omega_1} \frac{\partial u_1}{\partial x_j} \eta \, dx - \int_{\Omega_2} \frac{\partial u_2}{\partial x_j} \eta \, dx + \\ &+ \int_\Gamma (u_1(x) - u_2(x)) \, \eta \cos\left(\angle(\vec{n}, 0x_j)\right) dS(x) \end{aligned}$$

If $u_1 = u_2$ *on* Γ (we have **NO** jump on Γ), then the integral over Γ is equal to zero. In this case, there exists the weak derivative $\frac{\partial u}{\partial x_j}$ and

$$\frac{\partial u}{\partial x_j} = \begin{cases} \frac{\partial u_1}{\partial x_j} & \text{in } \Omega_1 \\ \frac{\partial u_2}{\partial x_j} & \text{in } \Omega_2 \end{cases}$$

Also, if $\cos\left(\angle(\vec{n}, 0x_j)\right) = 0$, then there exists $\frac{\partial u}{\partial x_j}$. For example, if Γ is parallel to the axis $0x_j$, then $\cos\left(\angle(\vec{n}, 0x_j)\right) = 0$.
$\Rightarrow$ Even if $u_1|_\Gamma \neq u_2|_\Gamma$, the *tangential* derivative exists.
If $\int_\Gamma (u_1(x) - u_2(x)) \, \eta \cos\left(\angle(\vec{n}, 0x_j)\right) dS(x) \neq 0$, then $\frac{\partial u}{\partial x_j}$ does not exist.

<u>Exercise</u>
Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, and let $u(x) = |x|^\alpha, \quad \alpha > -n + 1$.
<u>Prove that</u> there exist the weak derivatives

$$\frac{\partial u}{\partial x_j} \quad \text{and} \quad \frac{\partial u}{\partial x_j} = \alpha x_j \, |x|^{\alpha - 2}, \quad j = 1, \ldots, n.$$

For this consider continuous functions

$$u^{(\delta)}(x) = \begin{cases} |x|^\alpha & , |x| > \delta \\ \delta^\alpha & , |x| \leq \delta \end{cases}$$

From the previous example we know that

$$\exists \quad \frac{\partial u^{(\delta)}}{\partial x_j} = \begin{cases} \alpha x_j \, |x|^{\alpha - 2} & , |x| > \delta \\ 0 & , |x| \leq \delta \end{cases}$$

<u>Check that</u> $u^{(\delta)} \stackrel{L_1(\Omega)}{\longrightarrow} u$ and $\frac{\partial u^{(\delta)}}{\partial x_j} \stackrel{L_1(\Omega)}{\longrightarrow} v_j := \alpha x_j \, |x|^{\alpha - 2}$.
Then, by Theorem 3, it follows that $\exists \frac{\partial u}{\partial x_j} = v_j$

§5: The Sobolev spaces $W_p^l(\Omega)$ and $\overset{\circ}{W}{}_p^l(\Omega)$

1. Definition of $W_p^l(\Omega)$ $\quad (1 \le p < \infty, l \in \mathbb{Z}_+)$

Definition

Suppose that $u \in L_p(\Omega)$ and there exist weak derivatives $\partial^\alpha u$ for any α with $|\alpha| \le l$ (*all* derivatives up to order l), such that

$$\partial^\alpha u \in L_p(\Omega), \quad |\alpha| \le l.$$

Then we say that $u \in W_p^l(\Omega)$.
We introduce the (standard) norm in $W_p^l(\Omega)$:

$$\|u\|_{W_p^l(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \le l} |\partial^\alpha u|^p \, dx \right)^{1/p}.$$

Remark

1) The norm $\sum_{|\alpha| \le l} \|\partial^\alpha u\|_{p,\Omega}$ is equivalent to the standard norm.
2) $W_p^0(\Omega) = L_p(\Omega)$.

Proposition

$W_p^l(\Omega)$ is complete.
In other words, $W_p^l(\Omega)$ is a Banach space.

Proof
Let $\{u_m\}$ be a fundamental sequence in $W_p^l(\Omega)$. It is equivalent to the fact that all sequences $\{\partial^\alpha u_m\}$ for $|\alpha| \le l$ are fundamental sequences in $L_p(\Omega)$. Since the space $L_p(\Omega)$ *is complete* , there exist functions $u, v_\alpha \in L_p(\Omega)$ such that

$$u_m \xrightarrow{L_p(\Omega)} u, \quad \partial^\alpha u_m \xrightarrow{L_p(\Omega)} v_\alpha \quad \text{as } m \to \infty.$$

Then also $u_m \to u$, $\partial^\alpha u_m \to v_\alpha$ in $L_{1,loc}(\Omega)$.
By Theorem 3, $v_\alpha = \partial^\alpha u$. Hence,

$$u_m \xrightarrow{W_p^l(\Omega)} u \quad \text{as } m \to \infty.$$

■

If $p = 2$, the space $W_2^l(\Omega)$ *is a Hilbert space* with the inner product

$$(u, v)_{W_2^l(\Omega)} = \int_\Omega \sum_{|\alpha| \le l} \partial^\alpha u(x) \overline{\partial^\alpha v(x)} dx.$$

For $W_2^l(\Omega)$ another notation $H^l(\Omega)$ is often used: $W_2^l(\Omega) = H^l(\Omega)$.

Using the properties of weak derivatives (see section 4 „Change of variables“ in § 4), we can show that the class $W_p^l(\Omega)$ is invariant with respect to smooth (C^l-class) change of variables.

Theorem 7

Let $f : \Omega \to \tilde{\Omega}$ be a diffeomorphism of class C^l, so that $f \in C^l(\overline{\Omega}),\ \ f^{-1} \in C^l(\overline{\tilde{\Omega}})$.
Then, if $u \in W_p^l(\Omega)$, then $\tilde{u} = u \circ f^{-1} \in W_p^l(\tilde{\Omega})$, and

$$c_1 \|u\|_{W_p^l(\Omega)} \le \|\tilde{u}\|_{W_p^l(\tilde{\Omega})} \le c_2 \|u\|_{W_p^l(\Omega)} . \tag{17}$$

The constants c_1, c_2 do not depend on u; they depend only on $\|f\|_{C^l(\overline{\Omega})}$ and $\left\|f^{-1}\right\|_{C^l(\overline{\tilde{\Omega}})}$.

Proof:

For simplicity, let us prove Theorem 7 in the case $l = 1$. We have

$$u \in W_p^1(\Omega), \quad \tilde{u}(y) = u(f^{-1}(y)).$$

By section 4 in §4, there exist the weak derivatives

$$\frac{\partial \tilde{u}}{\partial y_k} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial y_k}.$$

Let us check that $\frac{\partial \tilde{u}}{\partial y_k} \in L_p(\tilde{\Omega})$:

$$\begin{aligned} \left(\int_{\tilde{\Omega}} \left| \frac{\partial \tilde{u}}{\partial y_k} \right|^p dy \right)^{1/p} &= \left(\int_{\Omega} \left| \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial y_k} \right|^p |J(x)| \, dx \right)^{1/p} \\ &\le \sum_{j=1}^{n} \left(\max_{x \in \Omega} \left| \frac{\partial x_j}{\partial y_k} \right| |J(x)|^{1/p} \right) \left(\int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right|^p dx \right)^{1/p} \\ &\le c \sum_{j=1}^{n} \left(\int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right|^p dx \right)^{1/p} . \end{aligned}$$

Here $J(x) = \det\ f'(x)$ and the constant $c := \max_{j,k} \left(\max_{x \in \Omega} \left| \frac{\partial x_j}{\partial y_k} \right| |J(x)|^{1/p} \right)$ depends only on the norms $\|f\|_{C^1(\overline{\Omega})}$ and $\left\|f^{-1}\right\|_{C^1(\overline{\tilde{\Omega}})}$.
Also we have

$$\int_{\tilde{\Omega}} |\tilde{u}(y)|^p \, dy = \int_{\Omega} |u(x)|^p \, |J(x)| \, dx \le (\max |J(x)|) \int_{\Omega} |u(x)|^p \, dx.$$

Thus, $\|\tilde{u}\|_{W_p^1(\tilde{\Omega})} \le c_2 \|u\|_{W_p^1(\Omega)}$ with the constant c_2 depending only on $\|f\|_{C^1}$ and $\left\|f^{-1}\right\|_{C^1}$. Prove the lower estimate in (17) yourself (for this change the roles of u and $\tilde{u}$ in the argument).

∎

2. Definition of $\overset{\circ}{W}{}^l_p(\Omega)$

Definition

The closure of $C_0^\infty(\Omega)$ *in the norm of* $W_p^l(\Omega)$ is denoted by $\overset{\circ}{W}{}^l_p(\Omega)$.

So, $\overset{\circ}{W}{}^l_p(\Omega)$ is a subspace in the space $W_p^l(\Omega)$.

Proposition

Let $u \in \overset{\circ}{W}{}^l_p(\Omega)$, and let

$$\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^n \backslash \Omega. \end{cases}$$

Then $\tilde{u} \in W_p^l(\Omega_1)$ for any Ω_1 such that $\Omega \subset \Omega_1$. In particular, $\tilde{u} \in W_p^l(\mathbb{R}^n)$.

Proof

By definition of $\overset{\circ}{W}{}^l_p(\Omega)$, there exists a sequence $u_m \in C_0^\infty(\Omega)$ such that $u_m \overset{W_p^l(\Omega)}{\longrightarrow} u$ as $m \to \infty$. We put

$$\tilde{u}_m(x) = \begin{cases} u_m(x) & x \in \Omega \\ 0 & \text{otherwise .} \end{cases}$$

Then $\tilde{u}_m \in C_0^\infty(\Omega_1)$ and $\tilde{u}_m \overset{W_p^l(\Omega_1)}{\longrightarrow} \tilde{u}$ as $m \to \infty$
(since $\|\tilde{u}_m - \tilde{u}\|_{W_p^l(\Omega_1)} = \|u_m - u\|_{W_p^l(\Omega)}$).
Hence, $\tilde{u} \in \overset{\circ}{W}{}^l_p(\Omega_1)$.

■

Theorem 8

Let $u \in \overset{\circ}{W}{}^l_p(\Omega)$ and let

$$\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^n \backslash \Omega. \end{cases}$$

Then for mollifications $u_\rho(x)$ we have $u_\rho \overset{\rho \to 0}{\longrightarrow} u$ in $W_p^l(\Omega)$.

Proof

We have already proved that $\tilde{u} \in W_p^l(\mathbb{R}^n)$. Then $\partial^\alpha \tilde{u} \in L_p(\mathbb{R}^n)$, $|\alpha| \le l$. By property 4) and 5) of $\partial^\alpha u$ (mollification of the weak derivative),

$$\partial^\alpha \tilde{u}_\rho \overset{\rho\to 0}{\longrightarrow} \partial^\alpha \tilde{u} \text{ in } L_p(\Omega), \quad |\alpha| \le l.$$

It means that $\tilde{u}_\rho \overset{\rho\to 0}{\longrightarrow} \tilde{u}$ in $W_p^l(\Omega)$. But, by definition of $\tilde{u}$ and definition of mollification, $\tilde{u} = u$ in Ω, and $\tilde{u}_\rho = u_\rho$. So, $u_\rho \overset{\rho\to 0}{\longrightarrow} u$ in $W_p^l(\Omega)$.

■

Remark

If $u(x)$ is an arbitrary function in $W_p^l(\Omega)$, and $\tilde{u}(x)$ is defined as above, then, in general, $\tilde{u}(x)$ does not have weak derivatives in $\mathbb{R}^n$. (See example 2 in Section 7 of §4). So, in general, $\overset{\circ}{W}{}_p^l(\Omega) \ne W_p^l(\Omega)$.

3. Integration by parts

Proposition

Let $u \in W_p^l(\Omega)$ and $v \in \overset{\circ}{W}{}_p^l(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\int_\Omega \partial^\alpha u v dx \quad = \quad (-1)^{|\alpha|} \int_\Omega u\ \partial^\alpha v dx, \quad |\alpha| \le l. \tag{18}$$

Proof

Let $v_m \in C_0^\infty(\Omega)$ and $v_m \to v$ as $m \to \infty$ in $\overset{\circ}{W}{}_p^l(\Omega)$. By Definition 1 of the weak derivative $\partial^\alpha u$, we have

$$\int_\Omega \partial^\alpha u v_m dx \quad = \quad (-1)^{|\alpha|} \int_\Omega u\ \partial^\alpha v_m dx. \tag{19}$$

Let us show that

$$\int_\Omega \partial^\alpha u\ v_m\ dx \overset{m\to\infty}{\longrightarrow} \int_\Omega \partial^\alpha u\ v\ dx,$$
$$\int_\Omega u\ \partial^\alpha v_m\ dx \overset{m\to\infty}{\longrightarrow} \int_\Omega u\ \partial^\alpha v\ dx.$$

We have

$$\begin{aligned} \left| \int_\Omega \partial^\alpha u\ (v_m - v)\ dx \right| &\le \left(\int_\Omega |\partial^\alpha u|^p\, dx \right)^{1/p} \left(\int_\Omega |v_m - v|^{p'}\, dx \right)^{1/p'} \\ &\le \|u\|_{W_p^l(\Omega)} \|v_m - v\|_{W_{p'}^l(\Omega)} \\ &\to 0 \quad \text{as } m \to \infty; \end{aligned}$$

$$
\begin{aligned}
\left|\int_\Omega u\ \left(\partial^\alpha v_m - \partial^\alpha v\right)\ dx\right| &\le \left(\int_\Omega |u|^p\,dx\right)^{1/p}\left(\int_\Omega |\partial^\alpha v_m - \partial^\alpha v|^{p'}\,dx\right)^{1/p'}\\
&\le \|u\|_{W_p^l(\Omega)}\,\|v_m - v\|_{W_{p'}^l(\Omega)}\\
&\to 0 \quad \text{as } m\to\infty.
\end{aligned}
$$

Tending to the limit in (19) as $m \to \infty$, we obtain (18).

■

4. Separability

By $V_p^l(\Omega)$ we denote the linear space of all vector–valued functions $v = \{v_\alpha\}_{|\alpha|\le l}$ such that $v_\alpha \in L_p(\Omega)$, $|\alpha| \le l$. We introduce the norm in $V_p^l(\Omega)$:

$$
\|v\|_{V_p^l(\Omega)} = \sum_{|\alpha|\le l} \|v_\alpha\|_{p,\Omega}\,.
$$

Then $V_p^l(\Omega)$ is the direct product of a finite number (equal to the number of multi–indices α with $|\alpha| \le l$) of $L_p(\Omega)$. We know that $L_p(\Omega)$ is a *separable* Banach space if $1 \le p < \infty$. Then so is $V_p^l(\Omega)$.
Now, consider the transformation J from $W_p^l(\Omega)$ (equipped with the norm $||u||_{W_p^l(\Omega)} = \sum_{|\alpha|\le l} \|\partial^\alpha u\|_{p,\Omega}$, which is equivalent to the standard norm) to $V_p^l(\Omega)$:

$$
J : W_p^l(\Omega) \to V_p^l(\Omega), \quad Ju = \{\partial^\alpha u\}_{|\alpha|\le l}\,.
$$

Then J is a linear operator; it preserves the norm: $\|Ju\|_{V_p^l(\Omega)} = ||u||_{W_p^l(\Omega)}$; and J is injective. Such an operator is called *isometric.*
The range Ran $J = \tilde{V}_p^l(\Omega)$ is a linear set in $V_p^l(\Omega)$ consisting of vector–valued functions v of the form $\quad v = \{\partial^\alpha u\}_{|\alpha|\le l}\,, \quad u \in W_p^l(\Omega)$.
From Theorem 3 it follows that $\tilde{V}_p^l(\Omega)$ is a *closed subspace* of $V_p^l(\Omega)$. Hence, $\tilde{V}_p^l(\Omega)$ is separable together with $V_p^l(\Omega)$. (Since any subspace of some separable space is also separable.)
Since J is isometric, we can identify $W_p^l(\Omega)$ with $\tilde{V}_p^l(\Omega)$. It follows that $W_p^l(\Omega)$ *is separable if* $1 \le p < \infty$.

5. The space $W_p^l(\mathbb{R}^n)$

Proposition

$$
\overset{\circ}{W}{}_p^l(\mathbb{R}^n) = W_p^l(\mathbb{R}^n)\ . \text{ In other words, } C_0^\infty(\mathbb{R}^n) \text{ is dense in } W_p^l(\mathbb{R}^n).
$$

Proof

Let $\zeta \in C^\infty(\mathbb{R}_+)$ be such that

$$0 \le \zeta(t) \le 1, \quad \zeta(t) = 1 \text{ if } 0 \le t \le 1, \quad \zeta(t) = 0 \text{ if } t \ge 2.$$

Let $u \in W_p^l(\mathbb{R}^n)$. We put $u^{(R)}(x) = u(x)\zeta\left(\frac{|x|}{R}\right)$. Then

$$u^{(R)}(x) = u(x) \text{ if } |x| \le R, \quad u^{(R)}(x) = 0 \text{ if } |x| \ge 2R.$$

Note that derivatives $\partial_x^\beta \zeta\left(\frac{|x|}{R}\right)$ are uniformly bounded with respect to $R \ge 1$. Calculating the derivatives of $u^{(R)}(x)$, we obtain the inequality

$$\left|\partial^\alpha u^{(R)}(x)\right| \le c \sum_{|\beta| \le |\alpha|} \left|\partial^\beta u(x)\right|, \quad a.e.\ x \in \mathbb{R}^n.$$

Then for $|\alpha| \le l$ we have

$$\begin{aligned}
\left\|\partial^\alpha u^{(R)} - \partial^\alpha u\right\|_{p,\mathbb{R}^n} &= \left(\int_{\mathbb{R}^n} \underbrace{\left|\partial^\alpha u^{(R)}(x) - \partial^\alpha u(x)\right|^p}_{=0 \text{ for } |x| \le R} dx\right)^{1/p} \\
&= \left(\int_{|x|>R} \left|\partial^\alpha u^{(R)}(x) - \partial^\alpha u(x)\right|^p dx\right)^{1/p} \\
&\le c \sum_{|\beta| \le |\alpha|} \left(\int_{|x|>R} \left|\partial^\beta u(x)\right|^p dx\right)^{1/p} \\
&\to 0 \quad \text{as } R \to \infty
\end{aligned}$$

This expression tends to zero as $R \to \infty$, since $u \in W_p^l(\mathbb{R}^n)$, and so, $\left|\partial^\beta u\right|^p \in L_1$. Thus, $u^{(R)} \to u$ as $R \to \infty$ in $W_p^l(\mathbb{R}^n)$.

Now, we consider mollification $u_\rho^{(R)}$ of $u^{(R)}$.

Then $u_\rho^{(R)} \in C_0^\infty(\mathbb{R}^n)$ and $u_\rho^{(R)} \to u^{(R)}$ as $\rho \to 0$ in $W_p^l(\mathbb{R}^n)$.

It follows that $C_0^\infty(\mathbb{R}^n)$ is dense in $W_p^l(\mathbb{R}^n)$. Indeed, let $u \in W_p^l(\mathbb{R}^n)$ and let $\varepsilon > 0$. We find R so large that $\left\|u^{(R)} - u\right\|_{W_p^l(\mathbb{R}^n)} < \frac{\varepsilon}{2}$. Next, we find ρ so small that $\left\|u_\rho^{(R)} - u^{(R)}\right\|_{W_p^l(\mathbb{R}^n)} < \frac{\varepsilon}{2}$. Then $\left\|u_\rho^{(R)} - u\right\|_{W_p^l(\mathbb{R}^n)} < \varepsilon$.

■

6. The Friedrichs inequality

Theorem 9

If Ω is a *bounded* domain in $\mathbb{R}^n$, then for any function $u \in \overset{\circ}{W}{}^l_p(\Omega)$ we have

$$\|u\|_{p,\Omega} \le (diam\Omega)^l \text{—}u\text{—}_{p,l,\Omega}. \tag{20}$$

Here

$$\text{—}u\text{—}_{p,l,\Omega} = \left(\sum_{|\alpha|=l} \|\partial^\alpha u\|^p_{p,\Omega} \right)^{1/p}. \tag{21}$$

Proof

Since $C_0^\infty(\Omega)$ is dense in $\overset{\circ}{W}{}^l_p(\Omega)$, it suffices to prove (20) for $u \in C_0^\infty(\Omega)$.

1) So, let $u \in C_0^\infty(\Omega)$. Let Q be a cube with the edge $d = diam\Omega$, such that $\Omega \subset Q$. We *extend* $u(x)$ *by zero* to $Q \backslash \Omega$. We can choose the coordinate system so that $Q = \{x : 0 < x_j < d, j = 1, \dots n\}$.
Obviously,

$$u(x) = \int_0^{x_n} \frac{\partial u}{\partial x_n}(x', y)dy, \quad x \in Q.$$

Here $x = (\underbrace{x_1, \dots, x_{n-1}}_{x'}, x_n) = (x', x_n)$.

Then, by the Hölder inequality,

$$|u(x)|^p \le \left(\int_0^{x_n} \left| \frac{\partial u}{\partial x_n}(x', y) \right|^p dy \right) \underbrace{\left(\int_0^{x_n} 1dy \right)^{p/p'}}_{\le d^{p/p'}} \le d^{p/p'} \int_0^d \left| \frac{\partial u(x', x_n)}{\partial x_n} \right|^p dx_n$$

Here $\frac{1}{p} + \frac{1}{p'} = 1$.
We integrate both sides of this inequality:

$$\int_\Omega |u|^p dx = \int_Q |u|^p dx \le d^{p/p'} \left(\int_0^d dx_n \right) \left(\int_Q \left| \frac{\partial u}{\partial x_n} \right|^p dx \right) \overset{\frac{p}{p'}+1=p}{=} d^p \int_\Omega \left| \frac{\partial u}{\partial x_n} \right|^p dx.$$

We have proved that

$$\|u\|_{p,\Omega} \le (diam\Omega) \left(\int_\Omega \left| \frac{\partial u}{\partial x_n} \right|^p dx \right)^{1/p} \le (diam\Omega)\text{—}u\text{—}_{p,1,\Omega}. \tag{22}$$

This is inequality (20) for $l = 1$.

2) In order to prove (20) with $l > 1$, we iterate (22):

$$\int_\Omega \left|\frac{\partial u}{\partial x_n}\right|^p dx \le d^p \int_\Omega \left|\frac{\partial^2 u}{\partial x_n^2}\right|^p dx, \quad etc.$$

$$\Rightarrow \quad \int_\Omega |u|^p dx \le d^{lp} \int_\Omega \left|\frac{\partial^l u}{\partial x_n^l}\right|^p dx \le d^{lp} — u —^p_{p,l,\Omega}.$$

■

Remark

Inequality (20) is not valid for all $u \in W_p^l(\Omega)$.

Example

If Ω is a bounded domain and $u(x) = P_{l-1}(x)(\neq 0)$ is a polynomial of order $\le l-1$, then $— u —_{p,l,\Omega} = 0$, but $\|u\|_{p,\Omega} \neq 0$.

§6. Domains of *star* type

A natural question:
Can we approximate functions in $W_p^l(\Omega)$ by smooth functions?
The answer depends on domain Ω. We'll consider the class of domains for which the answer is „YES, we can".

Definition

We say that a bounded domain Ω is of star type with respect to a point 0, if any half–line starting at point 0 intersects $\partial\Omega$ only in one point.

Theorem 10

Let Ω be a bounded domain of star type with respect to a point 0. Then $C^\infty(\overline{\Omega})$ is dense in $W_p^l(\Omega)$.

Proof
Let us use the coordinate system with origin 0. Consider a sequence of domains $\Omega_k = \left\{x : \frac{k-1}{k}x \in \Omega\right\}, \quad k \in \mathbb{N}$.
Then $\Omega_{k+1} \subset \Omega_k$ and $\Omega \subset \Omega_k$.
Let $u \in W_p^l(\Omega)$. We put $u_k(x) = u(\frac{k-1}{k}x)$.
Clearly, $u_k \in W_p^l(\Omega_k)$. Let us show that $\|u_k - u\|_{W_p^l(\Omega)} \to 0$ as $k \to \infty$.
We have

$$\|u_k - u\|_{p,\Omega} = \left(\int_\Omega \left|u\left(\frac{k-1}{k}x\right) - u(x)\right|^p dx\right)^{1/p} \longrightarrow 0 \text{ as } k \to \infty.$$

This follows from the property of L_p–functions: if $u \in L_p(\Omega)$, then

$$\sup_{|z(x)| \le \frac{c}{k}} \int_\Omega |u(x + z(x)) - u(x)|^p \, dx \overset{k\to\infty}{\longrightarrow} 0.$$

(In our case $z(x) = -\frac{x}{k}$ and $|x| \le diam\Omega = d \Rightarrow |z(x)| \le \frac{d}{k}$.)
Let α be a multi–index with $|\alpha| \le l$. Then

$$\begin{aligned}
\|\partial^\alpha u_k - \partial^\alpha u\|_{p,\Omega} &= \left(\int_\Omega \left|\left(\frac{k-1}{k}\right)^{|\alpha|} \partial^\alpha u\left(\frac{k-1}{k}x\right) - \partial^\alpha u(x)\right| dx\right)^{1/p} \\
&\le \underbrace{\left(1 - \left(\frac{k-1}{k}\right)^{|\alpha|}\right)}_{\to 0 \text{ as } k\to\infty} \underbrace{\left(\int_\Omega \left|\partial^\alpha u\left(\frac{k-1}{k}x\right)\right|^p dx\right)^{1/p}}_{\le c\|u\|_{W_p^l(\Omega)}} + \\
&+ \underbrace{\left(\int_\Omega \left|\partial^\alpha u\left(\frac{k-1}{k}x\right) - \partial^\alpha u(x)\right|^p dx\right)^{1/p}}_{\to 0 \text{ as } k\to\infty}.
\end{aligned}$$

Hence, $u_k \overset{k\to\infty}{\longrightarrow} u$ in $W_p^l(\Omega)$. Consider mollifications $u_{k,\rho}(x)$. Then $u_{k,\rho} \in C^\infty(\overline{\Omega})$ and $u_{k,\rho} \overset{\rho\to 0}{\longrightarrow} u_k$ in $W_p^l(\Omega)$ (since Ω is bounded and $\Omega \subset\subset \Omega_k$). We can choose a sequence $\{\rho_k\}$, so that $\rho_k \to 0$ as $k \to \infty$, and a sequence $\tilde{u}_k(x) := u_{k,\rho_k}(x)$ tends to $u(x)$ in $W_p^l(\Omega)$:
$\tilde{u}_k \in C^\infty(\overline{\Omega})$ and
$\|\tilde{u}_k - u\|_{W_p^l(\Omega)} \overset{k\to\infty}{\longrightarrow} 0$.

■

<u>Remark</u>

> Let $\Omega = \{x : |x| < 1, x_n > 0\}$ be a half–ball. Ω is of star type with respect to any interior point $0'$. Suppose that $u \in W_p^l(\Omega)$ and $u(x) = 0$ if $|x| > 1 - \varepsilon$. Then $\tilde{u_k} \in C^\infty(\overline{\Omega})$ and $\tilde{u}_k(x) = 0$ if $|x| > 1 - \frac{\varepsilon}{2}$ for sufficiently large k.

§7: Extension theorems

We can always extend a function $u \in \overset{\circ}{W}{}^l_p(\Omega)$ by zero and the extended function $\in W^l_p(\tilde\Omega)$ in $\tilde\Omega$ ($\supset \Omega$). It is a natural question if we can extend functions of class $W^l_p(\Omega)$. We start with the case $l = 1$.

Theorem 11

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain such that $\overline{\Omega}$ is a compact manifold of class C^1. Let $\tilde\Omega$ be a domain in $\mathbb{R}^n$ such that $\overline{\Omega} \subset \tilde\Omega$. Then there exists a linear bounded *extension operator*

$$\Pi : W^1_p(\Omega) \to \overset{\circ}{W}{}^1_p(\tilde\Omega) \text{ such that } (\Pi u)(x) = u(x), \quad x \in \Omega.$$

Proof

We proceed in three steps:

Step 1

Let $\Omega = K_+ = \{x : |x| < 1, x_n > 0\}$ be a half–ball, and let $u \in W^1_p(K_+)$ and $u(x) = 0$ near $\Sigma_+ = \{x \in \partial K_+ : |x| = 1\}$. We extend u to the left half–ball $K_- = \{x : |x| < 1, x_n < 0\}$ as follows:

$$v(x) = \begin{cases} u(x) & x \in K_+ \\ u(x', -x_n) & x \in K_-. \end{cases}$$

Let us show that $v \in W^1_p(K)$ and

$$\|v\|_{W^1_p(K)} = 2^{1/p} \|u\|_{W^1_p(K_+)}. \tag{23}$$

Here $K = \{x : |x| < 1\}$. Using construction of Theorem 10 (and Remark after Theorem 10), we can find a sequence $u_m(x)$ such that $u_m \in C^\infty(\overline{K_+})$, $u_m(x) = 0$ near Σ_+, and $\|u_m - u\|_{W^1_p(K_+)} \to 0$ as $m \to \infty$. We put

$$v_m(x) = \begin{cases} u_m(x) & x \in K_+ \\ u_m(x', -x_n) & x \in K_-. \end{cases}$$

Then $v_m \in C(\overline{K})$, $v_m(x) = 0$ near ∂K, $v_m \in C^\infty(\overline{K_+})$, $v_m \in C^\infty(\overline{K_-})$. It follows that $v_m \in \overset{\circ}{W}{}^1_p(K)$ (see §4, Subsection 7, Example 2). For the norm of v_m we have:

$$\begin{aligned} \|v_m\|^p_{W^1_p(K)} &= \int_K (|v_m(x)|^p + |\nabla v_m(x)|^p)\, dx \\ &= 2 \int_{K_+} (|u_m(x)|^p + |\nabla u_m(x)|^p)\, dx \end{aligned}$$

$$\Rightarrow \quad \|v_m\|_{W^1_p(K)} = 2^{1/p} \|u_m\|_{W^1_p(K_+)}. \tag{24}$$

Next,

$$\frac{\partial v_m(x)}{\partial x_j} = \begin{cases} \frac{\partial u_m(x)}{\partial x_j} & x \in K_+ \\ \frac{\partial}{\partial x_j}\left(u_m(x', -x_n)\right) & x \in K_-. \end{cases}$$

Since $u_m \overset{m\to\infty}{\longrightarrow} u$ in $W_p^1(K_+)$, it follows that $v_m \overset{m\to\infty}{\longrightarrow} v$ in $L_p(K)$ and $\frac{\partial v_m}{\partial x_j} \overset{m\to\infty}{\longrightarrow} w_j$ in $L_p(K)$, where

$$w_j(x) = \begin{cases} \frac{\partial u(x)}{\partial x_j} & x \in K_+ \\ \frac{\partial u}{\partial x_j}(x', -x_n) & x \in K_- \end{cases}, \quad j = 1, \dots, n-1;$$

$$w_n(x) = \begin{cases} \frac{\partial u(x)}{\partial x_n} & x \in K_+ \\ -\left(\frac{\partial u}{\partial x_n}\right)(x', -x_n) & x \in K_-. \end{cases}$$

By Theorem 3, there exist weak derivatives $\frac{\partial v}{\partial x_j}$ in K and $\frac{\partial v}{\partial x_j} = w_j$. Thus, $v_m \overset{m\to\infty}{\longrightarrow} v$ in $W_p^1(K)$. Relation (23) follows from (24) by the limit procedure (as $m \to \infty$).

Step 2

Suppose that $u \in W_p^1(\Omega)$ and $supp\, u \subset U$, where U is a neighbourhood of $x^0 \in \partial\Omega$, such that $\overline{U} \subset \tilde{\Omega}$ and $\exists$ diffeomorphism
$f : U \to K, \quad f \in C^1(\overline{U}), \quad f^{-1} \in C^1(\overline{K}), \quad f(U) = K,$
$f(U \cap \Omega) = K_+, \quad f(U \cap \partial\Omega) = \partial K_+ \backslash \Sigma_+.$
We consider the function $\tilde{u}(y) = u(f^{-1}(y)), \ y \in K_+$. Then $\tilde{u} \in W_p^1(K_+)$ and $\tilde{u}(y) = 0$ near Σ_+.
We extend $\tilde{u}(y)$ on K_- like in step 1:

$$\tilde{v}(y) = \begin{cases} \tilde{u}(y) & y \in K_+ \\ \tilde{u}(y', -y_n) & y \in K_-. \end{cases}$$

As it was proved in step 1, $\tilde{v} \in \overset{\circ}{W}{}_p^1(K)$, and

$$\|\tilde{v}\|_{W_p^1(K)} = 2^{1/p} \|\tilde{u}\|_{W_p^1(K_+)}.$$

Consider the function $v(x) = \tilde{v}(f(x)), \ x \in U$. Then $v \in \overset{\circ}{W}{}_p^1(U)$. We extend $v(x)$ by zero on $\tilde{\Omega}\backslash U$. Then $v \in \overset{\circ}{W}{}_p^1(\tilde{\Omega}), \ v|_\Omega = u$, and

$$\|v\|_{W_p^1(\tilde{\Omega})} = \|v\|_{W_p^1(U)} \le c_1 \|\tilde{v}\|_{W_p^1(K)} \le c_1 2^{1/p} \|\tilde{u}\|_{W_p^1(K_+)} \le \underbrace{c_2 c_1 2^{1/p}}_{=c} \|u\|_{W_p^1(\Omega)}.$$

The constant c depends on $\|f\|_{C^1}$, $\left\|f^{-1}\right\|_{C^1}$ and on p.

Step 3 (general case)

Let Ω be a bounded domain such that $\overline{\Omega}$ is a compact manifold of class

C^1 with boundary $\partial\Omega$. Then (by definition of such manifolds) there exists a finite number of open sets $U_1, U_2, \ldots, U_N$ such that either $\overline{U_j} \subset \Omega$ or U_j is a neighbourhood of some point $x^{(j)} \in \partial\Omega$, and $\exists$ a diffeomorphism $f_j \in C^1(\overline{U_j}), \quad f_j^{-1} \in C^1(\overline{K}), \quad f_j(U_j) = K, \quad f_j(U_j \cap \Omega) = K_+$,
$f_j(U_j \cap \partial\Omega) = \partial K_+ \backslash \Sigma_+$. Finally, $\quad \overline{\Omega} \subset \bigcup_{j=1}^N U_j$.
We can choose the sets $U_1, U_2, \ldots, U_N$ so that $\bigcup_{j=1}^N \overline{U_j} \subset \tilde{\Omega}$.
There exists a partition of unity $\{\zeta_j(x)\}_{j=1,\ldots,N}$ such that
$\zeta_j \in C_0^\infty(\mathbb{R}^n), \quad supp\ \zeta_j \subset U_j, \quad \sum_{j=1}^N \zeta_j(x) = 1, \ \ x \in \overline{\Omega}$.
Let $u \in W_p^1(\Omega)$. We represent $u(x)$ as $u(x) = \sum_{j=1}^N u_j(x)$, where
$u_j(x) = \zeta_j(x) u(x)$.
If $\overline{U_j} \subset \Omega$, then $u_j \in \overset{\circ}{W}{}_p^1(\Omega)$ (since $supp\ \zeta_j \subset U_j$) and we can extend $u_j(x)$ by zero to $\tilde{\Omega} \backslash \Omega$:

$$v_j(x) = \begin{cases} u_j(x) & x \in \Omega \\ 0 & x \in \tilde{\Omega} \backslash \Omega. \end{cases}$$

If $U \cap \partial\Omega \neq \emptyset$, then $u_j(x)$ satisfies assumptions of step 2. By the result of step 2, we can extend $u_j(x)$ to some function $v_j \in \overset{\circ}{W}{}_p^1(\tilde{\Omega})$ such that $v_j(x) = u_j(x), \quad x \in \Omega$, and $\quad \|v_j\|_{W_p^1(\tilde{\Omega})} \leq c_j \|u_j\|_{W_p^1(\Omega)}$.
The constant c_j depends on $\|f_j\|_{C^1}$, $\left\| f_j^{-1} \right\|_{C^1}$ and on p. We put

$$\Pi u = v = \sum_{j=1}^N v_j.$$

Then $v \in \overset{\circ}{W}{}_p^1(\tilde{\Omega}), \quad v(x) = \sum_{j=1}^N v_j(x) = \sum_{j=1}^N u_j(x) = u(x), \quad x \in \Omega$, and

$$\|v\|_{W_p^1(\tilde{\Omega})} \leq \sum_{j=1}^N \|v_j\|_{W_p^1(\tilde{\Omega})} \leq \sum_{j=1}^N c_j \|u_j\|_{W_p^1(\Omega)} \leq c \sum_{j=1}^N \|u_j\|_{W_p^1(\Omega)},$$

where $c = \max_{1 \leq j \leq N} \{c_j\}$. Finally,

$$\|u_j\|_{W_p^1(\Omega)} = \|\zeta_j u\|_{W_p^1(\Omega)} \leq \hat{c_j} \|u\|_{W_p^1(\Omega)}.$$

The constant $\hat{c_j}$ depends on $\|\zeta_j\|_{C^1}$. Hence,

$$\|v\|_{W_p^1(\tilde{\Omega})} \leq c\hat{c} \|u\|_{W_p^1(\Omega)}, \quad \hat{c} = \sum_{j=1}^N \hat{c_j}.$$

Thus, we constructed the linear continuous extension operator
$\Pi : W_p^1(\Omega) \to W_p^1(\tilde{\Omega})$.

■

Remark

It is clear from the proof that for the constructed extension operator Π we have

$$\|v\|_{L_p(\tilde{\Omega})} \leq c \|u\|_{L_p(\Omega)}, \quad v = \Pi u.$$

The constant c depends on p, Ω and $\tilde{\Omega}$.

2.

Similar extension theorem is true for unbounded domain $\Omega \subset \mathbb{R}^n$ satisfying the following condition. Suppose that there exist bounded open sets $\{U_j\}, j \in \mathbb{N}$, such that $\overline{\Omega} \subset \bigcup_{j=1}^{\infty} U_j$. Here either $\overline{U_j} \subset \Omega$ or U_j is a neighbourhood of a point $x^{(j)} \in \partial\Omega$ and $\exists$ a diffeomorphism $f_j \in C^1(\overline{U_j})$,
$f_j^{-1} \in C^1(\overline{K})$, $f_j(U_j) = K$, $f_j(\Omega \cap U_j) = K_+$, $f_j(\partial\Omega \cap U_j) = \partial K_+ \backslash \Sigma_+$.
Moreover, suppose that the norms $\|f_j\|_{C^1(\overline{U_j})}$ and $\left\|f_j^{-1}\right\|_{C^1(\overline{K})}$ are uniformly bounded for all $j \in \mathbb{N}$. Suppose also that each point $x \in \Omega$ belongs only to a finite number $N(x)$ of sets U_j, and that $N(x) \leq N < \infty, \quad \forall x \in \Omega$.
(This means that the multiplicity of covering is finite.)

Theorem 12

Under the above conditions on $\Omega \subset \mathbb{R}^n$, let $\tilde{\Omega}$ be a domain in $\mathbb{R}^n$ such that $\bigcup_{j=1}^{\infty} \overline{U_j} \subset \tilde{\Omega}$. Then there exists a linear bounded extension operator

$$\Pi : W_p^1(\Omega) \to \overset{\circ}{W}{}_p^1(\tilde{\Omega})$$

such that $(\Pi u)(x) = u(x), \quad x \in \Omega$.

We omit the proof.

3. Now we consider the case $l > 1$

Theorem 13

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain such that $\overline{\Omega}$ is a compact manifold of class C^l. Let $\tilde{\Omega}$ be a domain in $\mathbb{R}^n$ such that $\overline{\Omega} \subset \tilde{\Omega}$. Then there exists a linear bounded *extension operator*

$$\Pi : W_p^l(\Omega) \to \overset{\circ}{W}{}_p^l(\tilde{\Omega}), \text{ i. e., } (\Pi u)(x) = u(x), \text{ for } x \in \Omega$$

and $\quad \|\Pi u\|_{W_p^l(\tilde{\Omega})} \leq c_1 \|u\|_{W_p^l(\Omega)}$.
Besides, $\quad \|\Pi u\|_{L_p(\tilde{\Omega})} \leq c_2 \|u\|_{L_p(\Omega)}$.
The constants c_1, c_2 depend on l, p, Ω and $\tilde{\Omega}$.

Proof
Like in the proof of Theorem 11, the question reduces to the case, where $\Omega = K_+$ and $u(x) = 0$ near Σ_+. Moreover, it suffices to consider smooth functions $u \in C^\infty(\overline{K_+})$. So, let $u \in C^\infty(\overline{K_+})$ and $u(x) = 0$ near Σ_+. We extend $u(x)$ by zero to $\mathbb{R}^n_+ \backslash K_+$. We put

$$v(x) = \begin{cases} u(x) & x \in K_+ \\ \sum_{j=0}^{l-1} c_j \; u(x', -2^j x_n) & x \in K_-. \end{cases}$$

The constants $c_j, j = 0, \dots, l-1$, are chosen so that

$$\frac{\partial^m v}{\partial x_n^m}(x', +0) = \frac{\partial^m v}{\partial x_n^m}(x', -0), \quad m = 0, \dots, l-1.$$

These conditions are equivalent to the following system of linear equations for $c_0, c_1, \dots, c_{l-1}$:

$$\sum_{j=0}^{l-1} \left(-2^j\right)^m c_j = 1, \quad m = 0, 1, \dots, l-1.$$

The determinant of this system is not zero.
Hence, such constants $c_0, \dots, c_{l-1}$ exist. It is easy to check that $v \in W_p^l(K)$, and

$$\|\partial^\alpha v\|_{p,K} \le c_\alpha \, \|\partial^\alpha u\|_{p,K_+}, \quad |\alpha| \le l.$$

(We use that $v \in C^\infty(\overline{K_+}), v \in C^\infty(\overline{K_-})$ and $v \in C^{l-1}(\overline{K})$.
Then $v \in W_p^l(K)$.) Hence , $\|v\|_{W_p^l(K)} \le c_1 \, \|u\|_{W_p^l(K_+)}$.

Obviously, $v(x) = 0$ near ∂K. So, $v \in \overset{\circ}{W}{}_p^l(K)$.
Next, for arbitrary domain Ω, we use the covering $\overline{\Omega} \subset \bigcup_{j=1}^N U_j$ and the partition of unity. The argument is the same as in proof of Theorem 11. The only difference is that we consider diffeomorphisms of class C^l.

■

Remark

1) The conclusion of Theorem 13 remains true under weaker assumptions on the domain Ω. It suffices to assume that Ω is domain of class C^1 (for arbitrary l!) or, even that Ω is Lipschitz domain (it means that diffeomorphisms $f_j, f_j^{-1} \in Lip_1$).

2) Extension theorems allow us to reduce the study of functions in $W_p^l(\Omega)$ to the study of functions in $\overset{\circ}{W}{}_p^l(\tilde{\Omega})$. In particular, from the fact that $C_0^\infty(\tilde{\Omega})$ is dense in $\overset{\circ}{W}{}_p^l(\tilde{\Omega})$ it follows that *$C^\infty(\overline{\Omega})$ is dense in $W_p^l(\Omega)$, if domain Ω satisfies conditions of Theorem 13.*

Chapter 2: Embedding Theorems

Introduction

Embedding theorems give relations between different functional spaces.

Definition

Let B_1 and B_2 be two Banach spaces. We say that B_1 *is embedded into* B_2 and write $B_1 \hookrightarrow B_2$, if for any $u \in B_1$ we have $u \in B_2$ and $\|u\|_{B_2} \le c\,\|u\|_{B_1}$, where the constant c does not depend on $u \in B_1$.
We define the embedding operator $J : B_1 \to B_2$, which takes $u \in B_1$ into the same element u considered as an element of B_2.

The fact that $B_1 \hookrightarrow B_2$ is equivalent to the fact that the embedding operator $J : B_1 \to B_2$ is continuous linear operator.
If $\quad \|u\|_{B_2} \le c\,\|u\|_{B_1}, \quad \forall u \in B_1, \quad$ then $\|J\|_{B_1 \to B_2} \le c$.

Definition

If $B_1 \hookrightarrow B_2$ and the embedding operator $J : B_1 \to B_2$ *is a compact operator*, then we say that B_1 *is compactly embedded into* B_2.

The compactness of operator J is equivalent to the fact that *any bounded set in* B_1 *is a compact set in* B_2.
Some embeddings are obvious.
For example, it is obvious that $W_p^{l_1}(\Omega) \hookrightarrow W_p^{l_2}(\Omega)$, if $l_1 > l_2$. In particular, $W_p^l(\Omega) \hookrightarrow L_p(\Omega),\ l > 0$. But the fact that for *bounded* domain Ω, these embeddings are *compact*, is non–trivial. (This is the Rellich embedding theorem.)
More general is the Sobolev embedding theorem : $W_p^l(\Omega) \hookrightarrow W_q^r(\Omega)$ under some conditions on p, l, q, r (with $q > p$ and $r < l$).
Another embedding theorem is that, if $pl > n$, then a function $u \in W_p^l(\Omega)$ is continuous (precisely, $u(x)$ coincides with a continuous function for a. e. $x \in \Omega$).
The trace embedding theorems show that functions in $W_p^l(\Omega)$ have traces on some surfaces of lower dimension.
The embedding theorems are very important for the modern analysis and boundary value problems.

§1: Integral operators in $L_p(\Omega)$

In order to prove embedding theorems, we need some auxiliary material about integral operators.

1.

Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^m$ be some *bounded* domains. We consider the integral operator

$$(\mathcal{K}u)(x) = v(x) = \int_\Omega K(x,y)u(y)dy, \quad x \in \mathcal{D}, \quad u \in L_p(\Omega) \quad (1 \le p < \infty).$$

We'll show that under some conditions on the kernel $K(x,y)$, the operator $\mathcal{K}$ is continuous or, even, compact from $L_p(\Omega)$ to $L_q(\mathcal{D})$, or from $L_p(\Omega)$ to $C(\overline{\mathcal{D}})$.
We always assume that $K(x,y)$ is a measurable function on $\mathcal{D} \times \Omega$, and K satisfies one or several of the following conditions:

a)
$$\int_\Omega |K(x,y)|^t \, dy \le M \quad \text{for a. e. } x \in \mathcal{D}, \text{ where } t \ge 1.$$

b)
$$\int_\mathcal{D} |K(x,y)|^s \, dx \le N \quad \text{for a. e. } y \in \Omega, \text{ where } s > 0.$$

c)
$$ess \sup_{x\in\mathcal{D}, y\in\Omega} |K(x,y)| \le L < \infty \quad \text{(K is bounded)}.$$

d)
$$\sup_{\substack{x,z\in\mathcal{D} \\ |x-y|\le\rho}} \sup_{y\in\Omega} |K(x,y) - K(z,y)| \le \varepsilon(\rho) \to 0 \text{ as } \rho \to 0$$

(K is continuous in x).

Lemma 1

If $K(x,y)$ satisfies conditions c) and d) , then $\mathcal{K} : L_p(\Omega) \to C(\overline{\mathcal{D}})$ is *compact operator.* Here $1 \le p < \infty$.

Proof
Let $u \in L_p(\Omega)$ and $v(x) = (\mathcal{K}u)(x)$. Then, by condition c),

$$|v(x)| \le L \int_\Omega |u(y)| \, dy \le L \left(\int_\Omega |u(y)|^p \, dy \right)^{1/p} \left(\int_\Omega 1^{p'} dy \right)^{1/p'} = L|\Omega|^{1/p'} \|u\|_{p,\Omega}, \quad (1)$$

where $\frac{1}{p}+\frac{1}{p'}=1$. (If $p=1$ then (1) is also true with $p'=\infty$, $|\Omega|^{1/p'}=1$.)
Next, if $|x-z|\le\rho \quad (x,z\in\mathcal{D})$, then, by condition d),

$$\begin{aligned}|v(x)-v(z)| &= \left|\int_\Omega (K(x,y)-K(z,y))\,u(y)dy\right| \\ &\le \varepsilon(\rho)\int_\Omega |u(y)|\,dy \\ &\le \varepsilon(\rho)|\Omega|^{1/p'}\,\|u\|_{p,\Omega}\,. \end{aligned} \tag{2}$$

From (1) and (2) it follows that, if u belongs to some *bounded* set in $L_p(\Omega)$: $\|u\|_{p,\Omega}\le c$, then the set of functions $\{v\}$ is uniformly bounded ($\|v\|_{C(\overline{\mathcal{D}})}\le L|\Omega|^{1/p'}c$) and equicontinuous ($|v(x)-v(z)|\le\varepsilon(\rho)|\Omega|^{1/p'}c$, if $|x-z|\le\rho$).
By the Arzela Theorem, this set is *compact in* $C(\overline{\mathcal{D}})$. It means that the operator $\mathcal{K}:L_p(\Omega)\to C(\overline{\mathcal{D}})$ is *compact.*

■

Lemma 2

1) If $p>1, \frac{1}{p}+\frac{1}{p'}=1$, and $K(x,y)$ satisfies conditions a) and b) with some $t<p'$ and $\frac{s}{p}+\frac{t}{p'}\ge 1$, then $v=\mathcal{K}u \ \in L_q(\mathcal{D})$ (for $u\in L_p(\Omega)$), where $q\ge p$ is defined from the relation $\frac{s}{q}+\frac{t}{p'}=1$.
We have

$$\|v\|_{q,\mathcal{D}}\le M^{1/p'}N^{1/q}\,\|u\|_{p,\Omega}\,,\quad u\in L_p(\Omega). \tag{3}$$

2) If $p=1$, and $K(x,y)$ satisfies condition b) with $s=q\ge 1$. Then $v=\mathcal{K}u\in L_q(\mathcal{D})$ and

$$\|v\|_{q,\mathcal{D}}\le N^{1/q}\,\|u\|_{1,\Omega}\,,\quad u\in L_1(\Omega). \tag{4}$$

3) If $p>1$, and $K(x,y)$ satisfies condition a) with $t=p'$, then $v=\mathcal{K}u\in L_\infty(\mathcal{D})$ and

$$\|v\|_{\infty,\mathcal{D}}\le M^{1/p'}\,\|u\|_{p,\Omega}\,,\quad u\in L_p(\Omega). \tag{5}$$

4) If $p=1$, and $K(x,y)$ satisfies condition c), then $v=\mathcal{K}u\in L_\infty(\mathcal{D})$ and

$$\|v\|_{\infty,\mathcal{D}}\le L\,\|u\|_{1,\Omega}\,,\quad u\in L_1(\Omega). \tag{6}$$

Proof

1) Let $p > 1$. Using that $\frac{s}{q} + \frac{t}{p'} = 1$, we obtain:

$$|K(x,y)u(y)| = \left(|K(x,y)|^{s/q}\,|u(y)|^{p/q}\right) |u(y)|^{1-\frac{p}{q}}\,|K(x,y)|^{t/p'}$$

We apply the Hölder inequality for the product of three functions:

$$\int_\Omega |f_1(y)f_2(y)f_3(y)|\,dy \le \left(\int_\Omega |f_1|^{p_1}\,dy\right)^{\frac{1}{p_1}} \left(\int_\Omega |f_2|^{p_2}\,dy\right)^{\frac{1}{p_2}} \left(\int_\Omega |f_3|^{p_3}\,dy\right)^{\frac{1}{p_3}}$$

with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$.
We take $p_1 = q,\;\; p_2 = \frac{pq}{q-p},\;\; p_3 = p'$. Then

$$\begin{aligned}
|v(x)| &\le \int_\Omega |K(x,y)u(y)|\,dy \\
&\le \left(\int_\Omega |K(x,y)|^s\,|u(y)|^p\,dy\right)^{\frac{1}{q}} \left(\int_\Omega |u(y)|^p\,dy\right)^{\frac{1}{p}-\frac{1}{q}} \underbrace{\left(\int_\Omega |K(x,y)|^t\,dy\right)^{\frac{1}{p'}}}_{\le M^{1/p'} \text{ (by cond. a))}} \\
&\le M^{\frac{1}{p'}}\,\|u\|_{p,\Omega}^{1-\frac{p}{q}} \left(\int_\Omega |K(x,y)|^s\,|u(y)|^p\,dy\right)^{\frac{1}{q}}.
\end{aligned}$$

Note that in the case $q = p$, we simply apply the ordinary Hölder inequality and obtain the same result. We have

$$\begin{aligned}
|v(x)|^q &\le M^{q/p'}\,\|u\|_{p,\Omega}^{q-p} \int_\Omega |K(x,y)|^s\,|u(y)|^p\,dy \\
\Rightarrow \int_{\mathcal{D}} |v(x)|^q\,dx &\le M^{q/p'}\,\|u\|_{p,\Omega}^{q-p} \int_{\mathcal{D}} dx \int_\Omega |K(x,y)|^s\,|u(y)|^p\,dy \\
&= M^{q/p'}\,\|u\|_{p,\Omega}^{q-p} \int_\Omega |u(y)|^p\,dy \underbrace{\left(\int_{\mathcal{D}} |K(x,y)|^s\,dx\right)}_{\le N \text{(by cond b))}} \\
&\le N M^{q/p'}\,\|u\|_{p,\Omega}^{q}.
\end{aligned}$$

This gives estimate (3).

2) Let $p = 1$ and $s = q \ge 1$. If $q > 1$, we have

$$|K(x,y)u(y)| = \left(|K(x,y)|^q\,|u(y)|\right)^{1/q} |u(y)|^{1/q'}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Then, by the Hölder inequality,

$$\begin{aligned}
|v(x)| &\le \int_\Omega |K(x,y)u(y)|\,dy \\
&\le \left(\int_\Omega |K(x,y)|^q\,|u(y)|\,dy\right)^{1/q} \left(\int_\Omega |u(y)|\,dy\right)^{1/q'}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \int_{\mathcal{D}} |v(x)|^q\,dx &\le \left(\int_{\mathcal{D}} dx \int_\Omega |K(x,y)^q|\,|u(y)|\,dy\right) \|u\|_{1,\Omega}^{q/q'} \\
&= \int_\Omega |u(y)|\,dy \underbrace{\left(\int_{\mathcal{D}} |K(x,y)|^q\,dx\right)}_{\le N(\text{ by cond b) with } s=q)} \|u\|_{1,\Omega}^{q/q'} \\
&\le N\,\|u\|_{1,\Omega}^q\,.
\end{aligned}$$

This implies (4) (in the case $q > 1$).
If $q = 1$, then

$$\begin{aligned}
|v(x)| &\le \int_\Omega |K(x,y)|\,|u(y)|\,dy \\
\Rightarrow \int_{\mathcal{D}} |v(x)|\,dx &\le \int_{\mathcal{D}} dx \int_\Omega |K(x,y)|\,|u(y)|\,dy \\
&= \int_\Omega |u(y)|\,dy \underbrace{\int_{\mathcal{D}} |K(x,y)|\,dx}_{\le N} \\
&\le N\,\|u\|_{1,\Omega}\,.
\end{aligned}$$

This implies (4) (in the case $q = 1$).

3) Let $p > 1$, and condition a) is satisfied with $t = p'$. Then, by the Hölder inequality,

$$\begin{aligned}
|v(x)| &\le \int_\Omega |K(x,y)u(y)|\,dy \\
&\le \underbrace{\left(\int_\Omega |K(x,y)|^{p'}\,dy\right)^{1/p'}}_{\le M^{1/p'}} \left(\int_\Omega |u(y)|^p\right)^{1/p} \\
&\le M^{1/p'}\,\|u\|_{p,\Omega}\,.
\end{aligned}$$

This yields (5).

4) Let $p = 1$ and K satisfies condition c). Then

$$|v(x)| \le \int_\Omega |K(x,y)|\,|u(y)|\,dy \le L \int_\Omega |u(y)|\,dy = L\,\|u\|_{1,\Omega}\,,$$

which gives (6).

■

Remark

Statements 1) and 2) mean that the operator $\mathcal{K} : L_p(\Omega) \to L_q(\mathcal{D})$ is continuous, and

$$\begin{aligned} \|\mathcal{K}\|_{L_p(\Omega)\to L_q(\mathcal{D})} &\le M^{1/p'} N^{1/q}, \quad p > 1; && (7) \\ \|\mathcal{K}\|_{L_p(\Omega)\to L_q(\mathcal{D})} &\le N^{1/q}, \quad p = 1. && (8) \end{aligned}$$

Under conditions of 3) and 4) the operator $\mathcal{K} : L_p(\Omega) \to L_\infty(\mathcal{D})$ is continuous and

$$\begin{aligned} \|\mathcal{K}\|_{L_p(\Omega)\to L_\infty(\mathcal{D})} &\le M^{1/p'}, \quad p > 1; && (9) \\ \|\mathcal{K}\|_{L_p(\Omega)\to L_\infty(\mathcal{D})} &\le L, \quad p = 1. && (10) \end{aligned}$$

2.

Now, we'll show that under some additional assumptions on $K(x, y)$, the operator $\mathcal{K}$ is compact. We'll assume that $K(x, y)$ can be approximated by $K_h(x, y)$ (as $h \to 0$) and $K_h(x, y)$ are bounded and continuous in x.

Lemma 3

Suppose that $K_h(x, y)$, $0 < h < h_0$, satisfies conditions c) and d) (where $L = L(h)$ and $\varepsilon(\rho) = \varepsilon(\rho; h)$ depend on h).

1) Suppose that $K(x, y)$ and $K_h(x, y)$, $0 < h < h_0$, satisfy conditions of Lemma 2(1) with common t, s, M, N, and

$$\begin{aligned} \int_\Omega |K_h(x, y) - K(x, y)|^t \, dy &\le m_h \xrightarrow{h\to 0} 0, \text{ for a. e. } x \in \mathcal{D}; && (11) \\ \int_{\mathcal{D}} |K_h(x, y) - K(x, y)|^s \, dx &\le n_h \xrightarrow{h\to 0} 0, \text{ for a. e. } y \in \Omega. && (12) \end{aligned}$$

Then the operator $\mathcal{K} : L_p(\Omega) \to L_q(\mathcal{D})$ is compact.

2) Suppose that $K(x, y)$, $K_h(x, y)$, $0 < h < h_0$, satisfy conditions of Lemma 2(2) with common $s = q, N$, and

$$\int_{\mathcal{D}} |K_h(x, y) - K(x, y)|^q \, dx \le n_h \xrightarrow{h\to 0} 0, \text{ for a. e. } y \in \Omega. \quad (13)$$

Then the operator $\mathcal{K} : L_1(\Omega) \to L_q(\mathcal{D})$ is compact.

3) Suppose that $K(x, y), K_h(x, y)$ satisfy conditions of Lemma 2(3) with common $t = p', M$, and condition (11) is satisfied with $t = p'$. Then the operator $\mathcal{K} : L_p(\Omega) \to C(\overline{\mathcal{D}})$ is compact. (Here $p > 1$.)

Proof

We denote
$$(\mathcal{K}_h u)(x) = v_h(x) = \int_\Omega K_h(x,y)u(y)dy.$$
By Lemma 1, the operator $\mathcal{K}_h : L_p(\Omega) \to C(\overline{\mathcal{D}})$ is *compact*.
Obviously, the embedding $C(\overline{\mathcal{D}}) \hookrightarrow L_q(\mathcal{D})$ (for a *bounded domain* $\mathcal{D}$) is continuous. Hence, the operator $\mathcal{K}_h : L_p(\Omega) \to L_q(\mathcal{D})$ is also *compact*.

1) From conditions (11), (12) and the estimate (7) it follows that
$$\|\mathcal{K}_h - \mathcal{K}\|_{L_p(\Omega)\to L_q(\mathcal{D})} \le m_h^{1/p'} n_h^{1/q} \to 0 \text{ as } h \to 0.$$
Thus, $\mathcal{K}$ is the limit in the operator norm of *compact* operators $\mathcal{K}_h$. Hence, $\mathcal{K} : L_p(\Omega) \to L_q(\mathcal{D})$ is *compact*.

2) Similarly, if $p = 1$, from condition (13) and estimate (8) it follows that
$$\|\mathcal{K}_h - \mathcal{K}\|_{L_1(\Omega)\to L_q(\mathcal{D})} \le n_h^{1/q} \to 0 \text{ as } h \to 0.$$
It follows that $\mathcal{K} : L_1(\Omega) \to L_q(\mathcal{D})$ is compact.

3) From condition (11) with $t = p'$ and estimate (5), it follows that
$$\|v_h - v\|_{\infty,\mathcal{D}} \le m_h^{1/p'} \|u\|_{p,\Omega}, \quad u \in L_p(\Omega).$$
Hence, $\|v_h - v\|_{\infty,\mathcal{D}} \to 0$ as $h \to 0$. Since $(\mathcal{K}_h u)(x) = v_h(x)$ is *uniformly continuous*, then $v(x)$ is also *uniformly continuous*: $v \in C(\overline{\mathcal{D}})$. Thus, the operator $\mathcal{K}$ maps $L_p(\Omega)$ into $C(\overline{\mathcal{D}})$, and
$$\|\mathcal{K}_h - \mathcal{K}\|_{L_p(\Omega)\to C(\overline{\mathcal{D}})} \le m_h^{1/p'} \to 0 \text{ as } h \to 0.$$
Since $\mathcal{K}_h : L_p(\Omega) \to C(\overline{\mathcal{D}})$ is compact and $\mathcal{K}_h \xrightarrow{h\to 0} \mathcal{K}$ in the operator norm, then $\mathcal{K} : L_p(\Omega) \to C(\overline{\mathcal{D}})$ is also *compact operator.*

■

3.

Now we apply Lemmas 1–3 to the study of the operator
$$(\mathcal{K}_j u)(x) = \int_\Omega \frac{x_j - y_j}{|x-y|^n} u(y)dy, \quad j = 1, \dots, n.$$
Here $x \in \Omega$ or $x \in \Omega_m$, where Ω_m is some section of Ω by m–dimensional hyper–plane ($m < n$). So, either $\mathcal{D} = \Omega$ or $\mathcal{D} = \Omega_m$. If $m = n$, we agree that $\Omega_n \equiv \Omega$.

Lemma 4

1) Suppose that $1 \le p \le n,\ \ n-p < m \le n,\ \ q \ge 1$ and $1 - \frac{n}{p} + \frac{m}{q} > 0$. Then the operator $\mathcal{K}_j : L_p(\Omega) \to L_q(\Omega_m)$ is *compact.* (If $m = n$, then $\Omega_m = \Omega$; if $m < n$, then Ω_m is arbitrary section of Ω by m–dimensional hyper–plane.)

2) If $p > n$, then the operator $\mathcal{K}_j : L_p(\Omega) \to C(\overline{\Omega}_m)$ is *compact.* In particular, $\mathcal{K}_j : L_p(\Omega) \to C(\overline{\Omega})$ is compact.

<u>Proof</u>
The proof is based on Lemmas 2 and 3.

1) <u>Case $1 < p \le n$</u>
Suppose that conditions 1) are satisfied and, moreover, that <u>$q \ge p > 1$</u>. We put

$$\theta = 1 - \frac{n}{p} + \frac{m}{q},\ \ t = \frac{n}{n-1+\theta} = \frac{n}{\frac{n}{p'} + \frac{m}{q}};$$

$$s = \frac{m}{n-1+\theta} = \frac{m}{\frac{n}{p'} + \frac{m}{q}}. \quad \text{Then } \frac{s}{q} + \frac{t}{p'} = 1.$$

Since $q \ge p$, then $\frac{s}{p} + \frac{t}{p'} \ge 1$.
Clearly, $t < p'$. Since $m \le n,\ \ q \ge p$, then $\theta \le 1$.
Hence, $t \ge 1$. Thus, the numbers t and s satisfy conditions of Lemma 2(1). Let us check that $K_j(x,y) = \frac{x_j - y_j}{|x-y|^n}$ satisfy conditions a) and b) with these t and s.
Note that $t(n-1) < n$ (since $t(n-1) < t(n-1+\theta) = n$) and $s(n-1) < m$ (since $s(n-1) < s(n-1+\theta) = m$). We have

$$\int_\Omega |K_j(x,y)|^t\, dy = \int_\Omega \frac{|x_j - y_j|^t}{|x-y|^{tn}} dy \le \int_\Omega \frac{dy}{|x-y|^{t(n-1)}}.$$

This integral converges since $t(n-1) < n$. Let $d = diam \Omega$, $B(x) = \{y \in \mathbb{R}^n : |x-y| \le d\}$. Obviously, $\overline{\Omega} \subset B(x)$. Then

$$\begin{aligned} \int_\Omega |K_j(x,y)|^t\, dy \quad &\le \quad \int_{B(x)} \frac{dy}{|x-y|^{t(n-1)}} \\ &\overset{\substack{y=x+r\xi \\ \xi\in\mathbb{S}^{n-1}}}{=} \quad \kappa_n \int_0^d \frac{r^{n-1} dr}{r^{t(n-1)}} \\ &= \quad \frac{\kappa_n d^{n-t(n-1)}}{n - t(n-1)}, \end{aligned}$$

where κ_n is the square of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.
Thus, condition a) is satisfied with

$$M = \frac{\kappa_n d^{n-t(n-1)}}{n - t(n-1)} < \infty.$$

Let us check condition b):

$$\int_{\Omega_m} |K_j(x,y)|^s \, dx \le \int_{\Omega_m} \frac{dx}{|x-y|^{s(n-1)}} \le \int_{\Omega_m} \frac{dx}{|x-y'|^{s(n-1)}},$$

where y' is the projection of point y onto the hyper–plane Π_m (which contains $\Omega_m : \Omega_m \subset \Pi_m$). The integral is finite, since $s(n-1) < m$. Consider the m–dimensional ball $B_m(y') = \{x \in \Pi_m : |x-y'| \le d\}$. Clearly, $\overline{\Omega_m} \subset B_m(y')$. Then

$$\int_{\Omega_m} |K_j(x,y)|^s \, dx \le \int_{B_m(y')} \frac{dx}{|x-y'|^{s(n-1)}} = \frac{\kappa_m d^{m-s(n-1)}}{m-s(n-1)}.$$

Hence, condition b) is satisfied with

$$N = \frac{\kappa_m d^{m-s(n-1)}}{m-s(n-1)} < \infty.$$

The constant N depends on m and on $d = diam\Omega$, but *it does not depend on* Ω_m (it is one and the same for all sections Ω_m of dimension m). Thus, conditions of Lemma 2(1) are satisfied and, therefore, $\mathcal{K}_j : L_p(\Omega) \to L_q(\Omega_m)$ is *continuous.* We want to prove that this operator is *compact.* For this, we want to find the operators $\mathcal{K}_{jh}$ satisfying conditions of Lemma 3.
Let $\Psi(r)$, $r \in [0,\infty)$, be a smooth function such that $\Psi \in C^\infty([0,\infty))$, $\Psi(r) = 0$ if $0 \le r \le \frac{1}{2}$, $\Psi(r) = 1$ if $r \ge 1$, and $0 \le \Psi(r) \le 1$, $\forall r$.
We put $\Psi_h(r) = \Psi(\frac{r}{h})$. Then $\Psi_h(r) = 0$ if $0 \le r \le \frac{h}{2}$,
$\Psi_h(r) = 1$ if $r \ge h$. Consider the kernels

$$K_{jh}(x,y) = \frac{x_j - y_j}{|x-y|^n} \Psi_h(|x-y|).$$

Obviously, $|K_{jh}(x,y)| \le |K_j(x,y)|$, $\forall x, y$. Hence, K_{jh} satisfy conditions a) and b) together with K with the same constants t, s, M and N. Clearly, $K_{jh}(x,y)$ are bounded:

$$|K_{jh}(x,y)| \le \frac{\Psi_h(|x-y|)}{|x-y|^{n-1}} \le \frac{1}{\left(\frac{h}{2}\right)^{n-1}} = L(h).$$

So, K_{jh} satisfy condition c). And, finally, K_{jh} is uniformly continuous in both variables. So, condition d) for K_{jh} is also satisfied. Let us check condition (11):

$$\begin{aligned}
\int_\Omega |K_{jh}(x,y) - K_j(x,y)|^t\, dy &= \int_{|x-y|<h} \frac{|x_j - y_j|^t}{|x-y|^{tn}} \underbrace{(1-\Psi_h(|x-y|))^t}_{\le 1} dy \\
&\le \int_{|x-y|<h} \frac{dy}{|x-y|^{t(n-1)}} \\
&= \frac{\kappa_n h^{n-t(n-1)}}{n-t(n-1)} \\
&\to 0 \quad \text{as } h \to 0.
\end{aligned}$$

Thus, (11) is true. Next,

$$\begin{aligned}
\int_{\Omega_m} |K_{jh}(x,y) - K_j(x,y)|^s\, dx &= \int_{\substack{\Omega_m \\ |x-y|<h}} \frac{|x_j - y_j|^s}{|x-y|^{sn}} (1-\Psi_h(|x-y|))^s\, dx \\
&\le \int_{\{x\in\Omega_m : |x-y'|<h\}} \frac{dx}{|x-y'|^{s(n-1)}} \\
&= \frac{\kappa_m h^{m-s(n-1)}}{m-s(n-1)} \\
&\to 0 \quad \text{as } h \to 0.
\end{aligned}$$

Thus, (12) is satisfied. Then all conditions of Lemma 3(1) are satisfied. Hence, the operator $\mathcal{K}_j : L_p(\Omega) \to L_q(\Omega_m)$ is *compact*. (Recall that we assumed $q \ge p > 1$).

2) If $1 \le q < p$, then we apply the result that $\mathcal{K}_j : L_p(\Omega) \to L_p(\Omega_m)$ is compact (i. e. , we apply 1) with $p = q$; condition $1 - \frac{n}{p} + \frac{m}{p} > 0$ is true, since $m > n - p$).
Since Ω_m is a *bounded* domain, then $L_p(\Omega_m) \hookrightarrow L_q(\Omega_m)$ (if $q < p$), and any compact set in $L_p(\Omega_m)$ is also compact in $L_q(\Omega_m)$. It follows that the operator $\mathcal{K}_j : L_p(\Omega) \to L_q(\Omega_m)$ is compact.

3) Case p=1
Condition $n - p < m \le n$ means that $n - 1 < m \le n$. Then $m = n$, so, now $\Omega_m = \Omega$. Next, condition $1 - \frac{n}{p} + \frac{m}{q} > 0$ means that $1 - n + \frac{n}{q} > 0 \;\Rightarrow\; 1 \le q < \frac{n}{n-1}$. Let us check that condition b) with $s = q$ is true:

$$\int_\Omega |K_j(x,y)|^q\, dx = \int_\Omega \frac{|x_j - y_j|^q}{|x-y|^{nq}} dx \le \int_\Omega \frac{dx}{|x-y|^{(n-1)q}}.$$

The integral is finite since $(n-1)q < n$

$$\Rightarrow \quad \int_\Omega |K_j(x,y)|^q\, dx \le \frac{\kappa_n d^{n-q(n-1)}}{n-q(n-1)}.$$

Also, condition (13) is satisfied:

$$\int_\Omega |K_{jh}(x,y) - K_j(x,y)|^q \, dx \leq \frac{\kappa_n h^{n-q(n-1)}}{n-q(n-1)} \to 0 \text{ as } h \to 0.$$

By Lemma 3(2), the operator $\mathcal{K}_j : L_1(\Omega) \to L_q(\Omega)$ is compact.

4) Case p ¿ n

Let us check that conditions of Lemma 3(3) are satisfied. Indeed, the kernels $K_j(x,y)$ (and $K_{jh}(x,y)$ with it) satisfy condition a) with $t = p'$:

$$\int_\Omega |K_j(x,y)|^{p'} \, dy \leq \int_\Omega \frac{dy}{|x-y|^{p'(n-1)}} \leq \frac{\kappa_n d^{n-p'(n-1)}}{n-p'(n-1)} < \infty.$$

Since $p > n$, then $\frac{1}{p} < \frac{1}{n}$, $\frac{1}{p'} = 1 - \frac{1}{p} > 1 - \frac{1}{n} = \frac{n-1}{n}$.
Hence $n > p'(n-1)$. Next,

$$\begin{aligned} \int_\Omega |K_{jh}(x,y) - K_j(x,y)|^{p'} \, dy &\leq \int_{|x-y| \leq h} \frac{dy}{|x-y|^{p'(n-1)}} \\ &\leq \frac{\kappa_n h^{n-p'(n-1)}}{n-p'(n-1)} \\ &\to 0 \text{ as } h \to 0. \end{aligned}$$

Thus, conditions of Lemma 3(3) are satisfied. It follows that the operator $\mathcal{K}_j : L_p(\Omega) \to C(\overline{\Omega})$ is *compact.*

■

Lemma 5

If $1 < p < n$, $n - p < m \leq n$, then the operator $\mathcal{K}_j : L_p(\Omega) \to L_{q^*}(\Omega_m)$ is continuous (but **not** *compact*), where

$$1 - \frac{n}{p} + \frac{m}{q^*} = 0 \quad \left(\Leftrightarrow q^* = \frac{mp}{n-p} \right).$$

Without proof.

§2: Embedding theorems for $W_p^1(\Omega)$

1. The integral representation for functions in $\overset{\circ}{W}{}_p^1(\Omega)$

Lemma 6

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $u \in \overset{\circ}{W}{}_p^1(\Omega)$. Then

$$u(x) = \frac{1}{\kappa_n} \sum_{j=1}^{n} \int_\Omega \frac{x_j - y_j}{|x-y|^n} \frac{\partial u}{\partial y_j} dy, \qquad \text{for a. e. } x \in \Omega. \tag{14}$$

Proof

1) First, assume that $u \in C_0^\infty(\Omega)$. Consider the fundamental solution of the Poisson equation $\triangle \mathcal{E}(z) = \delta(z)$:

$$\mathcal{E}(z) = \begin{cases} -\frac{1}{\kappa_n (n-2)|z|^{n-2}} & n > 2 \\ \frac{1}{\kappa_2} \ln|z| & n = 2. \end{cases}$$

Then for any $u \in C_0^\infty(\Omega)$ we have

$$u(x) = \int_\Omega \mathcal{E}(x-y)(\triangle u)(y) dy.$$

The function $\mathcal{E}(z)$ has weak derivatives

$$\frac{\partial \mathcal{E}(z)}{\partial z_j} = \frac{1}{\kappa_n} \frac{z_j}{|z|^n}.$$

Then, $\frac{\partial}{\partial y_j} \mathcal{E}(x-y) = -\frac{1}{\kappa_n} \frac{x_j - y_j}{|x-y|^n}$, $j = 1, \ldots, n$.
By Definition 1 of weak derivatives, we have

$$\int_\Omega \mathcal{E}(x-y)(\triangle u)(y) dy = -\sum_{j=1}^{n} \int_\Omega \frac{\partial \mathcal{E}(x-y)}{\partial y_j} \frac{\partial u}{\partial y_j} dy.$$

Then,

$$u(x) = \frac{1}{\kappa_n} \sum_{j=1}^{n} \int_\Omega \frac{x_j - y_j}{|x-y|^n} \frac{\partial u}{\partial y_j} dy, \quad \forall u \in C_0^\infty(\Omega).$$

2) Now, let $u \in \overset{\circ}{W}{}_p^1(\Omega)$, and let $u_k \in C_0^\infty(\Omega)$, $u_k \overset{k\to\infty}{\longrightarrow} u$ in $W_p^1(\Omega)$. For u_k we have

$$u_k(x) = \frac{1}{\kappa_n} \sum_{j=1}^{n} \int_\Omega \frac{x_j - y_j}{|x-y|^n} \frac{\partial u_k}{\partial y_j} dy.$$

Thus, $\quad u_k = \frac{1}{\kappa_n} \sum_{j=1}^n \mathcal{K}_j \left(\frac{\partial u_k}{\partial y_j} \right)$.
By Lemma 4, each operator $\mathcal{K}_j$ is compact from $L_p(\Omega)$ to $L_p(\Omega)$. We know that $u_k \stackrel{k\to\infty}{\longrightarrow} u$ in $L_p(\Omega)$ and $\frac{\partial u_k}{\partial y_j} \stackrel{k\to\infty}{\longrightarrow} \frac{\partial u}{\partial y_j}$ in $L_p(\Omega)$. Then $\mathcal{K}_j(\frac{\partial u_k}{\partial y_j}) \stackrel{k\to\infty}{\longrightarrow} \mathcal{K}_j(\frac{\partial u}{\partial y_j})$ in $L_p(\Omega)$.
(since $\mathcal{K}_j : L_p(\Omega) \to L_p(\Omega)$ is continuous operator.)
Hence, by the limit procedure as $k \to \infty$ we obtain:

$$u(x) = \frac{1}{\kappa_n} \sum_{j=1}^n \int_\Omega \frac{x_j - y_j}{|x-y|^n} \frac{\partial u}{\partial y_j} dy, \quad \forall u \in \overset{\circ}{W}{}_p^1(\Omega).$$

■

2. Embedding theorems for $\overset{\circ}{W}{}_p^1(\Omega)$

Theorem 1

Let $\Omega \subset \mathbb{R}^n$ be a *bounded* domain.

1) If $1 \le p \le n,\ m > n-p,\ q < \infty$ and $1 - \frac{n}{p} + \frac{m}{q} \ge 0$, then $\overset{\circ}{W}{}_p^1(\Omega)$ *is embedded into* $L_q(\Omega_m)$, where $\Omega_m = \Omega$ (if $m = n$) and Ω_m is any section of Ω by m–dimensional plane (if $m < n$). In the case $1 - \frac{n}{p} + \frac{m}{q} > 0$, *this embedding is compact.*
2) If $p > n$, then $\overset{\circ}{W}{}_p^1(\Omega)$ *is compactly embedded into* $C(\overline{\Omega})$.

Comments

1) Let us distinguish the case $m = n$ ($\Omega_m = \Omega$):

   If $1 \le p \le n$, $q < \infty$ and $q \le \frac{np}{n-p} = q^*$, then $\overset{\circ}{W}{}_p^1(\Omega) \hookrightarrow L_q(\Omega)$. If $q < q^*$, then this embedding *is compact.*

2) What does it mean that $\overset{\circ}{W}{}_p^1(\Omega) \hookrightarrow L_q(\Omega_m)$ in the case $m < n$?

   A function $u \in \overset{\circ}{W}{}_p^1(\Omega)$ is a measurable function in Ω; it can be changed on any set of measure zero; Ω_m is a set of measure zero.
   First we consider $u \in C_0^\infty(\Omega)$, and put $Tu = u|_{\Omega_m}$.
   Then $T : C_0^\infty(\Omega) \to C_0^\infty(\Omega_m)$ is a linear operator. This linear operator *can be extended* by continuity *to a continuous operator*
   $T : \overset{\circ}{W}{}_p^1(\Omega) \to L_q(\Omega_m)$.
   We have the estimate

$$\|Tu\|_{q,\Omega_m} \le c \, \|u\|_{W_p^1(\Omega)}, \quad \forall u \in C_0^\infty(\Omega).$$

Let $u \in \overset{\circ}{W}{}^1_p(\Omega)$. Then $\exists\ u_k \in C_0^\infty(\Omega),\ \|u_k - u\|_{W_p^1(\Omega)} \overset{k\to\infty}{\longrightarrow} 0$.
Then $\|Tu_k - Tu_j\|_{q,\Omega_m} \leq c\,\|u_k - u_j\|_{W_p^1(\Omega)} \overset{k,j\to\infty}{\longrightarrow} 0$.
Hence $\{Tu_k\}$ is a Cauchy sequence in $L_q(\Omega_m)$. There exists limit $Tu_k \overset{k\to\infty}{\longrightarrow} w$ in $L_q(\Omega_m)$. *By definition*, $w = Tu$.

Proof of Theorem 1

By D_j we denote operators $D_j u = \frac{\partial u}{\partial x_j}$. Then $D_j : \overset{\circ}{W}{}^1_p(\Omega) \to L_p(\Omega)$ is *continuous* operator, $j = 1, \dots, n$. Then representation (14) can be written as

$$u = \frac{1}{\kappa_n} \sum_{j=1}^{n} \mathcal{K}_j D_j u. \tag{15}$$

1) Suppose that $1 \leq p \leq n$, $m > n - p$, $q < \infty$ and $1 - \frac{n}{p} + \frac{m}{q} > 0$. Then conditions of Lemma 4(1) are satisfied. So, operator
$\mathcal{K}_j : L_p(\Omega) \to L_q(\Omega_m)$ is *compact*. Hence, the *embedding operator*

$$J = \kappa_n^{-1} \sum_{j=1}^{n} \mathcal{K}_j D_j : \overset{\circ}{W}{}^1_p(\Omega) \to L_q(\Omega_m)$$

is *compact*.
(We use the fact that if $A_1 : B_1 \to B_2$ is *continuous* operator and $A_2 : B_2 \to B_3$ is *compact* operator, then $A_2 A_1 : B_1 \to B_3$ is *compact*. Here B_1, B_2, B_3 are Banach spaces.)
If $p > 1$ and $1 - \frac{n}{p} + \frac{m}{q} = 0$ (i. e., $q = q^*$), then, by Lemma 5, the operator $\mathcal{K}_j : L_p(\Omega) \to L_q(\Omega_m)$ is *continuous*. Hence, the embedding operator

$$J = \kappa_n^{-1} \sum_{j=1}^{n} \mathcal{K}_j D_j : \overset{\circ}{W}{}^1_p(\Omega) \to L_q(\Omega_m)$$

is *continuous*.
For $p = 1$ – without proof.

2) Let $p > n$. Then, by Lemma 4(2), operators $\mathcal{K}_j : L_p(\Omega) \to C(\overline{\Omega})$ are *compact*. Hence, the embedding operator

$$J = \kappa_n^{-1} \sum_{j=1}^{n} \mathcal{K}_j D_j : \overset{\circ}{W}{}^1_p(\Omega) \to C(\overline{\Omega})$$

is *compact*.

■

Remark

1) Under conditions of Theorem 1(1), we have the estimate

$$\begin{aligned}\|u\|_{q,\Omega_m} &\le \kappa_n^{-1}\sum_{j=1}^{n}\|\mathcal{K}_j\|_{L_p(\Omega)\to L_q(\Omega_m)}\|D_ju\|_{L_p(\Omega)}\\ &\le c'\sum_{j=1}^{n}\|\partial_j u\|_{p,\Omega}\\ &\le c\,\|u\|_{W_p^1(\Omega)}\,,\quad u\in \overset{\circ}{W}{}_p^1(\Omega).\end{aligned}\tag{16}$$

2) Under conditions of Theorem 1(2), we have

$$\begin{aligned}\|u\|_{C(\overline{\Omega})} &\le \kappa_n^{-1}\sum_{j=1}^{n}\|\mathcal{K}_j\|_{L_p(\Omega)\to C(\overline{\Omega})}\|\partial_j u\|_{p,\Omega}\\ &\le c'\sum_{j=1}^{n}\|\partial_j u\|_{p,\Omega}\\ &\le c\,\|u\|_{W_p^1(\Omega)}\,,\quad u\in \overset{\circ}{W}{}_p^1(\Omega).\end{aligned}\tag{17}$$

Using the estimates from Lemma 4 it is easy to see that the constants in estimates (16), (17) depend only on $diam\Omega, n, m, p, q$, but they *do not depend on* Ω_m (they are one and the same for any section Ω_m).

3. Embedding theorems for $W_p^1(\Omega)$

Theorem 2

Let $\Omega\subset\mathbb{R}^n$ be a *bounded* domain of class C^1. Then both statements of Theorem 1 are true for $W_p^1(\Omega)$.

1) If $1\le p\le n$, $m>n-p$, $q<\infty$ and $1-\frac{n}{p}+\frac{m}{q}\ge 0$, then $W_p^1(\Omega)$ *is embedded into* $L_q(\Omega_m)$. In the case $1-\frac{n}{p}+\frac{m}{q}>0$, *this embedding is compact.*
2) If $p>n$, then $W_p^1(\Omega)$ *is compactly embedded into* $C(\overline{\Omega})$.

Proof

Let $\tilde\Omega\subset\mathbb{R}^n$ be a *bounded* domain such that $\overline{\Omega}\subset\tilde\Omega$. (For example, $\tilde\Omega$ is a ball of sufficiently large diameter.)

By Theorem 11 (Chapter 1), there exists a linear continuous extension operator $\Pi: W_p^1(\Omega)\to \overset{\circ}{W}{}_p^1(\tilde\Omega)$. If $u\in W_p^1(\Omega)$, then $v=\Pi u\in \overset{\circ}{W}{}_p^1(\tilde\Omega)$, and

$$\|v\|_{W_p^1(\tilde\Omega)}\le c\,\|u\|_{W_p^1(\Omega)}\,,\quad c=\|\Pi\|\,.$$

1) Under conditions of part 1), by Theorem 1, $\overset{\circ}{W}{}_p^1(\tilde{\Omega}) \hookrightarrow L_q(\tilde{\Omega}_m)$; if $1 - \frac{n}{p} + \frac{m}{q} > 0$, this embedding is compact. (Here $\Omega_m \subset \Pi_m$, where Π_m is m–dimensional plane, and $\tilde{\Omega}_m$ is the section of $\tilde{\Omega}$ *by the same* Π_m.)

a) Let $1 - \frac{n}{p} + \frac{m}{q} > 0$. Let $\mathfrak{M}$ be some bounded set in $W_p^1(\Omega)$.

Then $\Pi\mathfrak{M} = \{v = \Pi u : u \in \mathfrak{M}\}$ is a bounded set in $\overset{\circ}{W}{}_p^1(\tilde{\Omega})$. Then, by Theorem 1, this set is *compact* in $L_q(\tilde{\Omega}_m)$. Then $\mathfrak{M}$ is *compact* in $L_q(\Omega_m)$ (because functions in $\mathfrak{M}$ are restrictions of functions in $\Pi\mathfrak{M}$ back to Ω). Hence, $W_p^1(\Omega)$ *compactly embedded into* $L_q(\Omega_m)$.

b) Let $1 - \frac{n}{p} + \frac{m}{q} = 0$. In this case embedding $\overset{\circ}{W}{}_p^1(\tilde{\Omega}) \hookrightarrow L_q(\tilde{\Omega}_m)$ is *continuous* (but not compact). By similar arguments, we show that embedding $W_p^1(\Omega) \hookrightarrow L_q(\Omega_m)$ is also continuous.

2) Under condition $p > n$, by Theorem 1(2), $\overset{\circ}{W}{}_p^1(\tilde{\Omega})$ is compactly embedded into $C(\overline{\tilde{\Omega}})$. If $\mathfrak{M}$ is a bounded set in $W_p^1(\Omega)$, then $\Pi\mathfrak{M}$ is bounded set in $\overset{\circ}{W}{}_p^1(\tilde{\Omega})$; $\Pi\mathfrak{M}$ is *compact* in $C(\overline{\tilde{\Omega}})$. Hence, $\mathfrak{M}$ is compact in $C(\overline{\Omega})$.

■

Comments

1) Under conditions of Theorem2(1), let $J_\Omega : W_p^1(\Omega) \to L_q(\Omega_m)$ be the embedding operator and let $J_{\tilde{\Omega}} : \overset{\circ}{W}{}_p^1(\tilde{\Omega}) \to L_q(\tilde{\Omega}_m)$ be the embedding operator; $\Pi : W_p^1(\Omega) \to \overset{\circ}{W}{}_p^1(\tilde{\Omega})$ is the extension operator; $R : L_q(\tilde{\Omega}_m) \to L_q(\Omega_m)$ is the restriction operator. Then $J_\Omega = RJ_{\tilde{\Omega}}\Pi$. We have the estimate for all $u \in W_p^1(\Omega)$.

$$\begin{aligned}
\|u\|_{q,\Omega_m} &= \left\|RJ_{\tilde{\Omega}}\Pi u\right\|_{q,\Omega} \\
&\leq \left\|J_{\tilde{\Omega}}\Pi u\right\|_{q,\tilde{\Omega}_m} \\
&\leq \underbrace{\left\|J_{\tilde{\Omega}}\right\|_{\overset{\circ}{W}{}_p^1(\tilde{\Omega}) \to L_q(\tilde{\Omega}_m)}}_{=c_1} \underbrace{\|\Pi\|_{W_p^1(\Omega) \to \overset{\circ}{W}{}_p^1(\tilde{\Omega})}}_{=c_2} \|u\|_{W_p^1(\Omega)}
\end{aligned}$$

$$\Rightarrow \|u\|_{q,\Omega_m} \leq c\,\|u\|_{W_p^1(\Omega)}, \quad \forall u \in W_p^1(\Omega). \tag{18}$$

Compare (18) with estimate (16): in the case $u \in \overset{\circ}{W}{}_p^1(\Omega)$ we can estimate $\|u\|_{q,\Omega_m}$ by the norms of derivatives $\sum_{j=1}^n \|\partial_j u\|_{p,\Omega}$. Now it is impossible. (It is clear for $u = const \neq 0 : \|u\|_{q,\Omega_m} \neq 0$, but $\partial_j u \equiv 0$.)

2) Similarly, under conditions of Theorem 2(2), we have the estimate

$$\|u\|_{C(\overline{\Omega})} \leq C\,\|u\|_{W_p^1(\Omega)}\,, \quad \forall u \in W_p^1(\Omega). \tag{19}$$

The constants in estimates (18), (19) depend on $\|\Pi\|$ and, so, on the properties of $\partial\Omega$. (While constants in estimates (16), (17) depend only on $diam\Omega$ and on p, q, m, n.)

Let us formulate the analog of Theorem 2 for unbounded domain.

Theorem 3

Suppose that $\Omega \subset \mathbb{R}^n$ is unbounded domain satisfying conditions of Theorem 12 (Chapter 1). Then

1) If $p \geq 1$, $m > n - p$, $p \leq q < \infty$ and $1 - \frac{n}{p} + \frac{m}{q} \geq 0$, then $W_p^1(\Omega) \hookrightarrow L_q(\Omega_m)$.
2) If $p > n$, then $W_p^1(\Omega) \hookrightarrow C(\overline{\Omega})$.

Remark

1) In Theorem 3 embeddings are continuous, but **not** compact.
2) In part 1) we have condition $q \geq p$ (we don't need this condition in Theorem 2.).
3) If Ω is bounded and $p > n$, then 1) follows from 2). Now 1) does not follow from 2).

4. Comments. Examples.

All conditions in Theorems 2, 3 are *precise*.

1) If $1 - \frac{n}{p} + \frac{m}{q} < 0$, then $W_p^1(\Omega) \not\hookrightarrow L_q(\Omega_m)$.
Example.
Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$. Let $u(x) = |x|^\lambda$ with $1 - \frac{n}{p} < \lambda < -\frac{m}{q}$.
Then $u \in W_p^1(\Omega)$, but $u \notin L_q(\Omega)$.
Indeed, $|\nabla u| \leq c\,|x|^{\lambda-1}$,

$$\begin{aligned} \int_\Omega |\nabla u|^p\,dx &\leq c\int_\Omega |x|^{p(\lambda-1)}dx \\ &= c\kappa_n \int_0^1 r^{n-1+p(\lambda-1)}dr \\ &< \infty, \quad \text{since } n-1+p(\lambda-1) > -1 \Leftrightarrow \lambda > 1 - \frac{n}{p}. \end{aligned}$$

Also, $\int_\Omega |u|^p dx < \infty$. However,

$$\begin{aligned}\int_{\Omega_m} |u(y)|^q \, dy &= \int_{\Omega_m} |y|^{q\lambda} \, dy \\ &= \kappa_m \int_0^1 r^{m-1+q\lambda} dr \\ &= \infty, \quad \text{since } m-1+q\lambda < -1 (\Leftrightarrow \lambda < -\frac{m}{q}).\end{aligned}$$

Here Ω_m is a section of Ω by some m–dimensional plane Π_m such that point $0 \in \Pi_m$.

2) For unbounded domains, if $p < q$, then $W_p^1(\Omega) \not\hookrightarrow L_q(\Omega)$.
Example
Let $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$, $u(x) = |x|^\lambda$. Let $-\frac{n}{q} < \lambda < -\frac{n}{p}$. Then $u \in W_p^1(\Omega)$, but $u \notin L_q(\Omega)$. Check yourself.

3) The „*critical exponent*“ q^* is defined by the relation $1 - \frac{n}{p} + \frac{m}{q^*} = 0$. ($q^* = \frac{mp}{n-p}$). Here $p < n$. We have $q^* > p$, since $m > n - p$.
$W_p^1(\Omega) \hookrightarrow L_q(\Omega_m)$ for $q \le q^*$, but **not** for $q > q^*$. If $p \ge n$, then $W_p^1(\Omega) \hookrightarrow L_q(\Omega_m)$ for all $q < \infty$ (if Ω is bounded) and all $p \le q < \infty$ (if Ω is unbounded). If $p > n$, then $W_p^1(\Omega) \hookrightarrow C(\overline{\Omega})$.
But for $p = n > 1$, $W_n^1(\Omega) \not\hookrightarrow C(\overline{\Omega})$ and *even* $W_n^1(\Omega) \not\hookrightarrow L_\infty(\Omega)$.
(Here $q^* = \infty$.)
Example
Let $\Omega = \left\{x \in \mathbb{R}^n : |x| < \frac{1}{e}\right\}$. Consider $u(x) = \ln|\ln|x||$.
Then $u \in W_n^1(\Omega)$, but $u \notin L_\infty(\Omega)$. Indeed, $|\nabla u(x)| \le \frac{1}{|x||\ln|x||}$. Then

$$\begin{aligned}\int_\Omega |\nabla u(x)|^n dx &\le \int_\Omega \frac{dx}{|x|^n |\ln|x||^n} \\ &= \kappa_n \int_0^{1/e} \frac{r^{n-1} dr}{r^n |\ln r|^n} \\ &= \kappa_n \int_0^{1/e} \frac{dr}{r|\ln r|^n} \\ &< \infty.\end{aligned}$$

Also, $\int_\Omega |u(x)|^n dx < \infty$. Then $u \in W_n^1(\Omega)$.

4) If $p = n = 1$, $\Omega = (a, b)$, then any function $u \in W_1^1(\Omega)$ is *absolutely continuous*. This follows from Theorem 5 (Chapter 1).

5) For unbounded domains embeddings from Theorem 3 are *not* compact.
Example
Let $u \in C_0^\infty(\mathbb{R}^n)$ and let $\left\{x^{(k)}\right\}$ be a sequence of points $x^{(k)} \in \mathbb{R}^n$ such

that $\left|x^{(k)}\right| \to \infty$ as $k \to \infty$. We put $u_k(x) = u(x - x^{(k)})$. Then the set $\{u_k\}$ is bounded in $W_p^1(\mathbb{R}^n)$. Here $p > n$. (Obviously, $\|u_k\|_{W_p^1(\mathbb{R}^n)} = \|u\|_{W_p^1(\mathbb{R}^n)} = const.$) But the set $\{u_k\}$ is *not compact* in $C(\mathbb{R}^n)$. Indeed, suppose that there exists a subsequence u_{k_j} such that $u_{k_j} \stackrel{j\to\infty}{\longrightarrow} u_0$ in $C(\mathbb{R}^n)$. Since $u_{k_j} \stackrel{j\to\infty}{\longrightarrow} 0$ in $C(\overline{\Omega})$ for any bounded domain Ω (simply $u_{k_j} \equiv 0$ in Ω for sufficiently large j), then $u_0(x) \equiv 0$. But $\left\|u_{k_j}\right\|_{C(\mathbb{R}^n)} = \|u\|_{C(\mathbb{R}^n)} \neq 0$. *Contradication.*

Example

Let $u \in C_0^\infty(\mathbb{R}^n)$, and $v_k(x) = k^{-\frac{n}{p}} u\left(\frac{x}{k}\right)$. Then $v_k \in W_p^1(\mathbb{R}^n)$ and $\{v_k\}$ is bounded in $W_p^1(\mathbb{R}^n)$. But $\{v_k\}$ is *not compact* in $L_p(\mathbb{R}^n)$. Thus, the embedding $W_p^1(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$ is *not compact.*

6) For bounded domains Ω and $q = q^*$ embedding $W_p^1(\Omega) \hookrightarrow L_{q^*}(\Omega)$ is *not compact.*

Example

$\Omega = \{x : |x| < 1\}$, $u \in C_0^\infty(\mathbb{R}^n)$, $w_k(x) = k^{\frac{n}{p}-1} u(kx)$, $p < n$. Then $\{w_k\}$ is bounded in $W_p^1(\Omega)$, but $\{w_k\}$ is not compact in $L_{q^*}(\Omega)$.

Check this yourself.

5. Embeddings on submanifolds

Instead of the section of Ω by m–dimensional planes we can consider sections of Ω by some m–dimensional manifolds.

Theorem 4

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1. Let $1 \leq p \leq n$, $m > n - p$, $1 \leq q < \infty$ and $1 - \frac{n}{p} + \frac{m}{q} \geq 0$. Let $\Gamma \subset \mathbb{R}^n$ be a manifold of class C^1, $dim\Gamma = m$. Let $\Omega_\Gamma = \Gamma \cap \overline{\Omega}$. Then $W_p^1(\Omega) \hookrightarrow L_q(\Omega_\Gamma)$. If $1 - \frac{n}{p} + \frac{m}{q} > 0$, then this embedding is compact.

Without proof

(The proof is based on Theorem 2 and using of covering $\bigcup U_j$, diffemorphisms f_j and patition of unity.)

Important case

$\Gamma = \partial\Omega$ (then also $\Omega_\Gamma = \partial\Omega$). $dim\Gamma = n - 1$.

Conditions: $m = n - 1 > n - p \Rightarrow p > 1$, $1 - \frac{n}{p} + \frac{n-1}{q} \geq 0 \Leftrightarrow q \leq \frac{(n-1)p}{n-p} = q^*$.

If $q^* < \infty$ $(1 < p < n)$, then $W_p^1(\Omega) \hookrightarrow L_q(\partial\Omega)$, $\forall q \leq q^*$.

For $q < q^*$ this embedding is *compact.* If $n = p > 1$, then $q^* = \infty$,

$W_n^1(\Omega) \hookrightarrow L_q(\partial\Omega)$, $\forall q < \infty$.

§3: Embedding theorems for $W^l_p(\Omega)$

Theorem 5

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1.

1) If $p \geq 1$, $1 \leq q < \infty$, $0 \leq r < l$, $l - r - \frac{n}{p} + \frac{n}{q} \geq 0$, then $W^l_p(\Omega) \hookrightarrow W^r_q(\Omega)$. If $l - r - \frac{n}{p} + \frac{n}{q} > 0$, then this embedding is compact.
2) If $p(l - r) > n$, then $W^l_p(\Omega) \hookrightarrow C^r(\overline{\Omega})$ and this embedding is compact.

Proof

1) We put $s = l - r$ and fix the numbers $q_0, q_1, \ldots, q_s$ such that $q_j \geq 1$, $q_0 = p$, $q_s = q$ and $1 - \frac{n}{q_j} + \frac{n}{q_{j+1}} \geq 0$. Such numbers exist due to condition $s - \frac{n}{p} + \frac{n}{q} \geq 0$. If $l - r - \frac{n}{p} + \frac{n}{q} = 0$, then $q_0, \ldots, q_s$ are defined uniquely from the equations $1 - \frac{n}{q_j} + \frac{n}{q_{j+1}} = 0$, $j = 0, \ldots, s-1$. If $\theta = s - \frac{n}{p} + \frac{n}{q} > 0$, such numbers exist (but they are not unique).
By Theorem 2(1), $W^1_{q_j}(\Omega) \hookrightarrow L_{q_{j+1}}(\Omega)$.
It follows that $W^{l-j}_{q_j}(\Omega) \hookrightarrow W^{l-j-1}_{q_{j+1}}(\Omega)$. Indeed, let $u \in W^{l-j}_{q_j}(\Omega)$. Then $\partial^\alpha u \in W^1_{q_j}(\Omega)$ for $|\alpha| \leq l - j - 1$. Since $W^1_{q_j}(\Omega) \hookrightarrow L_{q_{j+1}}(\Omega)$, then $\partial^\alpha u \in L_{q_{j+1}(\Omega)}, |\alpha| \leq l - j - 1$, and

$$\|\partial^\alpha u\|_{q_{j+1},\Omega} \leq c \|\partial^\alpha u\|_{W^1_{q_j}(\Omega)} \leq \tilde{c} \|u\|_{W^{l-j}_{q_j}(\Omega)}$$

for all α with $|\alpha| \leq l - j - 1$.

$$\Rightarrow u \in W^{l-j-1}_{q_{j+1}}(\Omega) \text{ and } \|u\|_{W^{l-j-1}_{q_{j+1}}(\Omega)} \leq c \|u\|_{W^{l-j}_{q_j}(\Omega)}.$$

We denote the embedding operator by J_j,

$$J_j : W^{l-j}_{q_j}(\Omega) \to W^{l-j-1}_{q_{j+1}}(\Omega), \; j = 0, 1, \ldots, s-1.$$

J_j is a continuous operator. We have:

$$W^l_p(\Omega) = W^l_{q_0}(\Omega) \stackrel{J_0}{\to} W^{l-1}_{q_1}(\Omega) \stackrel{J_1}{\to} W^{l-2}_{q_2}(\Omega) \stackrel{J_3}{\to} \ldots \stackrel{J_{s-1}}{\to} W^{l-s}_{q_s}(\Omega) = W^r_q(\Omega).$$

$\Rightarrow$ The embedding operator $J : W^l_p(\Omega) \to W^r_q(\Omega)$ is represented as $J = J_{s-1} \ldots J_1 J_0$. Each operator J_j is continuous , then J is also continuous. If $\theta > 0$, then at least one of J_j is *compact* (at least for one index j we have $1 - \frac{n}{q_j} + \frac{n}{q_{j+1}} > 0$). In this case J is also *compact*.

2) Let $p(l-r) > n \Leftrightarrow l-r-\frac{n}{p} > 0$
Case a) $r = l-1$
$l-l+1-\frac{n}{p} > 0 \Leftrightarrow p > n$. By Theorem 2(2), the embedding $W_p^1(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact. It follows that $W_p^l(\Omega) \hookrightarrow C^{l-1}(\overline{\Omega})$ and this embedding is compact. (If $u \in W_p^l(\Omega)$, then $\partial^\alpha u \in W_p^1(\Omega) \hookrightarrow C(\overline{\Omega})$ for $|\alpha| \le l-1$.)
Case b) $r < l-1$
Then there exists a number q such that $q > n$ and $l-(r+1)-\frac{n}{p}+\frac{n}{q} > 0$. (Indeed, $l-r-\frac{n}{p} =: \varepsilon > 0$. We can find $q > n$ such that $1-\frac{n}{q} < \varepsilon$, i. e., $n < q < \frac{n}{1-\varepsilon}$.)
Then we can represent the embedding operator $J : W_p^l(\Omega) \to C^r(\overline{\Omega})$ as $J = J_2 J_1$, where $J_1 : W_p^l(\Omega) \to W_q^{r+1}(\Omega)$ (J_1 is compact by part 1) of Theorem 5) and $J_2 : W_q^{r+1} \hookrightarrow C^r(\overline{\Omega})$ (J_2 is compact by case a), since $q > n$).
Hence, $J : W_p^l(\Omega) \to C^r(\overline{\Omega})$ is compact.

■

Particular cases

1) Let $r = 0$, $pl < n$. The critical exponent q^* is defined from the condition $l-\frac{n}{p}+\frac{n}{q^*} = 0 \Leftrightarrow q^* = \frac{np}{n-lp}$. Since $pl < n$, then $q^* < \infty$. Embedding $W_p^l(\Omega) \to L_q(\Omega)$ is compact for $q < q^*$, and continuous for $q = q^*$.

2) If $pl = n$, then $q^* = \infty$. In this case $W_p^l(\Omega) \hookrightarrow L_q(\Omega)$ $\forall q < \infty$ (and this embedding is compact).
But $W_p^l(\Omega) \not\hookrightarrow L_\infty(\Omega)$.

3) If $pl > n$, then $W_p^l(\Omega) \hookrightarrow C(\overline{\Omega})$ and this embedding is compact.

4) Let $q = p$, $r < l$. Then embedding $W_p^l(\Omega) \hookrightarrow W_p^r(\Omega)$ is compact. In particular, embedding $W_p^l(\Omega) \hookrightarrow L_p(\Omega)$ (for $l \ge 1$) is compact.

Remarks

1) The embedding theorem for Ω_m with $m < n$ ($W_p^1(\Omega) \hookrightarrow L_q(\Omega_m)$) can be also generalized for $W_p^l(\Omega)$. However, for the proof we need another integral representation for $u \in \overset{\circ}{W}{}_p^l(\Omega)$ (including derivatives of higher order).

2) The embedding theorems for $W_p^l(\Omega)$ can be also generalized for the case of unbounded domains.

Equivalent norms in Sobolev spaces $W_p^l(\Omega)$

(lecture by prof. M. Birman)

1. Finitedimensional linear spaces and norms in these spaces

Let X be a linear space, $\dim X = N < \infty$. It means that there exists a system of linear independent elements $x_1, \dots, x_N \in X$, such that any $x \in X$ can be represented as a linear combination of $x_1, \dots, x_N$:

$$x = \sum_{k=1}^{N} \xi^k x_k, \quad ,\xi^k \in \mathbb{C} \quad ,k = 1, \dots N. \tag{1}$$

There is a one–to–one correspondence of elements $x \in X$ and coordinates $\xi = \left\{\xi^k\right\}_{k=1}^{N}$. We denote $\|x\| = \left(\sum_{k=1}^{N} \left|\xi^k\right|^2\right)^{1/2}$. Check yourself, that this functional has all properties of the norm. X is a Banach space with respect to this norm (i. e., the space X with this norm is complete).
The mapping $x \mapsto \xi$ is an isometric isomorphism of X and $\mathbb{C}^N$ (with the standard norm).

Proposition

Any other norm $\langle x\rangle$ on X is equivalent to $\|x\|$. Therefore, all norms on X are equivalent to each other.

Proof

From (1) it follows that

$$\langle x\rangle \le \sum_{k=1}^{N} \left|\xi^k\right| \langle x_k\rangle \le \left(\sum_{k=1}^{N} \left|\xi^k\right|^2\right)^{1/2} \left(\sum_{k=1}^{N} \langle x_k\rangle^2\right)^{1/2},$$

$$\text{i. e. },\ \langle x\rangle \le \gamma \|x\|, \quad \gamma = \left(\sum_{k=1}^{N} \langle x_k\rangle^2\right)^{1/2} > 0. \tag{2}$$

Now, let us prove the opposite inequality. Let us check that the function $\langle x\rangle$ is continuous on X with respect to $\|x\|$. From (2) and from the triangle inequality it follows that

$$\left|\langle x\rangle - \langle x'\rangle\right| \le \langle x - x'\rangle \le \gamma \left\|x - x'\right\|.$$

Now we restrict the continuous function $\langle x\rangle$ to the unit sphere $\|x\| = 1$. Then $\langle x\rangle$ is a continuous function of ξ on the closed bounded set $\left\{\xi \in \mathbb{C}^N : |\xi| = 1\right\}$ in $\mathbb{C}^N$. Since $\langle x\rangle > 0$, then by the Weierstrass Theorem, $\langle x\rangle \ge \beta > 0$ for $\|x\| = 1$. Then

$$\langle y\rangle = \|y\| \left\langle \frac{y}{\|y\|} \right\rangle \ge \beta \|y\|, \ \forall\, y \in X. \quad \text{Thus,} \quad \langle x\rangle \asymp \|x\|, \quad \forall\, x \in X. \tag{3}$$

■

2. „Trivial“ equivalent norms in $W_p^l(\Omega)$.

The standard norm in $W_p^l(\Omega)$, $l \in \mathbb{N}$, $1 \le p < \infty$, is

$$\|u\|_{W_p^l(\Omega)} = \left(\sum_{|\alpha| \le l} \|\partial^\alpha u\|^p_{L_p(\Omega)} \right)^{1/p}, \quad \Omega \subseteq \mathbb{R}^d. \tag{4}$$

Let N be the number of all multi–indices α with $|\alpha| \le l$. In $\mathbb{C}^N$ we introduce the norm of l_p–type by the formula

$$\|\vec{\eta}\|^p_{\mathbb{C}^N} = \sum_{s=1}^{N} |\eta^s|^p, \quad \vec{\eta} \in \mathbb{C}^N. \tag{5}$$

Then we can rewrite (4) as

$$\|u\|^p_{W_p^l(\Omega)} = \|\vec{\eta}\|^p_{\mathbb{C}^n}, \tag{6}$$

where $\vec{\eta} = \left\{ \|\partial^\alpha u\|_{L_p(\Omega)} \right\}$, $|\alpha| \le l$.
If we replace the norm (5) in relation (6) by any other (equivalent!) norm of vector $\vec{\eta}$ in $\mathbb{C}^N$, then (6) will automatically define some norm in $W_p^l(\Omega)$, which is equivalent to the standard one. Such new norms in $W_p^l(\Omega)$ are trivial.
Example
The norm $\|u\|_{L_p(\Omega)} + \max_{1 \le |\alpha| \le l} \|\partial^\alpha u\|_{L_p(\Omega)}$ is equivalent to the standard norm in $W_p^l(\Omega)$. Give yourself several examples of new „trivial“ norms in $W_p^l(\Omega)$.

3. The notion of seminorm.

Definition

A functional φ on a linear space X is called a *seminorm* on X, if

1) $0 \le \varphi(x) < \infty, \quad \forall\, x \in X$,
2) $\varphi(cx) = |c|\varphi(x), \quad \forall\, x \in X, \ \forall\, c \in \mathbb{C}$,
3) $\varphi(x_1 + x_2) \le \varphi(x_1) + \varphi(x_2)$.

Thus, a seminorm φ has all properties of the norm besides one: from $\varphi(x) = 0$ it does not follow $x = 0$.
Example
$X = W_p^l(\Omega)$, $\varphi(x) = \left| \int_\Omega u(x)dx \right|$. This functional is equal to zero for any $u \in W_p^l(\Omega)$ with zero mean value.

4. General theorem about equivalent norms in $W_p^l(\Omega)$.

Assume that $\Omega \subset \mathbb{R}^n$ *is bounded and* $\partial\Omega \in C^1$. By $\mathcal{P}_l$ we denote the class of all polynomials in $\mathbb{R}^n$ of order $\leq l-1$. Let φ be a *seminorm* on $W_p^l(\Omega)$ which satisfies properties:

4) $\varphi(u) \leq c\,\|u\|_{W_p^l(\Omega)}$ (It means that φ is bounded, and, therefore, continuous in $W_p^l(\Omega)$.)

5) If $u \in \mathcal{P}_l$ and $\varphi(u) = 0$, then $u = 0$ (φ is non–degenerate on the subspace $\mathcal{P}_l \subset W_2^l(\Omega)$).

Theorem

Let φ be a functional on $W_p^l(\Omega)$ satisfying conditions 1) – 5). Then the functional

$$\text{—}u\text{—}_{W_p^l(\Omega)} = \left(\sum_{|\alpha|=l} \|\partial^\alpha u\|^p_{L_p(\Omega)} + \varphi(u)^p \right)^{1/p} \tag{7}$$

defines the norm in $W_p^l(\Omega)$, which is equivalent to the standard norm.

Proof

Obviously, functional (7) is homogeneous and satisfies the triangle inequality. Next, if $\text{—}u\text{—}_{W_p^l(\Omega)} = 0$, then $\partial^\alpha u = 0$ for $\forall\ \alpha$ with $|\alpha| = l$. Then it follows that $u \in \mathcal{P}_l$. Besides, $\varphi(u) = 0$, and, by property 5), u=0. Thus, functional (7) is a *norm* on $W_p^l(\Omega)$.
Taking account of property 4), it suffices to check that

$$\|u\|_{W_p^l(\Omega)} \leq C\text{—}u\text{—}_{W_p^l(\Omega)}, \quad u \in W_p^l(\Omega). \tag{8}$$

Suppose the opposite. Then for any $C > 0$, (8) is not true. Then there exists a sequence $\{u_m\}$, $u_m \in W_p^l(\Omega)$ such that

$$m\text{—}u_m\text{—}_{W_p^l(\Omega)} \leq \|u_m\|_{W_p^l(\Omega)}\,. \tag{9}$$

We put $v_m = \frac{u_m}{\|u_m\|_{W_p^l(\Omega)}}$. Then, by (9),

$$\|v_m\|_{W_p^l(\Omega)} = 1, \tag{10}$$

$$\text{—}v_m\text{—}_{W_p^l(\Omega)} \leq \frac{1}{m} \to 0 \text{ as } m \to \infty. \tag{11}$$

Since the embedding $W_p^l(\Omega) \hookrightarrow W_p^{l-1}(\Omega)$ is *compact* it follows from (10) that there exists a subsequence $\left\{v_{m_j}\right\}$, which converges in $W_p^{l-1}(\Omega)$ to some $v_0 \in W_p^{l-1}(\Omega)$:

$$\left\|v_{m_j} - v_0\right\|_{W_p^{l-1}(\Omega)} \to 0 \text{ as } j \to \infty \tag{12}$$

From (11) it follows that

$$\left\|\partial^\alpha v_{m_j}\right\|_{L_p(\Omega)} \stackrel{j\to\infty}{\longrightarrow} 0 \text{ for } \forall\, \alpha \text{ with } |\alpha| = l. \qquad (13)$$

Since the operator ∂^α is closed in $L_p(\Omega)$, $\partial^\alpha v_0 = 0$ for $\forall\ \alpha$ with $|\alpha| = l$. Then by (12) and (13), we have

$$v_{m_j} \stackrel{W_p^l(\Omega)}{\longrightarrow} v_0 \text{ as } j \to \infty, \quad v_0 \in \mathcal{P}_l. \qquad (14)$$

From (11) it follows that $\varphi(v_{m_j}) \to 0$ as $j \to \infty$. By (14) and property 4), $\varphi(v_{m_j}) \to \varphi(v_0)$ as $j \to \infty$. Thus, $\varphi(v_0) = 0$, $v_0 \in \mathcal{P}_l$. By property 5), $v_0 = 0$. Together with (14) this contradicts to (10).

■

Mention that, in the proof of inequality (8), we did not use any explicit construction and we did not obtain any upper bound for the constant C. However, we have proved rather general theorem, which in particular cases implies a number of concrete inequalities (proved before by special tricks).
Control question
Where did we use that Ω is bounded and $\partial\Omega \in C^1$?

5. Examples. Additions.

1.
Let $l \geq 2$ and $\varphi(u) = \|u\|_{L_p(\Omega)}$. Conditions 1) - 2) are obviously satisfies. By Theorem, the norm

$$—u—_{W_p^l(\Omega)} = \left(\sum_{|\alpha|=l} \|\partial^\alpha u\|^p_{L_p(\Omega)} + \|u\|^p_{L_p(\Omega)} \right)^{1/p} \qquad (15)$$

is equivalent to the standard one. It follows that $\|\partial^\alpha u\|_{L_p(\Omega)}$, $0 < |\alpha| < l$, is estimated by the norm (15).
Exercise
In the case $p = 2$, $l = 2$, prove this estimate using Fourier transform.
2.
Let $l = 1$, $\omega \subseteq \Omega$, ω is a measurable set such that $mes_n \omega > 0$. Now $\mathcal{P}_l$ consists of constants. Let $\varphi(u) = \left|\int_\omega u(x)dx\right|$. Clearly, conditions 1) – 5) are satisfied. Then the Theorem implies that

$$\|u\|^p_{L_p(\Omega)} \leq C \left(\int_\Omega |\nabla u|^p \, dx + \left| \int_\omega u(x)dx \right|^p \right).$$

For $\omega = \Omega$ and $p = 2$ this is the classical *Poincare inequality.*

3.
Let $l = 1$, $\Gamma \subset \partial\Omega$, $mes_{d-1}\Gamma > 0$. We put $\varphi(u) = \left|\int_\Gamma udS\right|$.
Properties 1) – 5) are satisfied. Condition 4) follows from the estimate

$$\int_{\partial\Omega} |u|^p dS \leq C \|u\|^p_{W^l_p(\Omega)},$$

i. e. , from the trace embedding theorem. By Theorem (on equivalent norms) we obtain

$$\|u\|^p_{L_p(\Omega)} \leq C \left(\int_\Omega |\nabla u|^p \, dx + \left| \int_\Gamma udS \right|^p \right).$$

This generalizes and strengthens the Friedrichs inequality

$$\int_\Omega |u|^2 dx \leq C \left(\int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} |u|^2 dS \right).$$

4.
Let $l = 2$. $\mathcal{P}_2$ consists of linear functions, i. e. , of linear combinations of the basis funcrtions $1, x^1, \ldots, x^n$. Let $\omega \subseteq \Omega$ be a measurable set, $mes_d\omega > 0$. We put

$$\varphi(u) = \left| \int_\omega u(x)dx \right| + \sum_{k=1}^n \left| \int_\omega x^k u(x)dx \right|. \tag{16}$$

We have to check condition 5).
Consider $\mathcal{P}_2$ as a finite–dimensional subspace in $L_2(\omega)$. If $\varphi(u) = 0$, then u is orthogonal in $L_2(\omega)$ to the basis in $\mathcal{P}_2$. Then, if $u \in \mathcal{P}_2$, it follows that $u = 0$. Thus, the norm (7) with $l = 2$ and such $\varphi(u)$ is equivalent to the standard norm in $W^2_p(\Omega)$.
5.
Let $l = 2$, $\Gamma \subset \partial\Omega$, $mes_{d-1}\Gamma > 0$. We put

$$\varphi(u) = \int_\Gamma |u|dS. \tag{17}$$

Condition 4) follows from the trace embedding theorem. Let us check 5): $\varphi(u) = 0 \Leftrightarrow u|_\Gamma = 0$.
If $u \in \mathcal{P}_2$ ($u(x)$ is a linear function), then condition $u|_\Gamma = 0$ and $u \neq 0$ is equivalent to the fact that Γ is a plane part of the boundary, and $u(x) = 0$ is *equation* of this plane. In the case where Γ does not lie in some plane, from $u \in \mathcal{P}_2$, $u|_\Gamma = 0$, it follows that $u = 0$. Then the norm (7) with $l = 2$ and $\varphi(u)$ given by (17) is equivalent to the standard norm in $W^2_p(\Omega)$. In particular, it is always so, if $\Gamma = \partial\Omega$.

6.
In conclusion, we discuss one example, which does not follow from Theorem. The norm

$$—u—_{W_p^l(\Omega)} = \left(\sum_{k=1}^{n} \left\| \frac{\partial^l u}{\partial (x^k)^l} \right\|_{L_p(\Omega)}^p + \|u\|_{L_p(\Omega)}^p \right)^{1/p}$$

is equivalent to the standard one.
For example, in $W_2^2(\mathbb{R}^d) = H^2(\mathbb{R}^d)$, this fact follows from the inequality $2|\xi^j \xi^k| \le |\xi^j|^2 + |\xi^k|^2$.

Chapter 3: Sobolev spaces $H^s(\mathbb{R}^n)$

§1: Classes $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$. Fourier transform.

Definition

$S(\mathbb{R}^n)$ is a class of functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that for any multi–index α and any $k \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^k |\partial^\alpha \varphi(x)| < \infty.$$

$S(\mathbb{R}^n)$ is called the *Schwartz class.*

For $\varphi \in S(\mathbb{R}^n)$ all derivatives $\partial^\alpha\varphi(x)$ are rapidly decreasing as $|x| \to \infty$. We can introduce topology in $S(\mathbb{R}^n)$.

Definition

We say that $\varphi_m \overset{m\to\infty}{\longrightarrow} \varphi$ in $S(\mathbb{R}^n)$, if

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^k |\partial^\alpha \varphi_m(x) - \partial^\alpha\varphi(x)| \overset{m\to\infty}{\longrightarrow} 0, \ \ \forall\alpha, \ \forall k.$$

$S(\mathbb{R}^n)$ is a topological space, but **not** Banach space.

Definition

Let $f \in S(\mathbb{R}^n)$. We define the transformation $\mathcal{F} : f \mapsto \widehat{f}$,

$$\widehat{f}(\xi) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} f(x)e^{-ix\xi}dx.$$

$\mathcal{F}$ is called the *Fourier transformation.*

It is known that $\widehat{f} \in S(\mathbb{R}^n)$, if $f \in S(\mathbb{R}^n)$. So, $\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is a linear operator. The inverse transformation $\mathcal{F}^{-1}$ is given by the formula

$$f(x) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\xi}d\xi,$$

$\mathcal{F}^{-1} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$.
It is known that the Fourier transform $\mathcal{F}$ can be extended by continuity to $L_2(\mathbb{R}^n)$, and $\mathcal{F}$ is *unitary* operator in $L_2(\mathbb{R}^n)$:

$$\mathcal{F} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n),$$

$$\int_{\mathbb{R}^n} |f(x)|^2 \, dx = \int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right|^2 d\xi, \ \ f \in L_2(\mathbb{R}^n).$$

Definition

> By $S'(\mathbb{R}^n)$ we denote the dual space to $S(\mathbb{R}^n)$ (i. e., the space of linear continuous functionals on $S(\mathbb{R}^n)$).

Sometimes, $S'(\mathbb{R}^n)$ is called the space of slowly increasing distributions.
If $v \in S'(\mathbb{R}^n)$, $\varphi \in S(\mathbb{R}^n)$, by $\langle v, \varphi \rangle$ we denote the meaning of functional v on function φ.
The Fourier transformation is extended to the class $S'(\mathbb{R}^n)$.

Definition

> Let $f \in S'(\mathbb{R}^n)$. A functional $\widehat{f} \in S'(\mathbb{R}^n)$ is called the Fourier image of f, if
>
> $$\left\langle \widehat{f}, \varphi \right\rangle := \langle f, \widehat{\varphi} \rangle, \quad \forall \varphi \in S(\mathbb{R}^n).$$
>
> It is known that $\mathcal{F} : S'(\mathbb{R}^n) \xrightarrow{onto} S'(\mathbb{R}^n)$, $\mathcal{F}^{-1} : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$.

§2: Spaces $H^s(\mathbb{R}^n)$

1. Definition of $H^s(\mathbb{R}^n)$

We know that the spaces $W_2^l(\Omega)$ ($l \in \mathbb{N}$) are *Hilbert* spaces : $W_2^l(\Omega) = H^l(\Omega)$. Let $\Omega = \mathbb{R}^n$. We can use the Fourier transform and express the norm in $W_2^l(\mathbb{R}^n) = H^l(\mathbb{R}^n)$ in terms of the Fourier image. Let $u \in H^l(\mathbb{R}^n)$. Consider the Fourier image

$$\widehat{u}(\xi) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} u(x) e^{-ix\xi} dx.$$

Then

$$u(x) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} \widehat{u}(\xi) e^{ix\xi} d\xi.$$

For the derivatives $\partial^\alpha u(x)$, we have

$$\widehat{\partial^\alpha u}(\xi) = (i\xi)^\alpha \, \widehat{u}(\xi) = i^{|\alpha|} \xi^\alpha \widehat{u}(\xi).$$

Then

$$\|u\|^2_{H^l(\mathbb{R}^n)} = \sum_{|\alpha| \le l} \int_{\mathbb{R}^n} |\partial^\alpha u|^2 \, dx = \int_{\mathbb{R}^n} \left(\sum_{|\alpha| \le l} |\xi^\alpha|^2 \right) |\widehat{u}(\xi)|^2 \, d\xi.$$

Since $c_1 (1 + |\xi|^2)^l \le \sum_{|\alpha| \le l} |\xi^\alpha|^2 \le c_2 (1 + |\xi|^2)^l$ (prove this!), then

$$c_1 \int_{\mathbb{R}^n} (1 + |\xi|^2)^l \, |\widehat{u}(\xi)|^2 \, d\xi \le \|u\|^2_{H^l(\mathbb{R}^n)} \le c_2 \int_{\mathbb{R}^n} (1 + |\xi|^2)^l \, |\widehat{u}(\xi)|^2 \, d\xi.$$

Thus, the norm $\left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^l \, |\widehat{u}(\xi)|^2 \, d\xi \right)^{1/2}$ is equivalent to the standard norm in $W_2^l(\mathbb{R}^n)$. We introduce the space with this norm; now we consider arbitrary l (not only $l \in \mathbb{N}$).

Definition

$H^s(\mathbb{R}^n) = \left\{ u \in S'(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \, |\widehat{u}(\xi)|^2 \, d\xi < \infty \right\}, \quad s \in \mathbb{R}^n.$

The inner product in $H^s(\mathbb{R}^n)$ is defined by

$$(u, v)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi.$$

Theorem 1

$H^s(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\|_{H^s}$.

Proof

1) Let us show that any $u \in H^s(\mathbb{R}^n)$ can be approximated by functions in $C_0^\infty(\mathbb{R}^n)$. If $u \in H^s(\mathbb{R}^n)$, then $u_*(\xi) = \widehat{u}(\xi)(1+|\xi|^2)^{s/2} \in L_2(\mathbb{R}^n)$. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L_2(\mathbb{R}^n)$, there exists a sequence $v_k \in C_0^\infty(\mathbb{R}^n)$ such that $v_k(\xi) \overset{k\to\infty}{\longrightarrow} u_*(\xi)$ in $L_2(\mathbb{R}^n)$. We put $v_k(\xi)(1+|\xi|^2)^{-s/2} = w_k(\xi)$. Then $w_k \in C_0^\infty(\mathbb{R}^n)$ and $w_k(\xi)(1+|\xi|^2)^{s/2} \to u_*(\xi)$ in $L_2(\mathbb{R}^n)$. Obviously, $w_k \in S(\mathbb{R}^n)$. We put $u_k = \mathcal{F}^{-1} w_k$. Then also $u_k \in S(\mathbb{R}^n)$ and $w_k = \widehat{u}_k$. Since $\widehat{u}_k(\xi)(1+|\xi|^2)^{s/2} \overset{k\to\infty}{\longrightarrow} u_*(\xi) = \widehat{u}(\xi)(1+|\xi|^2)^{s/2}$ in $L_2(\mathbb{R}^n)$, then $u_k \overset{k\to\infty}{\longrightarrow} u$ in $H^s(\mathbb{R}^n)$.
It remains to approximate functions $u_k \in S(\mathbb{R}^n)$ by functions $u_{kj} \in C_0^\infty(\mathbb{R}^n)$ (in the H^s–norm). For this, we fix $h \in C_0^\infty(\mathbb{R}^n)$ such that $h(x) = 1$ for $|x| \le 1$. We put $u_{kj}(x) = u_k(x) h\left(\frac{x}{j}\right)$. Then $u_{kj} \in C_0^\infty(\mathbb{R}^n)$ and

$$\begin{aligned}
\|u_{kj} - u_k\|^2_{H^s} &= \int_{\mathbb{R}^n} |\widehat{u}_{kj}(\xi) - \widehat{u}_k(\xi)|^2 \left(1+|\xi|^2\right)^s d\xi \\
&\overset{l\ge s, l\in\mathbb{N}}{\le} \int_{\mathbb{R}^n} |\widehat{u}_{kj}(\xi) - \widehat{u}_k(\xi)|^2 \left(1+|\xi|^2\right)^l d\xi \\
&= \|u_{kj} - u_k\|^2_{H^l} .
\end{aligned}$$

For $l \in \mathbb{N}$ we can use another norm (which is equivalent to the standard one):

$$\begin{aligned}
\|u_{kj} - u_k\|^2_{H^l} &\le c \sum_{|\alpha|\le l} \int_{\mathbb{R}^n} \left| \partial^\alpha u_k(x) \left(1 - h\left(\frac{x}{j}\right)\right)\right|^2 dx \\
&\le \tilde{c} \sum_{|\beta|\le l} \int_{|x|>j} |\partial^\alpha u_k(x)|^2 dx \quad \left(\text{since } h\left(\frac{x}{j}\right) = 1 \text{ for } |x| \le j.\right) \\
&\to 0 \text{ as } j \to \infty.
\end{aligned}$$

It follows that $u_{kj} \overset{j\to\infty}{\longrightarrow} u_k$ in $H^s(\mathbb{R}^n)$.

2) Let us show that *each element of the closure* of $C_0^\infty(\mathbb{R}^n)$ in H^s–norm *belongs* to $H^s(\mathbb{R}^n)$. Suppose that $u_m \in C_0^\infty(\mathbb{R}^n)$ and $\{u_m\}$ is the Cauchy sequence in $H^s(\mathbb{R}^n)$, i. e. , $\|u_m - u_l\|_{H^s(\mathbb{R}^n)} \to 0$ as $m, l \to \infty$. It means that $\widehat{u}_m(\xi)(1+|\xi|^2)^{s/2} =: u_m^*(\xi)$ is a fundamental sequence in $L_2(\mathbb{R}^n)$. Since $L_2(\mathbb{R}^n)$ is complete, there exists a limit $u_m^*(\xi) \overset{m\to\infty}{\longrightarrow} u_*(\xi)$ in $L_2(\mathbb{R}^n)$. We put $w(\xi) = u_*(\xi)(1+|\xi|^2)^{-s/2}$. Then $w \in S'(\mathbb{R}^n)$, and, therefore, $\mathcal{F}^{-1} w = u \in S'(\mathbb{R}^n)$. We have:

$$w(\xi) = \widehat{u}(\xi), \quad u_*(\xi) = \widehat{u}(\xi)(1+|\xi|^2)^{s/2} \in L_2(\mathbb{R}^n),$$

$$\widehat{u}_m(\xi)(1+|\xi|^2)^{s/2} \overset{m\to\infty}{\longrightarrow} \widehat{u}(\xi)(1+|\xi|^2)^{s/2} \text{ in } L_2(\mathbb{R}^n).$$

It means that $u_m \overset{m\to\infty}{\longrightarrow} u$ in $H^s(\mathbb{R}^n)$. Thus, each element of the closure of $C_0^\infty(\mathbb{R}^n)$ in $\|\cdot\|_{H^s}$ belongs to $H^s(\mathbb{R}^n)$.

■

2. Duality of H^s and H^{-s}.

Theorem 2

Let $u \in H^s(\mathbb{R}^n)$, $v \in H^{-s}(\mathbb{R}^n)$, and let $u_j, v_j \in C_0^\infty(\mathbb{R}^n)$, $u_j \overset{j\to\infty}{\longrightarrow} u$ in $H^s(\mathbb{R}^n)$, $v_j \overset{j\to\infty}{\longrightarrow} v$ in $H^{-s}(\mathbb{R}^n)$. Then there exists the limit

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} u_j(x)\overline{v_j(x)}dx.$$

We denote this limit by $\int_{\mathbb{R}^n} u(x)\overline{v(x)}dx$. We have

$$\left|\int_{\mathbb{R}^n} u\overline{v}dx\right| \le \|u\|_{H^s}\|v\|_{H^{-s}}.$$

Proof

We have

$$\begin{aligned}\int_{\mathbb{R}^n} u_j(x)\overline{v_j(x)}dx &= \int_{\mathbb{R}^n} \widehat{u}_j(\xi)\overline{\widehat{v}_j(\xi)}d\xi \\ &= \int_{\mathbb{R}^n} \widehat{u}_j(\xi)(1+|\xi|^2)^{s/2}\cdot\overline{\widehat{v}_j(\xi)}(1+|\xi|^2)^{-s/2}d\xi. \quad (1)\end{aligned}$$

Since $u_j \overset{j\to\infty}{\longrightarrow} u$ in $H^s(\mathbb{R}^n)$, it follows that

$$\widehat{u}_j(\xi)(1+|\xi|^2)^{s/2} \overset{j\to\infty}{\longrightarrow} \widehat{u}(\xi)(1+|\xi|^2)^{s/2} =: (A_s u)(\xi) \text{ in } L_2(\mathbb{R}^n).$$

The fact that $v_j \overset{j\to\infty}{\longrightarrow} v$ in $H^{-s}(\mathbb{R}^n)$ means that

$$\widehat{v}_j(\xi)(1+|\xi|^2)^{-s/2} \overset{j\to\infty}{\longrightarrow} \widehat{v}(\xi)(1+|\xi|^2)^{-s/2} =: (A_{-s} v)(\xi) \text{ in } L_2(\mathbb{R}^n).$$

Then, by (1), we have

$$\int_{\mathbb{R}^n} u_j(x)\overline{v_j(x)}dx \overset{j\to\infty}{\longrightarrow} \int_{\mathbb{R}^n} (A_s u)(\xi)\overline{(A_{-s}v)(\xi)}d\xi =: \int_{\mathbb{R}^n} u(x)\overline{v(x)}dx.$$

It is clear that the limit $\lim_{j\to\infty}\int_{\mathbb{R}^n} u_j\overline{v_j}dx$ does not depend on the choice of the sequences $\{u_j\}$ and $\{v_j\}$. We have:

$$\begin{aligned}\left|\int_{\mathbb{R}^n} u\overline{v}dx\right| &= \left|\int_{\mathbb{R}^n} (A_s u)(\xi)\overline{(A_{-s}v)(\xi)}d\xi\right| \\ &\le \|A_s u\|_{L_2(\mathbb{R}^n)}\|A_{-s}v\|_{L_2(\mathbb{R}^n)} \\ &= \|u\|_{H^s}\|v\|_{H^{-s}}.\end{aligned}$$

■

Theorem 3

If $v \in H^{-s}(\mathbb{R}^n)$, then

$$\begin{aligned} \|v\|_{H^{-s}} &= \sup_{0 \neq u \in H^s(\mathbb{R}^n)} \frac{\left|\int_{\mathbb{R}^n} u\overline{v}dx\right|}{\|u\|_{H^s}} \\ &= \sup_{0 \neq u \in C_0^\infty(\mathbb{R}^n)} \frac{\left|\int_{\mathbb{R}^n} u\overline{v}dx\right|}{\|u\|_{H^s}}. \end{aligned} \tag{2}$$

Proof

1) The mapping $A_s : H^s(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$, $(A_s u)(\xi) = \widehat{u}(\xi)\left(1 + |\xi|^2\right)^{s/2}$ is a one–to–one isometric mapping. Indeed, $\|A_s u\|_{L_2} = \|u\|_{H^s}$. The inverse mapping $A_s^{-1} : L_2(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is defined as follows: for $u_* \in L_2$ consider $w(\xi) = \frac{u_*(\xi)}{(1+|\xi|^2)^{s/2}}$, and put $u = \mathcal{F}^{-1}w$.
Then $\widehat{u}(\xi) = w(\xi)$ and $u_*(\xi) = \widehat{u}(\xi)(1 + |\xi|^2)^{s/2} = (A_s u)(\xi)$.
Thus, $A_s^{-1}u_* = \mathcal{F}^{-1}w = u$. The mapping $A_{-s} : H^{-s} \to L_2$ is defined similarly.

2) Let $v \in H^{-s}$ and $v_*(\xi) = (A_{-s}v)(\xi)$. Then $v_* \in L_2$. It is known that in L_2 we have

$$\|v_*\|_{L_2} = \sup_{0 \neq g \in L_2} \frac{\left|\int_{\mathbb{R}^n} g(\xi)\overline{v_*(\xi)}d\xi\right|}{\|g\|_{L_2}}.$$

We put $u = A_s^{-1}g$. Then $g(\xi) = (A_s u)(\xi)$, $\|g\|_{L_2} = \|u\|_{H^s}$. If g runs over L_2, then u runs over H^s. Thus, for $v_* = A_{-s}v$ we have

$$\begin{aligned} \|v\|_{H^{-s}} &= \|v_*\|_{L_2} \\ &= \sup_{0 \neq u \in H^s} \frac{\left|\int_{\mathbb{R}^n} (A_s u)(\xi)\overline{(A_{-s}v)(\xi)}d\xi\right|}{\|u\|_{H^s}} \\ &= \sup_{0 \neq u \in H^s} \frac{\left|\int_{\mathbb{R}^n} u(x)\overline{v(x)}dx\right|}{\|u\|_{H^s}} \end{aligned}$$

■

From Theorems 2 and 3 it follows that $l(u) = \int_{\mathbb{R}^n} u\overline{v}dx$ is a linear continuous functional on $u \in H^s(\mathbb{R}^n)$ (if $v \in H^{-s}(\mathbb{R}^n)$) and the norm of this functional is equal to $\|v\|_{H^{-s}}$:

$$\|l\| = \sup_{0 \neq u \in H^s} \frac{|l(u)|}{\|u\|_{H^s}} = \sup_{0 \neq u \in H^s} \frac{\left|\int_{\mathbb{R}^n} u\overline{v}dx\right|}{\|u\|_{H^s}} = \|v\|_{H^{-s}}.$$

Riesz Theorem

Let H be a Hilbert space and $l(u)$, $u \in H$, be a continuous linear functional on H. Then there exists such element $v \in H$ that $l(u) = (u, v)_H$. This element v is unique and $\|l\| = \|v\|_H$.

Proof

1) Let $N = Ker\, l = \{z \in H : l(z) = 0\}$. Then N is a *closed subspace* in H. Indeed, if $z_j \in N$ and $z_j \overset{j\to\infty}{\longrightarrow} z$ in H, then $l(z_j) \overset{j\to\infty}{\longrightarrow} l(z)$. Since $l(z_j) = 0$, it follows that $l(z) = 0$, i. e. , $z \in N$.

2) If $N = H$, then $l(u) = 0$, $\forall u \in H$. In this case $v = 0$.
If $N \neq H$, then $N^\perp \neq \{0\}$ (where $N^\perp$ is the orthogonal complement of N). So, there exists $v_0 \in N^\perp$, $v_0 \neq 0$. Then, $l(v_0) \neq 0$.

3) For $\forall u \in H$ consider $u - \frac{l(u)}{l(v_0)} v_0 \in N$.
$\left(\text{Indeed, } l\left(u - \frac{l(u)}{l(v_0)} v_0\right) = l(u) - \frac{l(u)}{l(v_0)} l(v_0) = 0. \right)$
Since $v_0 \in N^\perp$, we have
$$\left(u - \frac{l(u)}{l(v_0)} v_0, v_0\right) = 0 \Rightarrow (u, v_0) = l(u) \frac{\|v_0\|^2}{l(v_0)}.$$
Denote $\quad v = \frac{\overline{l(v_0)}}{\|v_0\|^2} v_0$. Then $l(u) = (u, v)$.

4) Uniqueness
If $(u, v) = (u, \tilde{v})$, $\forall u \in H$, then $v - \tilde{v} \perp H \Rightarrow v - \tilde{v} = 0$.

5) The norm of l.
$$\|l\| = \sup_{0 \neq u \in H} \frac{|l(u)|}{\|u\|_H} = \sup_{0 \neq u \in H} \frac{|(u, v)|}{\|u\|_H} = \|v\|_H .$$
Indeed,
$\frac{|(u,v)|}{\|u\|_H} \leq \|v\|_H$ for $\forall\, 0 \neq u \in H$, and for $u = v$ we have $\frac{|(u,v)|}{\|u\|_H} = \|v\|_H$.

■

Let $l(u)$ be a continuous linear functional on $H^s(\mathbb{R}^n)$.
It means that $l : H^s \to \mathbb{C}$,

a) $l(c_1 u_1 + c_2 u_2) = c_1 l(u_1) + c_2 l(u_2)$, $\forall\, u_1, u_2 \in H^s$, $\forall\, c_1, c_2 \in \mathbb{C}$,

b) $|l(u)| \leq c \|u\|_{H^s}$, $\forall u \in H^s(\mathbb{R}^n)$.

The norm $\|l\|$ of a functional l is defined by the formula
$$\|l\| = \sup_{0 \neq u \in H^s} \frac{|l(u)|}{\|u\|_{H^s}}.$$

Theorem 4

Let $l(u)$ be a linear continuous functional on $H^s(\mathbb{R}^n)$. Then there exists unique element $v \in H^{-s}(\mathbb{R}^n)$, such that

$$l(u) = \int_{\mathbb{R}^n} u\overline{v}dx, \quad \forall\, u \in H^s(\mathbb{R}^n), \tag{3}$$

and

$$\|l\| = \|v\|_{H^{-s}}. \tag{4}$$

Proof

Consider the mapping

$$A_s : H^s(\mathbb{R}^n) \to L_2(\mathbb{R}^n), \;\; (A_s u)(\xi) = u_*(\xi) = \widehat{u}(\xi)\left(1+|\xi|^2\right)^{s/2}.$$

Then $u = A_s^{-1}u_*$. We define the functional $\tilde{l}(u_*)$ on $L_2(\mathbb{R}^n)$ by the formula

$$\tilde{l}(u_*) = l(A_s^{-1}u_*) = l(u).$$

Then $\tilde{l}$ is a linear continuous functional on $L_2(\mathbb{R}^n)$.
By the Riesz theorem for the functional $\tilde{l}$ there exists unique function $w \in L_2(\mathbb{R}^n)$ such that

$$\tilde{l}(u_*) = \int_{\mathbb{R}^n} u_*(\xi)\overline{w(\xi)}d\xi, \text{ and } \left\|\tilde{l}\right\| = \|w\|_{L_2}.$$

Then $l(u) = \tilde{l}(u_*) = \int_{\mathbb{R}^n} \widehat{u}(\xi)\left(1+|\xi|^2\right)^{s/2}\overline{w(\xi)}d\xi$.
We denote $v(x) = \mathcal{F}^{-1}\left(w(\xi)(1+|\xi|^2)^{s/2}\right)$.
Then

$$\widehat{v}(\xi) = w(\xi)(1+|\xi|^2)^{s/2}; \quad \int_{\mathbb{R}^n} |\widehat{v}(\xi)|^2 (1+|\xi|^2)^{-s}d\xi = \int_{\mathbb{R}^n} |w(\xi)|^2 d\xi.$$

So, $v \in H^{-s}$, and $\|v\|_{H^{-s}} = \|w\|_{L_2}$.
We have $w(\xi) = (1+|\xi|^2)^{-s/2}\widehat{v}(\xi) = (A_{-s}v)(\xi)$,
$l(u) = \tilde{l}(u_*) = \int_{\mathbb{R}^n} (A_s u)(\xi)\overline{(A_{-s}v)(\xi)}d\xi = \int_{\mathbb{R}^n} u\overline{v}dx$.
For the norm of the functional l we have:

$$\|l\| = \sup_{0\neq u\in H^s} \frac{|l(u)|}{\|u\|_{H^s}} = \sup_{0\neq u_*\in L_2} \frac{\left|\tilde{l}(u_*)\right|}{\|u_*\|_{L_2}} = \left\|\tilde{l}\right\| = \|w\|_{L_2} = \|v\|_{H^{-s}}.$$

■

Remark

Theorem 4 means that $H^{-s}(\mathbb{R}^n)$ *is dual to* $H^s(\mathbb{R}^n)$ *with respect to* L_2*–duality.*

3. Mollifications in $H^s(\mathbb{R}^n)$

Let $\omega_\rho(x) = \rho^{-n}\omega\left(\frac{x}{\rho}\right)$ be a mollifier.
Recall that $\omega \in C_0^\infty(\mathbb{R}^n)$, $\omega(x) \geq 0$, $\int_{\mathbb{R}^n} \omega(x)dx = 1$.
For $u \in H^s(\mathbb{R}^n)$ consider mollifications: $u_\rho(x) = (\omega_\rho * u)(x)$, $\rho > 0$.

Theorem 5

If $u \in H^s(\mathbb{R}^n)$, then $\|u_\rho - u\|_{H^s} \to 0$ as $\rho \to 0$.

Proof

For the Fourier transform of the convolution $u_\rho = \omega_\rho * u$ we have

$$\widehat{u}_\rho(\xi) = (2\pi)^{n/2}\widehat{\omega}_\rho(\xi)\widehat{u}(\xi).$$

Next,

$$\begin{aligned}
\widehat{\omega}_\rho(\xi) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \rho^{-n}\omega\left(\frac{x}{\rho}\right)e^{-ix\xi}dx \\
&\overset{\frac{x}{\rho}=y}{=} (2\pi)^{-n/2}\int_{\mathbb{R}^n} \omega(y)e^{-iy\xi\rho}dy \\
&= \widehat{\omega}(\rho\xi).
\end{aligned}$$

Since $\omega \in C_0^\infty(\mathbb{R}^n)$, then $\widehat{\omega}(\xi)$ belongs to the Schwartz class $S(\mathbb{R}^n)$. Hence,

a) $|\widehat{\omega}(\rho\xi)| \leq c, \ \ \forall \xi \in \mathbb{R}^n$.

b) $\lim_{\rho\to 0} \widehat{\omega}(\rho\xi) = \widehat{\omega}(0) = (2\pi)^{-n/2}\underbrace{\int_{\mathbb{R}^n} \omega(y)dy}_{=1} = (2\pi)^{-n/2}$.

Let us estimate the norm $\|u_\rho - u\|_{H^s}$. We have

$$\widehat{u}_\rho(\xi) - \widehat{u}(\xi) = \left((2\pi)^{n/2}\widehat{\omega}(\rho\xi) - 1\right)\widehat{u}(\xi);$$

$$\|u_\rho - u\|_{H^s}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s\, |\widehat{u}(\xi)|^2 \underbrace{\left|(2\pi)^{n/2}\widehat{\omega}(\rho\xi) - 1\right|^2}_{\to 0 \text{ as } \rho\to 0 \forall\ \xi} d\xi.$$

The function under the integral is estimated by $C(1 + |\xi|^2)^s\, |\widehat{u}(\xi)|^2$, which is summable since $u \in H^s$. By the Lebesgue Theorem, $\|u_\rho - u\|_{H^s} \to 0$ as $\rho \to 0$.

■

4. Embedding $H^s \hookrightarrow C^r$

Theorem 6

Let $s > r + \frac{n}{2}$. Then $H^s(\mathbb{R}^n) \hookrightarrow C^r(\mathbb{R}^n)$.

Proof

1) Let $u \in C_0^\infty(\mathbb{R}^n)$. We have

$$u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{u}(\xi) e^{ix\xi} d\xi, \quad \partial^\alpha u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (i\xi)^\alpha \widehat{u}(\xi) e^{ix\xi} d\xi, \ \forall \alpha.$$

Then, by the Hölder inequality,

$$\begin{aligned} |\partial^\alpha u(x)| &\leq \int_{\mathbb{R}^n} |\xi^\alpha| \, |\widehat{u}(\xi)| \, d\xi \\ &\leq \left(\int_{\mathbb{R}^n} |\xi^\alpha|^2 \, (1 + |\xi|^2)^{-s} d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \, (1 + |\xi|^2)^{s} d\xi \right)^{1/2}. \end{aligned}$$

If $|\alpha| \leq r$, and $s - r > \frac{n}{2}$, then $\int_{\mathbb{R}^n} \frac{|\xi^\alpha|^2 d\xi}{(1+|\xi|^2)^s} < \infty$. Thus,

$$\begin{aligned} \max_{|\alpha|\leq r} \max_{x\in\mathbb{R}^n} |\partial^\alpha u(x)| &\leq C \, \|u\|_{H^s}, \quad u \in C_0^\infty(\mathbb{R}^n), \\ \text{i. e. , } \|u\|_{C^r} &\leq C \, \|u\|_{H^s}, \quad u \in C_0^\infty(\mathbb{R}^n). \qquad (5) \end{aligned}$$

2) Let $u \in H^s(\mathbb{R}^n)$. Then there exists a sequence $u_j \in C_0^\infty(\mathbb{R}^n)$, such that $u_j \overset{j\to\infty}{\longrightarrow} u$ in H^s. By (5), $\|u_j - u_l\|_{C^r} \leq C \, \|u_j - u_l\|_{H^s} \overset{j,l\to\infty}{\longrightarrow} 0$.
So, $\{u_j\}$ is the Cauchy sequence in $C^r(\mathbb{R}^n)$. There exists a limit $\tilde{u} \in C^r(\mathbb{R}^n)$: $\|u_j - \tilde{u}\|_{C^r} \overset{j\to\infty}{\longrightarrow} 0$.
In fact, $\tilde{u}(x) = u(x)$, for a.e. $x \in \mathbb{R}^n$ (check this!). We identify $\tilde{u} = u$.
We have proved that $H^s \hookrightarrow C^r$ and

$$\|u\|_{C^r} \leq C \, \|u\|_{H^s}, \ \ \forall \, u \in H^s.$$

■

Remark

Theorem 6 is generalization of the embedding theorem:
$W_2^l \hookrightarrow C^r$ if $2(l - r) > n$.

5. Equivalent norm in H^s with fractional $s > 0$

Theorem 7

If $0 < s < 1$, the norm $\|u\|_{H^s}$ is equivalent to the norm

$$\|u\|'_{H^s} = \left(\int_{\mathbb{R}^n} |u|^2 dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2 dxdy}{|x-y|^{n+2s}}\right)^{1/2}.$$

Proof Note that

$$\mathcal{F} : u(x) - u(x+z) \mapsto \widehat{u}(\xi)(1 - e^{iz\xi}).$$

We have

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2 dxdy}{|x-y|^{n+2s}} &\overset{y=x+z}{=} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(x+z)|^2 dxdz}{|z|^{n+2s}} \\
&\overset{Parseval}{=} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\widehat{u}(\xi)|^2 |1-e^{iz\xi}|^2 d\xi dz}{|z|^{n+2s}} \\
&= \int_{\mathbb{R}^n} g(\xi)|\widehat{u}(\xi)|^2 d\xi,
\end{aligned}$$

where $g(\xi) = \int_{\mathbb{R}^n} \frac{|1-e^{iz\xi}|^2 dz}{|z|^{n+2s}}$.
The function $g(\xi)$ is homogeneous in ξ of order $2s$:

$$g(t\xi) = \int_{\mathbb{R}^n} \frac{|1-e^{itz\xi}|^2 dz}{|z|^{n+2s}} = t^{2s}\int_{\mathbb{R}^n} \frac{|1-e^{itz\xi}|^2 d(tz)}{|tz|^{n+2s}} = t^{2s} g(\xi), \ \ \forall\, t > 0.$$

The function $g(\xi)$ depends only on $|\xi|$:

$$g(\xi) = \int_{\mathbb{R}^n} \frac{|1-e^{iz_1|\xi|}|^2 dz}{|z|^{n+2s}},$$

where the axis $0z_1$ has direction of vector ξ.
It follows that $g(\xi) = A|\xi|^{2s}$, $A > 0$.
Then

$$\|u\|'_{H^s} = \left(\int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 (1 + A|\xi|^{2s}) d\xi\right)^{1/2}.$$

Obviously, $c_1(1+|\xi|^2)^s \ \le\ 1 + A|\xi|^{2s} \ \le\ c_2(1+|\xi|^2)^s,\ \xi \in \mathbb{R}^n$.
Then $\|u\|'_{H^s} \asymp \|u\|_{H^s}$.

■

Corollary

If $s > 0$, $[s] = k$, $\{s\} > 0$, then the norm

$$\|u\|'_{H^s} = \left(\sum_{|\alpha|\le k}\int_{\mathbb{R}^n} |\partial^\alpha u|^2 dx + \sum_{|\alpha|=k}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2 dxdy}{|x-y|^{n+2\{s\}}}\right)^{1/2}$$

is equivalent to $\|u\|_{H^s}$.

Proof $u \in H^s \Leftrightarrow u \in H^k$ and $\partial^\alpha u \in H^{\{s\}}$ with $|\alpha| = k$. It is easy to check that

$$\|u\|_{H^s}^2 \overset{\text{check this!}}{\asymp} \|u\|_{H^k}^2 + \sum_{|\alpha|=k} \|\partial^\alpha u\|_{H^{\{s\}}}^2$$

$$\overset{\text{by Theorem 7}}{\asymp} \sum_{|\alpha|\le k} \int_{\mathbb{R}^n} |\partial^\alpha u|^2 \, dx + \sum_{|\alpha|=k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2 dx dy}{|x-y|^{n+2\{s\}}}.$$

■

6. „ε–inequalities“

Obviously, $H^{s_1}(\mathbb{R}^n) \hookrightarrow H^{s_2}(\mathbb{R}^n)$ for $s_1 > s_2$.
Proposition

Let $s_1 < s < s_2$. Then for $\forall\ \varepsilon > 0\ \exists\ C(\varepsilon) > 0$ such that

$$\|u\|_{H^s}^2 \le \varepsilon \|u\|_{H^{s_2}}^2 + C(\varepsilon) \|u\|_{H^{s_1}}^2 \tag{6}$$

Proof
(6) is equivalent to the inequality

$$\begin{aligned} \left(1+|\xi|^2\right)^s &\le \varepsilon \left(1+|\xi|^2\right)^{s_2} + C(\varepsilon) \left(1+|\xi|^2\right)^{s_1} \\ \Leftrightarrow\ \rho^s &\le \varepsilon \rho^{s_2} + C(\varepsilon)\rho^{s_1}, \quad \rho \ge 1. \\ \Leftrightarrow\ 1 &\le \varepsilon \rho^{s_2-s} + C(\varepsilon)\rho^{-(s-s_1)}, \quad \rho \ge 1. \end{aligned}$$

We denote $\lambda = \varepsilon^{\frac{1}{s_2-s}} > 0$, and put $C(\varepsilon) = \lambda^{-(s-s_1)} = \varepsilon^{-\frac{s-s_1}{s_2-s}}$. Then

$$\varepsilon \rho^{s_2-s} + C(\varepsilon)\rho^{-(s-s_1)} = (\lambda\rho)^{s_2-s} + (\lambda\rho)^{-(s-s_1)} \overset{\textit{obviously}}{\ge} 1$$

■

§3: Trace embedding theorems

We write $x \in \mathbb{R}^n$ as $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1})$. Consider the traces of functions on the hyper–plane $x_n = 0$. We define the trace operator

$$\gamma_0 : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^{n-1}), \quad (\gamma_0 u)(x') = u(x', 0).$$

Theorem 8

Let $s > \frac{1}{2}$. Then the trace operator $\gamma_0 : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^{n-1})$ can be extended by continuity to the linear continuous operator $\gamma_0 : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. We have

$$\|\gamma_0 u\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C \|u\|_{H^s(\mathbb{R}^n)} . \tag{7}$$

Proof

1) Let $u \in C_0^\infty(\mathbb{R}^n)$. Then

$$u(x) = u(x', x_n) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{u}(\xi', \xi_n) e^{ix_n\xi_n} e^{ix'\xi'} d\xi' d\xi_n;$$

$$(\gamma_0 u)(x') = u(x', 0) = (2\pi)^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} d\xi' e^{ix'\xi'} \underbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{u}(\xi', \xi_n) d\xi_n \right)}_{=\widehat{\gamma_0 u}(\xi')}$$

$$\Rightarrow \quad \widehat{\gamma_0 u}(\xi') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{u}(\xi', \xi_n) d\xi_n.$$

Then

$$\left|\widehat{\gamma_0 u}(\xi')\right|^2 \leq \left(\int_{-\infty}^{\infty} |\widehat{u}(\xi)|^2 (1 + |\xi|^2)^s d\xi_n \right) \left(\int_{-\infty}^{\infty} (\underbrace{1 + |\xi'|^2}_{=a^2} + \xi_n^2)^{-s} d\xi_n \right). \tag{8}$$

Here the second integral is finite, since $s > \frac{1}{2}$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{d\xi_n}{(a^2 + \xi_n^2)^s} &= \frac{a}{a^{2s}} \int_{-\infty}^{\infty} \frac{d\left(\frac{\xi_n}{a}\right)}{\left(1 + \left(\frac{\xi_n}{a}\right)^2\right)^s} \\
&= a^{1-2s} \underbrace{\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^s}}_{=c_s} \\
&= c_s a^{1-2s} \\
\Rightarrow \int_{-\infty}^{\infty} \frac{d\xi_n}{(1 + |\xi'|^2 + \xi_n^2)^s} &= c_s (1 + |\xi'|^2)^{\frac{1}{2} - s}. \qquad (9)
\end{aligned}$$

Thus, from (8) and (9) it follows that

$$\left(1+|\xi'|^2\right)^{s-\frac{1}{2}}\left|\widehat{\gamma_0 u}(\xi')\right|^2 \le c_s \int_{-\infty}^{\infty} |\widehat{u}(\xi)|^2 (1+|\xi|^2)^s d\xi_n.$$

Integrate over $\mathbb{R}^{n-1}$:

$$\int_{\mathbb{R}^{n-1}} \left(1+|\xi'|^2\right)^{s-\frac{1}{2}}\left|\widehat{\gamma_0 u}(\xi')\right|^2 d\xi' \quad \le \quad c_s \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 (1+|\xi|^2)^s d\xi,$$

$$\text{i. e.} \qquad \|\gamma_0 u\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \quad \le \quad c_s \|u\|^2_{H^s(\mathbb{R}^n)}, \;\; u \in C_0^\infty. \qquad (10)$$

2) $C_0^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.
Let $u \in H^s(\mathbb{R}^n)$. Then $\exists\ \{u_j\}$, $u_j \in C_0^\infty(\mathbb{R}^n)$, $\|u_j - u\|_{H^s(\mathbb{R}^n)} \overset{j\to\infty}{\longrightarrow} 0$.
By (10),

$$\|\gamma_0 u_j - \gamma_0 u_l\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \le c_s \|u_j - u_l\|^2_{H^s(\mathbb{R}^n)} \overset{j,l\to\infty}{\longrightarrow} 0.$$

So, $\{\gamma_0 u_j\}$ is a Cauchy sequence in $H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. Then there exists a limit:

$$\gamma_0 u_j \overset{j\to\infty}{\longrightarrow} v \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \text{ in } H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}).$$

By definition , $v = \gamma_0 u$. By the limit procedure, the estimate (10) is extended to all $u \in H^s(\mathbb{R}^n)$.

■

Corollary

Let $k \in \mathbb{N}$ and $s > k + \frac{1}{2}$. Then the trace operators
$\gamma_j = \gamma_0 \circ \partial^j_{x_n} : H^s(\mathbb{R}^n) \to H^{s-j-\frac{1}{2}}(\mathbb{R}^{n-1})$ are continuous for
$j = 0, 1, \ldots, k$. We have

$$\|\gamma_j u\|_{H^{s-j-\frac{1}{2}}(\mathbb{R}^{n-1})} \le c\,\|u\|_{H^s(\mathbb{R}^n)}.$$

Theorem 9 (extension theorem)

Let $k \in \mathbb{Z}_+$, $s > k + \frac{1}{2}$.
Denote $H^{\langle s-\frac{1}{2}\rangle}(\mathbb{R}^{n-1}) = H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \times H^{s-\frac{3}{2}}(\mathbb{R}^{n-1}) \times \ldots \times H^{s-k-\frac{1}{2}}(\mathbb{R}^{n-1})$.
There exists a linear continuous operator

$$P : H^{\langle s-\frac{1}{2}\rangle}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^n),$$

such that, if $\boldsymbol{\varphi} = (\varphi_0, \varphi_1, \ldots, \varphi_k) \in H^{\langle s-\frac{1}{2}\rangle}(\mathbb{R}^{n-1})$, $u = P\boldsymbol{\varphi}$ ($\in H^s(\mathbb{R}^n)$), then $\varphi_j = \gamma_j u$, $j = 0, 1, \ldots, k$. We have

$$\|u\|^2_{H^s(\mathbb{R}^n)} \le c\,\|\boldsymbol{\varphi}\|^2_{H^{\langle s-\frac{1}{2}\rangle}(\mathbb{R}^{n-1})} = c \sum_{j=0}^{k} \|\varphi_j\|^2_{H^{s-j-\frac{1}{2}}(\mathbb{R}^n)}.$$

Proof
Let $h \in C_0^\infty(\mathbb{R})$, $h(t) = 1$ for $|t| \le 1$, $0 \le h(t) \le 1$. We put

$$V(\xi', x_n) = \sum_{j=0}^{k} \frac{1}{j!} x_n^j \widehat{\varphi}_j(\xi') h\left(x_n\sqrt{1+|\xi'|^2}\right), \quad \xi' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}.$$

Here $\widehat{\varphi}_j(\xi')$ is the Fourier image of $\varphi_j(x')$. Clearly, $V(\xi', 0) = \widehat{\varphi}_0(\xi')$, $\partial_{x_n}^j V(\xi', 0) = \widehat{\varphi}_j(\xi')$, $j = 1, \ldots, k$. Let us show that $V(\xi', x_n)$ is the Fourier image of the function $u(x', x_n)$ such that $u \in H^s(\mathbb{R}^n)$. We put $\widehat{u}(\xi', \xi_n) = \widehat{V}(\xi', \xi_n)$, where $\widehat{V}(\xi', \xi_n)$ is the Fourier image (in one variable $x_n \mapsto \xi_n$) of $V(\xi', x_n)$. Note that

$$\begin{aligned} x_n^j g(x_n) &\overset{\mathcal{F}}{\mapsto} i^j \widehat{g}^{(j)}(\xi_n) = i^j \frac{d^j}{d\xi_n^j}\widehat{g}(\xi_n), \\ g(\rho x_n) &\overset{\mathcal{F}}{\mapsto} \frac{1}{\rho}\widehat{g}\left(\frac{\xi_n}{\rho}\right), \\ x_n^j h(\rho x_n) &\overset{\mathcal{F}}{\mapsto} i^j \frac{1}{\rho^{j+1}}\widehat{h}^{(j)}\left(\frac{\xi_n}{\rho}\right). \end{aligned}$$

Then

$$\widehat{u}(\xi) = \sum_{j=0}^{k} \frac{i^j}{j!} \widehat{\varphi}_j(\xi')(1+|\xi'|^2)^{-\frac{j+1}{2}} \widehat{h}^{(j)}\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right).$$

We have:

$$\begin{aligned} \|u\|^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 (1+|\xi|^2)^s d\xi \\ &\le C\sum_{j=0}^{k} \int_{\mathbb{R}^n} \left|\widehat{\varphi}_j(\xi')\right|^2 (1+|\xi'|^2)^{-j-1} \left|\widehat{h}^{(j)}\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right)\right|^2 (1+|\xi|^2)^s d\xi. \end{aligned}$$

We write the integral as $\int_{\mathbb{R}^{n-1}} d\xi' \int_{\mathbb{R}} d\xi_n \ldots$, and in the internal integral change variable: $\tau = \frac{\xi_n}{\sqrt{1+|\xi'|^2}}$. Then

$$1 + |\xi|^2 = 1 + |\xi'|^2 + \xi_n^2 = (1+|\xi'|^2)(1+\tau^2);$$

$$\|u\|^2_{H^s(\mathbb{R}^n)} \le C\sum_{j=0}^{k} \int_{\mathbb{R}^{n-1}} d\xi' \left|\widehat{\varphi}_j(\xi')\right|^2 (1+|\xi'|^2)^{s-j-\frac{1}{2}} \int_{\mathbb{R}} \left|\widehat{h}^{(j)}(\tau)\right|^2 (1+\tau^2)^s d\tau.$$

Since $h \in C_0^\infty(\mathbb{R})$, then $\widehat{h} \in S(\mathbb{R})$ and, so,

$$\int_{\mathbb{R}} \left|\widehat{h}^{(j)}(\tau)\right|^2 (1+\tau^2)^s d\tau = C(j, s) < \infty.$$

$$\begin{aligned} \Rightarrow \quad \|u\|^2_{H^s(\mathbb{R}^n)} &\leq C\sum_{j=0}^{k}\int_{\mathbb{R}^{n-1}} \left|\widehat{\varphi}_j(\xi')\right|^2 (1+|\xi'|^2)^{s-j-\frac{1}{2}} d\xi'. \\ &= c\sum_{j=0}^{k} \|\varphi_j\|^2_{H^{s-j-\frac{1}{2}}(\mathbb{R}^{n-1})}. \end{aligned}$$

So, the operator $P : \boldsymbol{\varphi} = (\varphi_0, \varphi_1, \ldots, \varphi_k) \mapsto u$ is a linear continuous operator from $H^{\langle s-\frac{1}{2}\rangle}(\mathbb{R}^{n-1})$ to $H^s(\mathbb{R}^n)$, and $\gamma_j u = \varphi_j,\ j = 0, \ldots, k$.

■

§4: Spaces $H^s(\Omega)$ (survey)

1. Definition of $H^s(\Omega)$

Let $\Omega \subset \mathbb{R}^n$ be a domain. There are different ways of definition of the Sobolev spaces $H^s(\Omega)$.

Approach I.

Definition 1

$H^s(\Omega)$ is the class of restrictions to Ω of functions in $H^s(\mathbb{R}^n)$:

$$u \in H^s(\Omega) \quad \Leftrightarrow \quad \exists\, v \in H^s(\mathbb{R}^n), \quad v|_\Omega = u.$$

Approach II.

Case $s \geq 0$.

Definition 2

$H^s(\Omega)$ is the set of functions in $L_2(\Omega)$, such that their weak derivatives up to order $k = [s]$ also belong to $L_2(\Omega)$, and the following norm is finite: $\|u\|_{H^s} < \infty$,

$$\|u\|^2_{H^s} \stackrel{def}{=} \begin{cases} \sum_{|\alpha| \leq s} \int_\Omega |\partial^\alpha u|^2\, dx, \quad \text{if } s = [s] \\ \sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u|^2\, dx + \sum_{|\alpha| = k} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2 dx dy}{|x-y|^{n+2\{s\}}}, \\ \text{if } s \neq [s] = k,\ \{s\} = s - k. \end{cases} \quad (11)$$

Comments

1) If $\Omega \subset \mathbb{R}^n$ is a bounded domain of Lipschitz class, then both definitions give the same space: Def 1 $\Leftrightarrow$ Def 2.
   If $H^s(\Omega)$, $s \geq 0$, is the Sobolev space in the sense of Def 2, there exists a linear continuous extension operator $\Pi : H^s(\Omega) \to H^s(\mathbb{R}^n)$.

2) The spaces $W^s_p(\Omega)$ with fractional $s \geq 0$ and $p \neq 2$ can be defined by analogy with Def 2 (with "2" replaced by "p").

3) The embedding theorems can be generalized for spaces of fractional order.

Next, the space $\overset{\circ}{H}{}^s(\Omega)$ is defined.

Definition 3

$\overset{\circ}{H}{}^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (11).

Definition 4

Let $s > 0$. Then, by definition, $H^{-s}(\Omega) = \left(\overset{\circ}{H}{}^s(\Omega)\right)^*$, i. e. , $H^{-s}(\Omega)$ is the space of linear continuous functionals on $\overset{\circ}{H}{}^s(\Omega)$ with the norm

$$\|u\|_{H^{-s}} = \sup_{0 \neq \varphi \in \overset{\circ}{H}{}^s(\Omega)} \frac{|\langle u, \varphi \rangle|}{\|\varphi\|_{H^s(\Omega)}}.$$

By analogy with $H^s(\mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$, for $u \in H^{-s}(\Omega)$, $\varphi \in \overset{\circ}{H}{}^s(\Omega)$, we denote

$$\langle u, \varphi \rangle = \int_\Omega u(x)\varphi(x)dx.$$

Comments

1) $\overset{\circ}{H}{}^s(\Omega) = H^s(\Omega)$ for $s < \frac{1}{2}$.

2) Let $u \in \overset{\circ}{H}{}^s(\Omega)$, $P_0 u(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \backslash \Omega \end{cases}$.

   Then $P_0 : \overset{\circ}{H}{}^s(\Omega) \to H^s(\mathbb{R}^n)$ is continuous, if $s \neq m + \frac{1}{2}$, $m \in \mathbb{Z}_+$.

3) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Then $H^{-s}(\Omega)$ coincides with the space of restrictions to Ω of distributions $\in H^{-s}(\mathbb{R}^n)$, if $s \neq m + \frac{1}{2}$, $m \in \mathbb{Z}_+$.

4) $H^s(\Omega)$ is invariant with respect to diffeomorphisms of class C^l, $l \geq |s|, l \in \mathbb{N}$.

2. Trace embedding theorems

Theorems 8 and 9 can be extended to the case of bounded domain Ω with smooth boundary. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^l. Then there exists a covering $\{U_j\}_{j=1,\ldots,N}$ such that $\overline{\Omega} \subset \bigcup_{j=1}^N U_j$, either $\overline{U_j} \subset \Omega$, or $U_j \cap \partial\Omega \neq \emptyset$, then $\exists$ diffeomorphism $f_j : U_j \to K$, $f_j, f_j^{-1} \in C^l$, $f_j(U_j \cap \Omega) = K_+$, $f_j(U_j \cap \partial\Omega) = \partial K_+ \backslash \Sigma_+ = \Gamma$. Suppose that domains $U_1, U_2, \ldots, U_M$ are of second kind, and $U_{M+1}, \ldots, U_N$ are strictly interior. There exists a partition of unity $\{\zeta_j\}$, such that $\zeta_j \in C_0^\infty(\mathbb{R}^n)$, *supp* $\zeta_j \subset U_j$, $\sum_{j=1}^N \zeta_j(x) = 1$, $x \in \overline{\Omega}$.
For $x \in \partial\Omega$ we have $\sum_{j=1}^M \zeta_j(x) = 1$. (This is true in some neighbourhood of $\partial\Omega$.) Let $u \in C^l(\partial\Omega)$, $u_j(x) = u(x)\zeta_j(x)$, $v_j = u_j \circ f_j^{-1}$. Then $v_j \in C^l(\Gamma)$, *supp* $v_j \subset\subset \Gamma$. We extend v_j by zero to $\mathbb{R}^{n-1} \backslash \Gamma$:

$$\tilde{v}_j(y) = \begin{cases} v_j(y), & y \in \Gamma \\ 0, & y \in \mathbb{R}^{n-1} \backslash \Gamma \end{cases}.$$

Then $\tilde{v}_j \in C^l(\mathbb{R}^{n-1})$, *supp* $\tilde{v}_j \subset \Gamma$. Consider

$$\left(\sum_{j=1}^{M} \|\tilde{v}_j\|^2_{H^s(\mathbb{R}^{n-1})} \right)^{1/2} \stackrel{def}{=} \|u\|_{H^s(\partial\Omega)} , \quad s \le l. \tag{12}$$

Definition

$H^s(\partial\Omega)$ is the closure of $C^l(\partial\Omega)$ with respect to the norm (12).

This norm depends on the choice of covering $\{U_j\}$, diffeomorphisms $\{f_j\}$, and partition of unity $\{\zeta_j\}$. It can be proved that all such norms (for different $\{U_j\}, \{f_j\}, \{\zeta_j\}$) are equivalent to each other. So, the class $H^s(\partial\Omega)$ is well–defined.

Theorem 10 (trace embedding theorem)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^l, $l \in \mathbb{N}$. Let $k \in \mathbb{Z}_+$, $s > k + \frac{1}{2}$, $s \le l$. Let $\gamma_j : C^l(\overline{\Omega}) \to C^{l-j}(\partial\Omega)$ be the trace operator: $\gamma_j u = \left.\frac{\partial^j u}{\partial \nu^j}\right|_{\partial\Omega}$, $j = 0, \dots, k$ (where $\frac{\partial^j}{\partial \nu^j}$ are „normal" derivatives of u). Then the operator γ_j can be extended (uniquely) to linear continuous operator $\gamma_j : H^s(\Omega) \to H^{s-j-\frac{1}{2}}(\partial\Omega)$, $j = 0, 1, \dots, k$.

The proof is based on Theorem 8, and using covering $\{U_j\}$, diffeomorphisms $\{f_j\}$ and partition of unity $\{\zeta_j\}$.

Theorem 11 (extension theorem)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^l, $s \le l$, $s > k + \frac{1}{2}$, where $k \in \mathbb{Z}_+$. We denote

$$H^{\langle s-\frac{1}{2} \rangle}(\partial\Omega) = H^{s-\frac{1}{2}}(\partial\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \times \dots \times H^{s-k-\frac{1}{2}}(\partial\Omega).$$

There exists a linear continuous operator

$$P_\Omega : H^{\langle s-\frac{1}{2} \rangle}(\partial\Omega) \to H^s(\Omega)$$

such that, if $\boldsymbol{\varphi} = (\varphi_0, \varphi_1, \dots, \varphi_k)$, $\varphi_j \in H^{s-j-\frac{1}{2}}(\partial\Omega)$, and $u = P_\Omega \boldsymbol{\varphi}$, then $\gamma_j u = \varphi_j$, $j = 0, 1, \dots, k$, and

$$\|u\|^2_{H^s(\Omega)} \le c \sum_{j=0}^{k} \|\varphi_j\|^2_{H^{s-j-\frac{1}{2}}(\partial\Omega)} = c \, \|\boldsymbol{\varphi}\|^2_{H^{\langle s-\frac{1}{2} \rangle}(\partial\Omega)} .$$

Theorem 12

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^l, $l \in \mathbb{N}$. Then $u \in \overset{\circ}{H}{}^l(\Omega) = \overset{\circ}{W}{}^l_2(\Omega)$ if and only if $u \in H^l(\Omega)$ and $\gamma_0 u = \gamma_1 u = \dots = \gamma_{l-1} u = 0$.

Proof

For simplicity we prove Theorem 12 in the case $l = 1$.

$$u \in \overset{\circ}{H}{}^1(\Omega) \quad \Leftrightarrow \quad u \in H^1(\Omega) \text{ and } \gamma_0 u = 0.$$

"$\Rightarrow$" Obvious.

"$\Leftarrow$"

Using covering, diffeomorphisms and partition of unity, we reduce the question to the following. Let $K_+ = \{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$ be the half–ball. Suppose that $u \in H^1(K_+)$, $u(x) = 0$ near Σ_+, $\gamma_0 u = u|_\Gamma = 0$. We have to prove that $u \in \overset{\circ}{H}{}^1(K_+)$. We have the following representation for $u(x)$:

$$u(x', x_n) = \int_0^{x_n} \frac{\partial u}{\partial x_n}(x', t)dt, \text{ for a. e. } (x', x_n) \in K_+. \tag{13}$$

We fix a cut–off function $h(t)$ such that $h \in C^\infty(\mathbb{R}_+)$, $h(t) = 0$, $0 \le t \le \frac{1}{2}$, $h(t) = 1$ for $t \ge 1$, and $0 \le h(t) \le 1$. We put $h_m(t) = h(mt)$, then $h_m(t) = 1$ for $t \ge \frac{1}{m}$. Consider $u_m(x) = u(x', x_n)h_m(x_n)$. Then $u_m(x) = 0$ near ∂K_+, $u_m \in H^1(K_+)$.

Let us check that $\|u_m - u\|_{H^1(K_+)} \overset{m\to\infty}{\longrightarrow} 0$. We have:

$$\begin{aligned}
u(x', x_n) - u_m(x', x_n) &= (1 - h_m(x_n))u(x', x_n), \\
\frac{\partial}{\partial x_j}(u(x) - u_m(x)) &= (1 - h_m(x_n))\frac{\partial u(x)}{\partial x_j}, \quad j = 1, \dots, n-1, \\
\frac{\partial}{\partial x_n}(u(x) - u_m(x)) &= (1 - h_m(x_n))\frac{\partial u(x)}{\partial x_n} - \frac{\partial h_m}{\partial x_n}u(x).
\end{aligned}$$

Then

$$\begin{aligned}
\int_{K_+} |u(x) - u_m(x)|^2 dx &= \int_{K_+} \underbrace{|1 - h_m(x_n)|^2}_{\le 1} |u(x)|^2 dx \\
&\le \int_{K_+ \cap \{0 < x_n < \frac{1}{m}\}} |u(x)|^2 dx \\
&\to 0 \text{ as } m \to \infty; \\
\int_{K_+} \left|\frac{\partial}{\partial x_j}(u - u_m)\right|^2 dx &= \int_{K_+} |1 - h_m(x_n)|^2 \left|\frac{\partial u}{\partial x_j}\right|^2 dx \\
&\le \int_{K_+ \cap \{0 < x_n < \frac{1}{m}\}} \left|\frac{\partial u}{\partial x_j}\right|^2 dx \\
&\to 0 \text{ as } m \to \infty;
\end{aligned}$$

$$\left(\int_{K_+}\left|\frac{\partial}{\partial x_n}(u-u_m)\right|^2 dx\right)^{1/2} = \underbrace{\left(\int_{K_+}|1-h_m(x_n)|^2\left|\frac{\partial u}{\partial x_n}\right|^2 dx\right)^{1/2}}_{\to 0 \text{ as } m\to\infty} + \underbrace{\left(\int_{K_+}\left|\frac{\partial h_m}{\partial x_n}\right|^2 |u|^2\, dx\right)^{1/2}}_{=J_m[u]}.$$

It remains to show that $J_m[u]\to 0$ as $m\to\infty$. We have:

$$\frac{\partial h_m(x)}{\partial x_n} = \frac{\partial}{\partial x_n}(h(mx_n)) = mh'(mx_n).$$

Using (13), we obtain:

$$\begin{aligned}
J_m[u] &= m^2\int_{K_+}\left|h'(mx_n)\right|^2\left|\int_0^{x_n}\frac{\partial u}{\partial x_n}(x',t)dt\right|^2 dx\\
&\le m^2\int_{K_+}\underbrace{\left|h'(mx_n)\right|^2}_{\le C}\left(\int_0^{x_n}\left|\frac{\partial u}{\partial x_n}(x',t)\right|^2 dt\right)\underbrace{\left(\int_0^{x_n}1^2dt\right)}_{=x_n}dx\\
&\le cm^2\int_{K_+\cap\left\{0<x_n<\frac{1}{m}\right\}} x_n\left(\int_0^{x_n}\left|\frac{\partial u}{\partial x_n}(x',t)\right|^2 dt\right)dx_n dx'\\
&\le cm^2\frac{1}{m^2}\int_{K_+\cap\left\{0<x_n<\frac{1}{m}\right\}}\left|\frac{\partial u}{\partial x_n}\right|^2 dx\\
&\to 0 \text{ as } m\to\infty.
\end{aligned}$$

Next step: consider mollifications of $u_m(x)$: $(u_m)_\rho$, $\rho>0$. Then $(u_m)_\rho\in C_0^\infty(K_+)$ for sufficiently small ρ and $\|(u_m)_\rho - u_m\|_{H^1(K_+)}\xrightarrow{\rho\to 0} 0$.
Thus, we can approximate function $u(x)$ by functions $(u_m)_\rho\in C_0^\infty(K_+)$ in $H^1(K_+)$–norm $\Rightarrow u\in \overset{\circ}{H}{}^1(K_+)$.

■

§5: Application to elliptic boundary value problems

1. Dirichlet problem for the Poisson equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Consider the classical Dirichlet problem:

$$\left.\begin{array}{l} -\triangle u = F, \quad x \in \Omega \\ u|_{\partial\Omega} = g. \end{array}\right\} \tag{1}$$

If $\Phi(x)$ is arbitrary function in Ω such that $\Phi|_{\partial\Omega} = g$, then the function $v(x) = u(x) - \Phi(x)$ is solution of the problem

$$\left.\begin{array}{l} -\triangle v = f, \quad x \in \Omega \\ v|_{\partial\Omega} = 0 \end{array}\right\} \tag{2}$$

where $f(x) = F(x) + \triangle\Phi(x)$. First, we'll study problem (2) with homogeneous boundary condition. In the classical setting of problem (2), the boundary is sufficiently smooth, $f \in C(\overline{\Omega})$ and solution $v \in C^2(\overline{\Omega})$.
Now we want to define „weak" solution of problem (2) under wide conditions on $\partial\Omega$ and f. Let us *formally* multiply equation $-\triangle v = f$ by the test function $\varphi \in C_0^\infty(\Omega)$ and integrate over Ω. Then $v(x)$ satisfies the integral identity

$$\int_\Omega \nabla v \overline{\nabla\varphi} dx = \int_\Omega f\overline{\varphi} dx, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{3}$$

The left–hand side is well–defined for any $v \in \overset{\circ}{H}{}^1(\Omega) = \overset{\circ}{W}{}_2^1(\Omega)$, $\varphi \in \overset{\circ}{H}{}^1(\Omega)$; and the right–hand side is well–defined for $f \in H^{-1}(\Omega)$, $\varphi \in \overset{\circ}{H}{}^1(\Omega)$ (since $H^{-1}(\Omega)$ is the dual space to $\overset{\circ}{H}{}^1(\Omega)$ with respect to L_2–duality). The boundary condition $v|_{\partial\Omega} = 0$ we understand in the sense that $v \in \overset{\circ}{H}{}^1(\Omega)$. Then we can consider *arbitrary* domains.
Definition

> Let $\Omega \subset \mathbb{R}^n$ be arbitrary *bounded* domain. A function $v \in \overset{\circ}{H}{}^1(\Omega)$ is called a <u>weak solution</u> of the Dirichlet problem (2) with $f \in H^{-1}(\Omega)$, if v satisfies the identity (3) for any $\varphi \in \overset{\circ}{H}{}^1(\Omega)$.

<u>Theorem 1</u>

> Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, for any $f \in H^{-1}(\Omega)$, *there exists unique* (weak) *solution* $v \in \overset{\circ}{H}{}^1(\Omega)$ of the Dirichlet problem (2). We have $\|v\|_{H^1(\Omega)} \le C\,\|f\|_{H^{-1}(\Omega)}$.

Proof

1) The form

$$[v,\varphi] := \int_\Omega \nabla v \overline{\nabla \varphi} dx, \quad v,\varphi \in \overset{\circ}{H}{}^1(\Omega),$$

defines an inner product in the space $\overset{\circ}{H}{}^1(\Omega)$. The corresponding norm $[v,v]^{1/2}$ is equivalent to the standard norm $\|v\|_{H^1(\Omega)} = \left(\int_\Omega \left(|v|^2 + |\nabla v|^2\right) dx\right)^{1/2}$. This follows from the Friedrichs inequality:

$$\int_\Omega |v|^2 dx \le C_\Omega \int_\Omega |\nabla v|^2 dx, \quad \forall v \in \overset{\circ}{H}{}^1(\Omega)$$

(here it is important that Ω is *bounded*).

2) The right–hand side of (3) is

$$l_f(\varphi) = \int_\Omega f \overline{\varphi} dx.$$

$l_f(\varphi)$ is antilinear continuous functional on $\varphi \in \overset{\circ}{H}{}^1(\Omega)$:

$$|l_f(\varphi)| \le \|f\|_{H^{-1}(\Omega)} \|\varphi\|_{H^1(\Omega)}.$$

We rewrite (3) in the following form:

$$[v,\varphi] = l_f(\varphi). \tag{4}$$

By the Riesz Theorem, for antilinear continuous functional l_f on $\overset{\circ}{H}{}^1(\Omega)$ *there exists unique function* $v \in \overset{\circ}{H}{}^1(\Omega)$ such that $l_f(\varphi) = [v,\varphi]$, and the norm of this functional is equal to the norm of v. (Now we consider $\overset{\circ}{H}{}^1(\Omega)$ as the Hilbert space with the inner product $[\cdot,\cdot]$.) Then, by the Riesz Theorem,

$$\|l_f\| = \sup_{0 \ne \varphi \in \overset{\circ}{H}{}^1(\Omega)} \frac{|l_f(\varphi)|}{[\varphi,\varphi]^{1/2}} = [v,v]^{1/2}. \tag{5}$$

Thus, v is the *unique solution* of (4) ($\Leftrightarrow$ (3)). Since, by definition of the class $H^{-1}(\Omega)$,

$$\|f\|_{H^{-1}(\Omega)} = \sup_{0 \ne \varphi \in \overset{\circ}{H}{}^1(\Omega)} \frac{|l_f(\varphi)|}{\|\varphi\|_{H^1(\Omega)}},$$

and $\|\varphi\|_{H^1(\Omega)} \asymp [\varphi,\varphi]^{1/2}$, it follows from (5) that

$$\|v\|_{H^1(\Omega)} \le C \|f\|_{H^{-1}(\Omega)}.$$

■

2.

Now we return to the problem (1) with non–homogeneous boundary condition $u|_{\partial\Omega} = g$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain of class C^1. Then, by Theorem 10 (trace embedding theorem) the trace operator γ_0 ($\gamma_0 u = u|_{\partial\Omega}$) is continuous from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$:

$$\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega).$$

Consider the problem

$$\left.\begin{aligned} -\triangle u = F, \qquad & x \in \Omega, \\ \gamma_0 u = u|_{\partial\Omega} = g(x), & \end{aligned}\right\}$$

for given $F \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. We look for solution $u \in H^1(\Omega)$. Equation $-\triangle u = F$ in Ω is understood in the sense of distributions: $u(x)$ is a *weak solution* of (1), if $u \in H^1(\Omega)$, $u(x)$ satisfies the identity

$$\int_\Omega \nabla u \overline{\nabla \varphi} dx = \int_\Omega F \overline{\varphi} dx, \quad \forall \varphi \in C_0^\infty(\Omega),$$

and $\gamma_0 u = g$.

Theorem 2

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1. Let $F \in H^{-1}(\Omega)$, $g \in H^{1/2}(\partial\Omega)$. Then there exists unique weak solution $u \in H^1(\Omega)$ of problem (1). We have

$$\|u\|_{H^1(\Omega)} \le C \left(\|F\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)} \right). \tag{6}$$

Proof

1) By Theorem 11 (extension theorem), for $g \in H^{1/2}(\partial\Omega)$, there exists extension $G = P_\Omega g \in H^1(\Omega)$ such that $\gamma_0 G = g$ and

$$\|G\|_{H^1(\Omega)} \le C_1 \|g\|_{H^{1/2}(\partial\Omega)}. \tag{7}$$

If $u \in H^1(\Omega)$ and $\gamma_0 u = g$. Then $v = u - G \in H^1(\Omega)$ and $\gamma_0 v = 0$. This is equivalent to the fact that $v \in \overset{\circ}{H}{}^1(\Omega)$. Function v is solution of the problem

$$\left.\begin{aligned} -\triangle v &= f \\ v|_{\partial\Omega} &= 0, \end{aligned}\right\} \tag{8}$$

where $f = F + \triangle G$. From $G \in H^1(\Omega)$ it follows that $\triangle G \in H^{-1}(\Omega)$ and $\|\triangle G\|_{H^{-1}(\Omega)} \le C_2 \|G\|_{H^1(\Omega)}$. Then $f \in H^{-1}(\Omega)$ and

$$\|f\|_{H^{-1}} \le \|F\|_{H^{-1}} + C_2 \|G\|_{H^1(\Omega)} \le \|F\|_{H^{-1}} + C_1 C_2 \|g\|_{H^{1/2}}.$$

By Theorem 1, there exists unique solution $v \in \overset{\circ}{H}{}^1(\Omega)$ of the problem (8), and $\|v\|_{H^1(\Omega)} \leq C_3 \|f\|_{H^{-1}}$. Then $u = v + G$ is unique solution of the problem (1), and

$$\begin{aligned} \|u\|_{H^1} &\leq \|v\|_{H^1} + \|G\|_{H^1} \\ &\leq C_3 \|f\|_{H^{-1}} + C_1 \|g\|_{H^{1/2}(\partial\Omega)} \\ &\leq C_3 \|F\|_{H^{-1}} + (C_1 C_2 C_3 + C_1) \|g\|_{H^{1/2}(\partial\Omega)} . \end{aligned}$$

■

3. Dirichlet problem with spectral parameter

Now we consider the problem

$$\left.\begin{aligned} -\triangle u &= \lambda u + f(x), \quad x \in \Omega \\ u|_{\partial\Omega} &= 0, \end{aligned}\right\} \tag{9}$$

with spectral parameter λ. Here Ω is *bounded.*

Definition

Let $\Omega \subset \mathbb{R}^n$ be arbitrary bounded domain. Let $f \in H^{-1}(\Omega)$. A function $u \in \overset{\circ}{H}{}^1(\Omega)$ satisfying identity

$$\int_\Omega \nabla u \overline{\nabla \varphi} dx = \lambda \int_\Omega u \overline{\varphi} dx + \int_\Omega f(x) \overline{\varphi} dx, \quad \forall \varphi \in \overset{\circ}{H}{}^1(\Omega), \tag{10}$$

is called a *weak solution* of problem (9).

As before, we denote $[u, \varphi] = \int_\Omega \nabla u \overline{\nabla \varphi} dx$. This is inner product in $\overset{\circ}{H}{}^1(\Omega)$. The form $\int_\Omega u \overline{\varphi} dx$, $u, \varphi \in \overset{\circ}{H}{}^1(\Omega)$ is continuous sesquilinear form in $\overset{\circ}{H}{}^1(\Omega)$. By the Riesz theorem for such forms it can be represented as $[Au, \varphi]$, where A is a linear continuous operator in $\overset{\circ}{H}{}^1(\Omega)$.

Obviously, $\int_\Omega u \overline{\varphi} dx = \overline{\left(\int_\Omega \varphi \overline{u} dx\right)}$, so $[Au, \varphi] = \overline{[A\varphi, u]} = [u, A\varphi]$, $\forall u, \varphi \in \overset{\circ}{H}{}^1(\Omega)$. It follows that $A = A^*$.

Next, $[Au, u] = \int_\Omega |u|^2 dx > 0$ if $u \neq 0$. So, $A > 0$.

Lemma

The operator A is *compact* operator in $\overset{\circ}{H}{}^1(\Omega)$.

Proof

This follows from the embedding theorem: $\overset{\circ}{H}{}^1(\Omega)$ is *compactly* embedded in $L_2(\Omega)$.

We'll use the following property of compact operators: T is a compact operator in the Hilbert space $\mathcal{H}$, if and only if for any sequence $\{u_k\}$ which converges *weakly* in $\mathcal{H}$, the sequence $\{Tu_k\}$ *converges strongly* in $\mathcal{H}$.

Let $\{u_k\}$ be a weakly convergent sequence in $\overset{\circ}{H}{}^1(\Omega)$. Since the embedding operator $J : \overset{\circ}{H}{}^1(\Omega) \hookrightarrow L_2(\Omega)$ is *compact*, $\{u_k\}$ converges strongly in $L_2(\Omega)$. We want to check that $\{Au_k\}$ converges strongly in $L_2(\Omega)$. Since $\{u_k\}$ weakly converges in $\overset{\circ}{H}{}^1(\Omega)$, it follows that $\|u_k\|_{H^1(\Omega)}$ is uniformly bounded. A is a continuous operator; then also $\|Au_k\|_{H^1(\Omega)}$ is uniformly bounded. We have

$$\begin{aligned}
[A(u_k - u_l), A(u_k - u_l)] &= \int_\Omega (u_k - u_l)\overline{(Au_k - Au_l)}dx \\
&\leq \underbrace{\|u_k - u_l\|_{L_2(\Omega)}}_{\to 0} \underbrace{\|Au_k - Au_l\|_{L_2(\Omega)}}_{\leq C} \\
&\to 0 \quad \text{as } k, l \to \infty.
\end{aligned}$$

$\{Au_k\}$ converges strongly in $\overset{\circ}{H}{}^1(\Omega)$. It follows that A is *compact* operator.

■

As before, the functional $l_f(\varphi) = \int_\Omega f\overline{\varphi}dx$ (where $f \in H^{-1}(\Omega)$) is continuous antilinear functional on $\varphi \in \overset{\circ}{H}{}^1(\Omega)$. By the Riesz Theorem, there exists unique element $v \in \overset{\circ}{H}{}^1(\Omega)$ such that $\int_\Omega f\overline{\varphi}dx = [v, \varphi]$, $\forall \varphi \in \overset{\circ}{H}{}^1(\Omega)$, and $\|f\|_{H^{-1}(\Omega)} \asymp \|v\|_{H^1(\Omega)}$.
Now, we can rewrite identity (10) in the form

$$[u, \varphi] = \lambda\,[Au, \varphi] + [v, \varphi]\,, \quad \forall \varphi \in \overset{\circ}{H}{}^1(\Omega), \tag{11}$$

which is equivalent to the equation

$$u - \lambda Au = v, \tag{12}$$

where $v \in \overset{\circ}{H}{}^1(\Omega)$ is given, and we are looking for solution $u \in \overset{\circ}{H}{}^1(\Omega)$. Thus, we reduced the problem (9) to the abstract equation (12) with *compact* operator A in the Hilbert space $\overset{\circ}{H}{}^1(\Omega)$.
We analyse equation (12), using the properties of compact operators.

<u>The case $v = 0$</u> (which corresponds to $f = 0$):

$$\left.\begin{aligned}
-\triangle u &= \lambda u \quad x \in \Omega \\
u|_{\partial\Omega} &= 0
\end{aligned}\right\} \tag{13}$$

$$\Leftrightarrow \quad u - \lambda A u = 0 \quad \Leftrightarrow \quad Au = \mu u \text{ (where } \mu = \frac{1}{\lambda})$$

It is known that the *spectrum of a compact operator is discrete*: it consists of eigenvalues $\mu_j, j \in \mathbb{N}$, that may accumulate only to point $\mu = 0$; each eigenvalue is of finite multiplicity (i. e. , $\dim ker(A - \mu_j I) < \infty$). In our case $A = A^* > 0$, then all eigenvalues μ_j are positive: $\mu_j > 0$. We enumerate eigenvalues in non–increasing order counting multiplicities $\mu_1 \geq \mu_2 \geq \dots$
Then each eigenvalue corresponds to one eigenfunction $u_j : Au_j = \mu_j u_j$, $j \in \mathbb{N}$. Eigenfunctions $\{u_j\}$ are linearly independent. We have : $\mu_j \to 0$ as $j \to \infty$.
Then for the *eigenvalues* $\lambda_j = \frac{1}{\mu_j}$ *of the Dirichlet problem* (13) we have the following properties: $0 < \lambda_1 \leq \lambda_2 \leq \dots$, $\lambda_j \to \infty$ as $j \to \infty$.
Thus, we have the following theorem.

Theorem 3

> The spectrum of the Dirichlet problem (13) is discrete. There exists non–trivial solution only if $\lambda = \lambda_j$, $j \in \mathbb{N}$. All eigenvalues are positive and have finite multiplicities. The only accumulation point is infinity: $\lambda_j \to \infty$ as $j \to \infty$.

The case $v \neq 0$ ($f \neq 0$)

$$\left.\begin{array}{rcl} -\triangle u &=& \lambda u + f \quad x \in \Omega \\ u|_{\partial\Omega} &=& 0 \end{array}\right\} \tag{14}$$

$$\Leftrightarrow \quad u - \lambda A u = v$$

For compact operator A it is known that, if $\lambda \neq \lambda_j (= \frac{1}{\mu_j})$, $\forall j \in \mathbb{N}$, then the operator $(I - \lambda A)^{-1}$ is bounded. We can find unique solution

$$u = (I - \lambda A)^{-1} v,$$

and

$$\|u\|_{H^1(\Omega)} \leq \underbrace{\left\|(I - \lambda A)^{-1}\right\|}_{=C_\lambda} \|v\|_{H^1(\Omega)} .$$

Since $\|v\|_{H^1(\Omega)} \asymp \|f\|_{H^{-1}(\Omega)}$, we arrive at the following theorem.

Theorem 4

> If $\lambda \notin \{\lambda_j\}_{j\in\mathbb{N}}$ (λ is not eigenvalue), then for any $f \in H^{-1}(\Omega)$ there exists unique (weak) solution $u \in \overset{\circ}{H}{}^1(\Omega)$ of the problem (14), and
>
> $$\|u\|_{H^1(\Omega)} \leq C_\lambda \|f\|_{H^{-1}(\Omega)} .$$

Now, suppose that $\lambda = \lambda_j$, and $v \neq 0$ ($f \neq 0$). Then, solution of the equation $u - \lambda_j Au = v$ exists, if v satisfies the solvability condition: $v \perp ker(I - \lambda_j A)$. It means that v is orthogonal (with respect to the inner product $[\cdot, \cdot]$) in $\overset{\circ}{H}{}^1(\Omega)$ to all eigenfunctions $\varphi_j^{(k)}, k = 1, \dots, p$, corresponding to the eigenvalue λ_j (here p is the multiplicity of λ_j). Since $[v, \varphi] = \int_\Omega f(x)\overline{\varphi(x)}dx$, this solvability condition is equivalent to:

$$\int_\Omega f(x)\overline{\varphi_j^{(k)}(x)}dx = 0, \quad k = 1, \dots, p. \tag{15}$$

The solution $u(x)$ is not unique, but is defined up to a summand $\sum_{j=1}^p c_j \varphi_j^{(k)}$ with arbitrary constants c_j.

Theorem 5

If $\lambda = \lambda_j$ is eigenvalue of the Dirichlet problem, and $\varphi_j^{(k)}, k = 1, \dots, p$, are corresponding (linearly independent) eigenfunctions, then problem (14) has solution for any $f \in H^{-1}(\Omega)$, which satisfies the solvability conditions (15). Solution is not unique and is represented as

$$u = u_0 + \sum_{j=1}^p c_j \varphi_j^{(k)},$$

where u_0 is a fixed solution, and c_j are arbitrary constants.

4. Hilbert–Schmidt Theorem

Finally, we can apply the Hilbert–Schmidt Theorem for compact operators and obtain the following result.

Let $\lambda_1 \leq \lambda_2 \leq \dots$ be eigenvalues of the Dirichlet problem. Here we repeat each λ_j according to its multiplicity. There exists an *orthogonal system of eigenfunctions* $\{\varphi_j\}_{j \in \mathbb{N}}$:

$$\varphi_j - \lambda_j A\varphi_j = 0, \;\; j \in \mathbb{N}, \quad [\varphi_j, \varphi_l] = 0, \;\; j \neq l.$$

By the Hilbert–Schmidt Theorem, $\{\varphi_j\}_{j \in \mathbb{N}}$ is *orthogonal basis* in $\overset{\circ}{H}{}^1(\Omega)$, i. e. , for any $F \in \overset{\circ}{H}{}^1(\Omega)$,

$$F = \sum_{j=1}^\infty \frac{[F, \varphi_j]}{[\varphi_j, \varphi_j]}\varphi_j$$

It is important that $\varphi_j \perp \varphi_l$ also in $L_2(\Omega)$. Indeed, $[Au, \varphi] = \int_\Omega u\overline{\varphi}dx$ (by Definition of operator A). Next, $A\varphi_j = \mu_j \varphi_j$ (where $\mu_j = \frac{1}{\lambda_j}$). Thus,

$$[A\varphi_j, \varphi_l] = \int_\Omega \varphi_j \overline{\varphi_l} dx = \mu_j [\varphi_j, \varphi_l] = 0, \;\; j \neq l.$$

We have

$$\begin{aligned}
[F,\varphi_j] &= \lambda_j\,[F,A\varphi_j] = \lambda_j \int_\Omega F\overline{\varphi_j}dx, \\
[\varphi_j,\varphi_j] &= \lambda_j\,[A\varphi_j,\varphi_j] = \lambda_j \int_\Omega |\varphi_j|^2 dx.
\end{aligned}$$

Then

$$\frac{[F,\varphi_j]}{[\varphi_j,\varphi_j]} = \frac{\int_\Omega F\overline{\varphi_j}dx}{\int_\Omega |\varphi_j|^2 dx} = \frac{(F,\varphi_j)_{L_2(\Omega)}}{\|\varphi_j\|^2_{L_2(\Omega)}}, \text{ and}$$

$$F = \sum_{j=1}^{\infty} \frac{(F,\varphi_j)_{L_2(\Omega)}}{\|\varphi_j\|^2_{L_2(\Omega)}}\varphi_j.$$

The last fomula can be extended to all $F \in L_2(\Omega)$.

Theorem 6

> Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then there exists an ortogonal system of eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$ of the Dirichlet problem. This system forms an orthogonal basis in $L_2(\Omega)$ and in $\overset{\circ}{H}{}^1(\Omega)$ (with respect to the inner product $[\cdot,\cdot]$).