Lecture Notes on Sobolev Spaces

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1 Distributions and weak derivatives

We denote by $\mathbf{L}^1_{loc}(I\!R)$ the space of **locally integrable functions** $f : I\!R \mapsto I\!R$. These are the Lebesgue measurable functions which are integrable over every bounded interval.

The **support** of a function ϕ, denoted by $Supp(\phi)$, is the closure of the set $\{x\,;\ \ \phi(x) \neq 0\}$ where ϕ does not vanish. By $\mathcal{C}^\infty_c(I\!R)$ we denote the space of continuous functions with compact support, having continuous derivatives of every order.

Every locally integrable function $f \in \mathbf{L}^1_{loc}(I\!R)$ determines a linear functional $\Lambda_f : \mathcal{C}^\infty_c(I\!R) \mapsto I\!R$, namely

$$\Lambda_f(\phi) \ \doteq\ \int_{I\!R} f(x)\phi(x)\,dx\,. \tag{1.1}$$

Notice that this integral is well defined for all $\phi \in \mathcal{C}^\infty_c(I\!R)$, because ϕ vanishes outside a compact set. Moreover, if $Supp(\phi) \subseteq [a,b]$, we have the estimate

$$|\Lambda_f(\phi)| \ \leq\ \left(\int_a^b |f(x)|\,dx\right) \|\phi\|_{\mathcal{C}^0}\,. \tag{1.2}$$

Next, assume that f is continuously differentiable. Then its derivative

$$f'(x) \ =\ \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is continuous, hence locally integrable. In turn, f' also determines a linear functional on $\mathcal{C}^\infty_c(I\!R)$, namely

$$\Lambda_{f'}(\phi) \ \doteq\ \int_{I\!R} f'(x)\,\phi(x)\,dx \ =\ -\int_{I\!R} f(x)\,\phi'(x)\,dx\,. \tag{1.3}$$

At this stage, a key observation is that the first integral in (1.3) is defined only if $f'(x)$ exists for a.e. x, and is locally integrable. However, the second integral is well defined for every locally integrable function f, even if f does not have a pointwise derivative at any point. Moreover, if $Supp(\phi) \subseteq [a,b]$, we have the estimate

$$|\Lambda_{f'}(\phi)| \ \leq\ \left(\int_a^b |f(x)|\,dx\right) \|\phi\|_{\mathcal{C}^1}\,.$$

This construction can be performed also for higher order derivatives.

Definition 1.1 *Given an integer $k \geq 1$, the* **distributional derivative** *of order k of $f \in \mathbf{L}^1_{loc}$ is the linear functional*

$$\Lambda_{D^k f}(\phi) \doteq (-1)^k \int_{I\!R} f(x) D^k \phi(x)\, dx\,.$$

If there exists a locally integrable function g such that $\Lambda_{D^k f} = \Lambda_g$, namely

$$\int_{I\!R} g(x)\phi(x)\, dx = (-1)^k \int_{I\!R} f(x) D^k \phi(x)\, dx \qquad \text{for all} \quad \phi \in \mathcal{C}^\infty_c(I\!R),$$

then we say that g is the **weak derivative** *of order k of f.*

Remark 1.1 Classical derivatives are defined pointwise, as limits of difference quotients. On the other hand, weak derivatives are defined only in an integral sense, up to a set of measure zero. By arbitrarily changing the function f on a set of measure zero we do not affect its weak derivatives in any way.

Example 1. Consider the function

$$f(x) \doteq \begin{cases} 0 & \text{if} \quad x \leq 0, \\ x & \text{if} \quad x > 0\,. \end{cases}$$

Its distributional derivative is the map

$$\Lambda(\phi) = -\int_0^\infty x \cdot \phi'(x)\, dx = \int_0^\infty \phi(x)\, dx = \int_{I\!R} H(x)\, \phi(x)\, dx\,,$$

where

$$H(x) = \begin{cases} 0 & \text{if} \quad x \leq 0, \\ 1 & \text{if} \quad x > 0\,. \end{cases} \tag{1.4}$$

In this case, the Heaviside function H in (1.4) is the weak derivative of f.

Example 2. The function H in (1.4) is locally integrable. Its distributional derivative is the linear functional

$$\Lambda(\phi) \doteq -\int_{I\!R} H(x)\, \phi'(x)\, dx \; - \int_0^\infty \phi(x)\, dx = \phi(0)\,.$$

This corresponds to the Dirac measure, concentrating a unit mass at the origin. We claim that the function H does not have any weak derivative. Indeed, assume that, for some locally integrable function g, one has

$$\int g(x)\phi(x)\, dx = \phi(0) \qquad \text{for all} \quad \phi \in \mathcal{C}^\infty_c\,.$$

By the Lebesgue Dominated Convergence Theorem,

$$\lim_{h\to 0} \int_{-h}^{h} |g(x)|\, dx = 0.$$

Hence we can choose $\delta > 0$ so that $\int_{-\delta}^{\delta} |g(x)|\,dx \leq 1/2$. Let $\phi : I\!\!R \mapsto [0,1]$ be a smooth function, with $\phi(0) = 1$ and with support contained in the interval $[-\delta,\, \delta]$. We now reach a contradiction by writing

$$1 = \phi(0) \;=\; \Lambda(\phi) \;=\; \int_{I\!\!R} g(x)\phi(x)\,dx \;=\; \int_{-\delta}^{\delta} g(x)\phi(x)\,dx \;\leq\; \max_x |\phi(x)| \cdot \int_{-\delta}^{\delta} |g(x)|\,dx \;\leq\; \frac{1}{2}\,.$$

Example 3. Consider the function

$$f(x) \;\doteq\; \begin{cases} 0 & \text{if } x \text{ is rational}, \\ 2 + \sin x & \text{if } x \text{ is irrational}\,. \end{cases}$$

Clearly f is discontinuous at every point x. Hence it is not differentiable at any point. On the other hand, the function $g(x) = \cos x$ provides a weak derivative for f. Indeed, the behavior of f on the set of rational points (having measure zero) is irrelevant. We thus have

$$-\int f(x)\phi'(x)\,dx \;=\; -\int (2 + \sin x)\,\phi'(x)\,dx \;=\; \int (\cos x)\,\phi(x)\,dx.$$

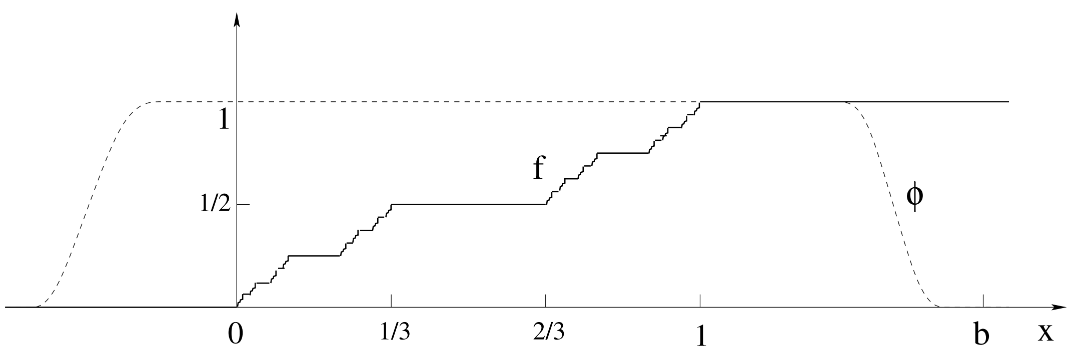

Figure 1: The Cantor function f and a test function ϕ showing that $g(x) \equiv 0$ cannot be the weak derivative of f.

Example 4. Consider the Cantor function $f : I\!\!R \mapsto [0,1]$, defined by

$$f(x) \;=\; \begin{cases} 0 & \text{if} \quad x \leq 0\,, \\ 1 & \text{if} \quad x \geq 1\,, \\ 1/2 & \text{if} \quad x \in [1/3,\, 2/3]\,, \\ 1/4 & \text{if} \quad x \in [1/9,\, 2/9]\,, \\ 3/4 & \text{if} \quad x \in [7/9,\, 8/9]\,, \\ & \cdots \end{cases} \tag{1.5}$$

This provides a classical example of a continuous function which is not absolutely continuous. We claim that f does not have a weak derivative. Indeed, let $g \in \mathbf{L}^1_{loc}$ be a weak derivative of f. Since f is constant on each of the open sets

$$]-\infty, 0[\,, \qquad]1, +\infty[\,, \qquad \Big]\frac{1}{3}, \frac{2}{3}\Big[\,, \qquad \Big]\frac{1}{9}, \frac{2}{9}\Big[\,, \qquad \Big]\frac{7}{9}, \frac{8}{9}\Big[\,, \quad \ldots$$

we must have $g(x) = f'(x) = 0$ on the union of these open intervals. Hence $g(x) = 0$ for a.e. $x \in \mathbb{R}$. To obtain a contradiction, it remains to show that the function $g \equiv 0$ is NOT the weak derivative of f. As shown in Fig. 1, let $\phi \in \mathcal{C}_c^\infty$ be a test function such that $\phi(x) = 1$ for $x \in [0,1]$ while $\phi(x) = 0$ for $x \geq b$. Then

$$\int g(x)\phi(x)\,dx \;=\; 0 \;\neq\; 1 \;=\; -\int f(x)\phi'(x)\,dx\,.$$

1.1 Distributions

The construction described in the previous section can be extended to any open domain in a multi-dimensional space. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. By $\mathbf{L}^1_{loc}(\Omega)$ we denote the space of locally integrable functions on Ω. These are the measurable functions $f : \Omega \mapsto \mathbb{R}$ which are integrable restricted to every compact subset $K \subset \Omega$.

Example 5. The functions e^x, and $\ln|x|$ are in $\mathbf{L}^1_{loc}(\mathbb{R})$, while $x^{-1} \notin \mathbf{L}^1_{loc}(\mathbb{R})$. On the other hand, the function $f(x) = x^\gamma$ is in $\mathbf{L}^1_{loc}(]0,\infty[)$ for every (positive or negative) exponent $\gamma \in \mathbb{R}$. In several space dimensions, the function $f(x) = |x|^{-\gamma}$ is in $\mathbf{L}^1_{loc}(\mathbb{R}^n)$ provided that $\gamma < n$. One should keep in mind that the pointwise values of a function $f \in \mathbf{L}^1_{loc}$ on a set of measure zero are irrelevant.

By $\mathcal{C}_c^\infty(\Omega)$ we denote the space of continuous functions $\phi : \Omega \mapsto \mathbb{R}$, having continuous partial derivatives of all orders, and whose support is a compact subset of Ω. Functions $\phi \in \mathcal{C}_c^\infty(\Omega)$ are usually called *"test functions"*. We recall that the **support** of a function ϕ is the closure of the set where ϕ does not vanish:

$$Supp\,(\phi) \;\doteq\; \overline{\{x \in \Omega\,; \quad \phi(x) \neq 0\}}.$$

We shall need an efficient way to denote higher order derivatives of a function f. A **multi-index** $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an n-tuple of non-negative integer numbers. Its length is defined as

$$|\alpha| \;\doteq\; \alpha_1 + \alpha_2 + \cdots + \alpha_n\,.$$

Each multi-index α determines a partial differential operator of order $|\alpha|$, namely

$$D^\alpha f \;=\; \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f\,.$$

Definition 1.2 *By a* **distribution** *on the open set $\Omega \subseteq \mathbb{R}^n$ we mean a linear functional $\Lambda : \mathcal{C}_c^\infty(\Omega) \mapsto \mathbb{R}$ such that the following boundedness property holds.*

- *For every compact $K \subset \Omega$ there exist an integer $N \geq 0$ and a constant C such that*

$$|\Lambda(\phi)| \;\leq\; C\,\|\phi\|_{\mathcal{C}^N} \qquad \text{for every } \phi \in \mathcal{C}^\infty \text{ with support contained inside } K. \tag{1.6}$$

In other words, for all test functions ϕ which vanish outside a given compact set K, the value $\Lambda(\phi)$ should be bounded in terms of the maximum value of derivatives of ϕ, up to a certain order N.

Notice that here both N and C depend on the compact subset K. If there exists an integer $N \geq 0$ independent of K such that (1.6) holds (with $C = C_K$ possibly still depending on K), we say that the distribution has finite order. The smallest such integer N is called the **order of the distribution**.

Example 6. Let Ω be an open subset of $I\!R^n$ and consider any function $f \in \mathbf{L}^1_{loc}(\Omega)$. Then the linear map $\Lambda_f : \mathcal{C}^\infty_c(\Omega) \mapsto I\!R$ defined by

$$\Lambda_f(\phi) \doteq \int_\Omega f\,\phi\,dx\,, \tag{1.7}$$

is a distribution. Indeed, it is clear that Λ_f is well defined and linear. Given a compact subset $K \subset \Omega$, for every test function ϕ with $Supp(\phi) \subseteq K$ we have the estimate

$$|\Lambda_f(\phi)| = \left|\int_K f\,\phi\,dx\right| \leq \int_K |f(x)|\,dx \cdot \max_{x\in K}|\phi(x)| \leq C\,\|\phi\|_{\mathcal{C}^0}\,.$$

Hence the estimate (1.6) holds with $C = \int_K |f|\,dx$ and $N = 0$. This provides an example of a distribution of order zero.

The family of all distributions on Ω is clearly a vector space. A remarkable fact is that, while a function f may not admit a derivative (in the classical sense), for a distribution Λ an appropriate notion of derivative can always be defined.

Definition 1.3 *Given a distribution Λ and a multi-index α, we define the distribution $D^\alpha\Lambda$ by setting*

$$D^\alpha\Lambda(\phi) \doteq (-1)^{|\alpha|}\Lambda(D^\alpha\phi)\,. \tag{1.8}$$

It is easy to check that $D^\alpha\Lambda$ is itself a distribution. Indeed, the linearity of the map $\phi \mapsto D^\alpha\Lambda(\phi)$ is clear. Next, let K be a compact subset of Ω and let ϕ is a test function with support contained in K. By assumption, there exists a constant C and an integer $N \geq 0$ such that (1.6) holds. In turn, this implies

$$|D^\alpha\Lambda(\phi)| = |\Lambda(D^\alpha\phi)| \leq C\,\|D^\alpha\phi\|_{\mathcal{C}^N} \leq C\,\|\phi\|_{\mathcal{C}^{N+|\alpha|}}.$$

Hence $D^\alpha\Lambda$ also satisfies (1.6), with N replaced by $N + |\alpha|$.

Notice that, if Λ_f is the distribution at (1.7) corresponding to a function f which is $|\alpha|$-times continuously differentiable, then we can integrate by parts and obtain

$$D^\alpha\Lambda_f(\phi) = (-1)^{|\alpha|}\Lambda_f(D^\alpha\phi) = (-1)^{|\alpha|}\int f(x)D^\alpha\phi(x)\,dx = \int D^\alpha f(x)\phi(x)\,dx = \Lambda_{D^\alpha f}(\phi)\,.$$

This justifies the formula (1.8).

1.2 Weak derivatives

For every locally integrable function f and every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, the distribution Λ_f always admits a distributional derivative $D^\alpha\Lambda_f$, defined acording to (1.8). In some cases, one can find a locally integrable function g such that the distribution $D^\alpha\Lambda_f$ coincides with the distribution Λ_g. This leads to the concept of weak derivative.

Definition 1.4 *Let* $f \in \mathbf{L}^1_{loc}(\Omega)$ *be a locally integrable function on the open set* $\Omega \subseteq I\!\!R^n$ *and let* Λ_f *be the corresponding distribution, as in (1.7). Given a multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n)$*, if there exists a locally integrable function* $g \in \mathbf{L}^1_{loc}(\Omega)$ *such that* $D^\alpha \Lambda_f = \Lambda_g$*, i.e.*

$$\int f\, D^\alpha \phi \, dx \;=\; (-1)^{|\alpha|} \int g\, \phi \, dx \qquad \textit{for all test functions } \; \phi \in \mathcal{C}^\infty_c(\Omega), \tag{1.9}$$

then we say that g *is the* **weak** α**-th derivative of** f*, and write* $g = D^\alpha f$*.*

In general, a weak derivative may not exist. In particular, Example 2 shows that the Heaviside function does not admit a weak derivative. Indeed, its distributional derivative is a Dirac measure (concentrating a unit mass at the origin), not a locally integrable function. On the other hand, if a weak derivative does exist, then it is unique (up to a set of measure zero).

Lemma 1.1 (uniqueness of weak derivatives). *Assume* $f \in \mathbf{L}^1_{loc}(\Omega)$ *and let* $g, \tilde g \in \mathbf{L}^1_{loc}(\Omega)$ *be weak* α*-th derivatives of* f*, so that*

$$\int f\, D^\alpha \phi \, dx \;=\; (-1)^{|\alpha|} \int g\, \phi \, dx \;=\; (-1)^{|\alpha|} \int \tilde g\, \phi \, dx$$

for all test functions $\phi \in \mathcal{C}^\infty_c(\Omega)$*. Then* $g(x) = \tilde g(x)$ *for a.e.* $x \in \Omega$*.*

Proof. By the assumptions, the function $(g - \tilde g) \in \mathbf{L}^1_{loc}(\Omega)$ satisfies

$$\int (g - \tilde g)\phi \, dx \;=\; 0 \qquad \text{for all test functions } \; \phi \in \mathcal{C}^\infty_c(\Omega).$$

By Corollary A.1 in the Appendix, we thus have $g(x) - \tilde g(x) = 0$ for a.e. $x \in \Omega$. □

If a function f is twice continuously differentiable, a basic theorem of Calculus states that partial derivatives commute: $f_{x_j x_k} = f_{x_k x_j}$. This property remains valid for weak derivatives. To state this result in full generality, we recall that the sum of two multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ is defined as $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

Lemma 1.2 (weak derivatives commute). *Assume that* $f \in \mathbf{L}^1_{loc}(\Omega)$ *has weak derivatives* $D^\alpha f$ *for every* $|\alpha| \le k$*. Then, for every pair of multi-indices* α, β *with* $|\alpha| + |\beta| \le k$ *one has*

$$D^\alpha(D^\beta f) \;=\; D^\beta(D^\alpha f) \;=\; D^{\alpha+\beta} f\,. \tag{1.10}$$

Proof. Consider any test function $\phi \in \mathcal{C}^\infty_c(\Omega)$. Using the fact that $D^\beta \phi \in \mathcal{C}^\infty_c(\Omega)$ is a test function as well, we obtain

$$\begin{aligned}
\int_\Omega D^\alpha f\, D^\beta \phi \, dx \;&=\; (-1)^{|\alpha|} \textstyle\int_\Omega f\, (D^{\alpha+\beta}\phi)\, dx \\
&=\; (-1)^{|\alpha|}(-1)^{|\alpha+\beta|} \displaystyle\int_\Omega (D^{\alpha+\beta} f)\, \phi \, dx \\
&=\; (-1)^{|\beta|} \int_\Omega (D^{\alpha+\beta} f)\, \phi \, dx\,.
\end{aligned}$$

By definition, this means that $D^{\alpha+\beta}f = D^\beta(D^\alpha f)$. Exchanging the roles of the multi-indices α and β in the previous computation one obtains $D^{\alpha+\beta}f = D^\alpha(D^\beta f)$, completing the proof.
□

The next lemma extends another familiar result, stating that the weak derivative of a limit coincides with the limit of the weak derivatives.

Lemma 1.3 (convergence of weak derivatives). *Consider a sequence of functions $f_n \in \mathbf{L}^1_{loc}(\Omega)$. For a fixed multi-index α, assume that each f_n admits the weak derivative $g_n = D^\alpha f_n$. If $f_n \to f$ and $g_n \to g$ in $\mathbf{L}^1_{loc}(\Omega)$, then $g = D^\alpha f$.*

Proof. For every test function $\phi \in \mathcal{C}^\infty_c(\Omega)$, a direct computation yields

$$\int_\Omega g\,\phi\,dx \;=\; \lim_{n\to\infty}\int_\Omega g_n\,\phi\,dx \;=\; \lim_{n\to\infty}(-1)^{|\alpha|}\int_\Omega f_n\,D^\alpha\phi\,dx \;=\; (-1)^{|\alpha|}\int_\Omega f\,D^\alpha\phi\,dx\,.$$

By definition, this means that g is the α-th weak derivative of f. □

2 Mollifications

As usual, let $\Omega \subseteq I\!\!R^n$ be an open set. For a given $\varepsilon > 0$, define the open subset

$$\Omega_\varepsilon \doteq \{x \in I\!\!R^n\,;\;\; \overline{B}(x,\varepsilon) \subset \Omega\}\,. \tag{2.1}$$

Then for every $u \in \mathbf{L}^1_{loc}(\Omega)$ the mollification

$$u_\varepsilon(x) \;\doteq\; (J_\varepsilon * u)(x) \;=\; \int_{B(x,\varepsilon)} J_\varepsilon(x-y)u(y)\,dy$$

is well defined for every $x \in \Omega_\varepsilon$. Moreover, $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$. A very useful property of the mollification operator is that it commutes with weak differentiation.

Lemma 2.1 (mollifications). *Let $\Omega_\varepsilon \subset \Omega$ be as in (2.1). Assume that a function $u \in \mathbf{L}^1_{loc}(\Omega)$ admits a weak derivative $D^\alpha u$, for some multi-index α. Then the derivative of the mollification (which exists in the classical sense) coincides with the mollification of the weak derivative:*

$$D^\alpha(J_\varepsilon * u) \;=\; J_\varepsilon * D^\alpha u \qquad\qquad \textit{for all}\;\; x \in \Omega_\varepsilon\,. \tag{2.2}$$

Proof. Observe that, for each fixed $x \in \Omega_\varepsilon$, the function $\phi(y) \doteq J_\varepsilon(x-y)$ is in $\mathcal{C}^\infty_c(\Omega)$. Hence we can apply the definition of weak derivative $D^\alpha u$, using ϕ as a test function. Writing D^α_x

and D^α_y to distinguish differentiation w.r.t. the variables x or y, we thus obtain

$$\begin{aligned} D^\alpha u_\varepsilon(x) &= D^\alpha_x \Big(\int_\Omega J_\varepsilon(x-y)\, u(y)\, dy\Big) \\ &= \int_\Omega D^\alpha_x J_\varepsilon(x-y)\, u(y)\, dy \\ &= (-1)^{|\alpha|} \int_\Omega D^\alpha_y J_\varepsilon(x-y)\, u(y)\, dy \\ &= (-1)^{|\alpha|+|\alpha|} \int_\Omega J_\varepsilon(x-y)\, D^\alpha_y u(y)\, dy \\ &= \Big(J_\varepsilon * D^\alpha u\Big)(x). \end{aligned}$$

□

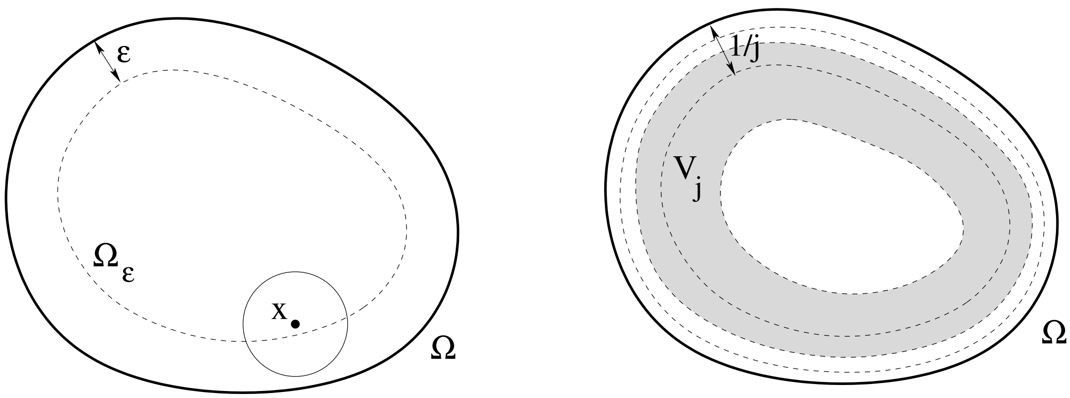

Figure 2: Left: the open subset $\Omega_\varepsilon \subset \Omega$ of points having distance $> \varepsilon$ from the boundary. Right: the domain Ω can be covered by countably many open subdomains $V_j = \Omega_{1/(j-1)} \setminus \overline{\Omega}_{1/(j+1)}$.

This property of mollifications stated in Lemma2.1 provides the key tool to relate weak derivatives with partial derivatives in the classical sense. As a first application, we prove

Corollary 2.1 (constant functions). *Let $\Omega \subseteq \mathbb{R}^n$ be an open, connected set, and assume $u \in L^1_{loc}(\Omega)$. If the first order weak derivatives of u satisfy*

$$D_{x_i} u(x) \;=\; 0 \qquad\qquad \textit{for} \;\; i = 1, 2, \ldots, n \quad \textit{and a.e.} \;\; x \in \Omega\,,$$

then u coincides a.e. with a constant function.

Proof. 1. For $\varepsilon > 0$, consider the mollified function $u_\varepsilon = J_\varepsilon * u$. By the previous analysis, $u_\varepsilon : \Omega_\varepsilon \mapsto \mathbb{R}$ is a smooth function, whose derivatives $D_{x_i} u_\varepsilon$ vanish identically on Ω_ε. Therefore, u_ε must be constant on each connected component of Ω_ε.

2. Now consider any two points $x, y \in \Omega$. Since the open set Ω is connected, there exists a polygonal path Γ joining x with y and remaining inside Ω. Let $\delta \doteq \min_{z\in\Gamma} d(z, \partial\Omega)$ be the minimum distance of points in Γ to the boundary of Ω. Then for every $\varepsilon < \delta$ the whole polygonal curve Γ is in Ω_ε. Hence x, y lie in the same connected component of Ω_ε. In particular, $u_\varepsilon(x) = u_\varepsilon(y)$.

3. Call $\tilde{u}(x) \doteq \lim_{\varepsilon\to 0} u_\varepsilon(x)$. By the previous step, $\tilde{u}$ is a constant function on Ω. Moreover, $\tilde{u}(x) = u(x)$ for every Lebesgue point of u, hence almost everywhere on Ω. This concludes the proof. □

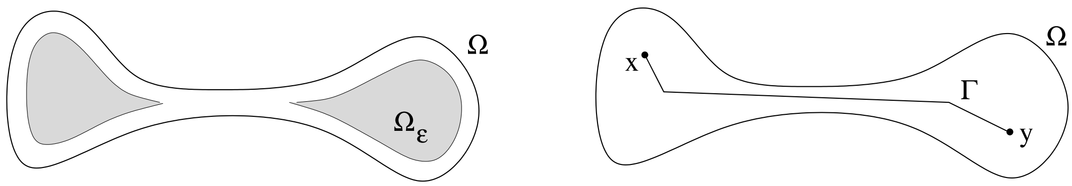

Figure 3: Left: even if Ω is connected, the subdomain $\Omega_\varepsilon \doteq \{x \in \Omega;\ \overline{B}(x,\varepsilon) \subseteq \Omega\}$ may not be connected. Right: any two points $x, y \in \Omega$ can be connected by a polygonal path Γ remaining inside Ω. Hence, if $\varepsilon > 0$ is sufficiently small, x and y belong to the same connected component of Ω_ε.

In the one-dimensional case, relying again on Lemma 2.1, we now characterize the set of functions having a weak derivative in $\mathbf{L}^1$.

Corollary 2.2 (absolutely continuous functions). *Consider an open interval* $]a,b[$ *and assume that* $u \in \mathbf{L}^1_{loc}(]a,b[)$ *has a weak derivative* $v \in \mathbf{L}^1(]a,b[)$. *Then there exists an absolutely continuous function* $\tilde{u}$ *such that*

$$\tilde{u}(x) \;=\; u(x) \qquad \text{for a.e. } x \in]a,b[\,, \tag{2.3}$$

$$v(x) \;=\; \lim_{h\to 0} \frac{\tilde{u}(x+h) - \tilde{u}(x)}{h} \qquad \text{for a.e. } x \in]a,b[\,. \tag{2.4}$$

Proof. Let $x_0 \in]a,b[$ be a Lebesgue point of u, and define

$$\tilde{u}(x) \;\doteq\; u(x_0) + \int_{x_0}^{x} v(y)\, dy$$

Clearly $\tilde{u}$ is absolutely continuous and satisfies (2.4).

In order to prove (2.3), let J_ε be the standard mollifier and call $u_\varepsilon \doteq J_\varepsilon * u$, $v_\varepsilon \doteq J_\varepsilon * v$. Then $u_\varepsilon, v_\varepsilon \in \mathcal{C}^\infty(]a+\varepsilon\,,\ b-\varepsilon[)$. Moreover, Lemma 2.1 yields

$$u_\varepsilon(x) \;=\; u_\varepsilon(x_0) + \int_{x_0}^{x} v_\varepsilon(y)\, dy \qquad \text{for all } x \in]a+\varepsilon\,,\ b-\varepsilon[\,. \tag{2.5}$$

Letting $\varepsilon \to 0$ we have $u_\varepsilon(x_0) \to u(x_0)$ because x_0 is a Lebesgue point. Moreover, the right hand side of (2.5) converges to $\tilde{u}(x)$ for every $x \in]a,b[$, while the left hand side converges to $u(x)$ for every Lebesgue point of u (and hence almost everywhere). Therefore (2.3) holds. □

If $f, g \in \mathbf{L}^1_{loc}(\Omega)$ are weakly differentiable functions, for any constants $a, b \in I\!R$ it is clear that the linear combination $af + bg$ is also weakly differentiable. Indeed, it satisfies

$$D_{x_i}(af + bg) \;=\; a\, D_{x_i} f + b\, D_{x_i} g\,. \tag{2.6}$$

We now consider products and compositions of weakly differentiable functions. One should be aware that, in general, the product of two functions $f, g \in \mathbf{L}^1_{loc}$ may not be locally integrable. Similarly, the product of two weakly differentiable functions on $I\!R^n$ may not be weakly differentiable (see problem 20). For this reason, in the next lemma we shall assume that one of the two functions is continuously differentiable with uniformly bounded derivatives.

Given two multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, we recall that the notation $\beta \le \alpha$ means $\beta_i \le \alpha_i$ for every $i = 1, \ldots, n$. Moreover,

$$\binom{\alpha}{\beta} \;\doteq\; \frac{\alpha!}{\beta!\,(\alpha-\beta)!} \;\doteq\; \frac{\alpha_1!}{\beta_1!\,(\alpha_1-\beta_1)!}\cdot\frac{\alpha_2!}{\beta_2!\,(\alpha_2-\beta_2)!}\;\cdots\;\frac{\alpha_2!}{\beta_2!\,(\alpha_2-\beta_2)!}\,.$$

Lemma 2.2 (products and compositions of weakly differentiable functions). *Let $\Omega \subseteq I\!R^n$ be any open set and consider a function $u \in \mathbf{L}^1_{loc}(\Omega)$ having weak derivatives $D^\alpha u$ of every order $|\alpha| \le k$.*

(i) If $\eta \in \mathcal{C}^k(\Omega)$, then the product ηu admits weak derivatives up to order k. These are given by the Leibniz formula

$$D^\alpha(\eta u) \;=\; \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta \eta\, D^{\alpha-\beta} u\,. \tag{2.7}$$

(ii) Let $\Omega' \subseteq I\!R^n$ be an open set and let $\varphi : \Omega' \mapsto \Omega$ be a $\mathcal{C}^k$ bijection whose Jacobian matrix has a uniformly bounded inverse. Then the composition $u \circ \varphi$ is a function in $\mathbf{L}^1_{loc}(\Omega')$ which admits weak derivatives up to order k.

Proof. To prove (i), let J_ε be the standard mollifier and set $u_\varepsilon \doteq J_\varepsilon * u$. Since the Leibniz formula holds for the product of smooth functions, for every $\varepsilon > 0$ we obtain

$$D^\alpha(\eta u_\varepsilon) \;=\; \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta \eta\, D^{\alpha-\beta} u_\varepsilon\,. \tag{2.8}$$

For every test function $\phi \in \mathcal{C}^\infty_c(\Omega)$, we thus have

$$(-1)^{|\alpha|}\int_\Omega (\eta u_\varepsilon) D^\alpha \phi\, dx \;=\; \int_\Omega D^\alpha(\eta u_\varepsilon)\phi\, dx \;=\; \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \int_\Omega (D^\beta \eta\, D^{\alpha-\beta} u_\varepsilon)\,\phi\, dx\,.$$

Notice that, if $\varepsilon > 0$ is small enough so that $Supp(\phi) \subset \Omega_\varepsilon$, then the above integrals are well defined. Letting $\varepsilon \to 0$ we obtain

$$(-1)^{|\alpha|}\int_\Omega (\eta u) D^\alpha \phi\, dx \;=\; \int_\Omega \left(\sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta \eta\, D^{\alpha-\beta} u\right) \phi\, dx\,.$$

By definition of weak derivative, (2.7) holds.

2. We prove (ii) by induction on k. Call y the variable in Ω' and $x = \varphi(y)$ the variable in Ω. By assumption, the $n \times n$ Jacobian matrix $\left(\frac{\partial \varphi_i}{\partial y_j}\right)_{i,j=1,\ldots,n}$ has a uniformly bounded inverse. Hence the composition $u \circ \varphi$ lies in $\mathbf{L}^1_{loc}(\Omega')$, proving the theorem in the case $k = 0$.

Next, assume that the result is true for all weak derivatives of order $|\alpha| \leq k-1$. Consider any test function $\phi \in \mathcal{C}^\infty_c(\Omega')$. and define the mollification $u_\varepsilon \doteq J_\varepsilon * u$. For any $\varepsilon > 0$ small enough so that $\varphi\Big(Supp(\phi)\Big) \subset \Omega_\varepsilon$, we have

$$\begin{aligned} -\int_{\Omega'} (u_\varepsilon \circ \varphi) \cdot D_{y_i}\phi \, dy &= \int_{\Omega'} D_{y_i}(u_\varepsilon \circ \varphi) \cdot \phi \, dy \\ &= \int_{\Omega'} \left(\sum_{j=1}^n D_{x_j} u_\varepsilon(\varphi(y)) \cdot D_{y_i}\varphi_j(y) \right) \cdot \phi(y) \, dy \,. \end{aligned}$$

Letting $\varepsilon \to 0$ we conclude that the composition $u \circ \varphi$ admits a first order weak derivative, given by

$$D_{y_i}(u \circ \varphi)(y) = \sum_{j=1}^n D_{x_j} u(\varphi(y)) \cdot D_{y_i}\varphi_j(y). \tag{2.9}$$

By the inductive assumption, each function $D_{x_j}(u \circ \varphi)$ admits weak derivatives up to order $k-1$, while $D_{y_i}\varphi_j \in \mathcal{C}^{k-1}(\Omega')$. By part (i) of the theorem, all the products on the right hand side of (2.9) have weak derivatives up to order $k-1$. Using Lemma 1.2 we conclude that the composition $u \circ \varphi$ admits weak derivatives up to order k. By induction, this concludes the proof. □

3 Sobolev spaces

Consider an open set $\Omega \subseteq I\!\!R^n$, fix $p \in [1, \infty]$ and let k be a non-negative integer. We say that an open set Ω' is *compactly contained in* Ω if the closure $\overline{\Omega'}$ is a compact subset of Ω.

Definition 3.1 *(i) The* **Sobolev space** $W^{k,p}(\Omega)$ *is the space of all locally summable functions* $u : \Omega \mapsto I\!\!R$ *such that, for every multi-index* α *with* $|\alpha| \leq k$*, the weak derivative* $D^\alpha u$ *exists and belongs to* $\mathbf{L}^p(\Omega)$.

On $W^{k,p}$ *we shall use the norm*

$$\|u\|_{W^{k,p}} \doteq \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, dx \right)^{1/p} \qquad \textit{if } 1 \leq p < \infty, \tag{3.1}$$

$$\|u\|_{W^{k,\infty}} \doteq \sum_{|\alpha| \leq k} \text{ess-}\sup_{x \in \Omega} |D^\alpha u| \qquad \textit{if } p = \infty\,. \tag{3.2}$$

(ii) The subspace $W^{k,p}_0(\Omega) \subseteq W^{k,p}(\Omega)$ *is defined as the closure of* $\mathcal{C}^\infty_c(\Omega)$ *in* $W^{k,p}(\Omega)$*. More precisely,* $u \in W^{k,p}_0(\Omega)$ *if and only if there exists a sequence of functions* $u_n \in \mathcal{C}^\infty_c(\Omega)$ *such that*

$$\|u - u_n\|_{W^{k,p}} \to 0.$$

(iii) By $W^{k,p}_{loc}(\Omega)$ we mean the space of functions which are locally in $W^{k,p}$. These are the functions $u : \Omega \mapsto I\!R$ satisfying the following property. If Ω' is an open set compactly contained in Ω, then the restriction of u to Ω' is in $W^{k,p}(\Omega')$.

Intuitively, one can think of the closed subspace $W^{1,p}_0(\Omega)$ as the space of all functions $u \in W^{1,p}(\Omega)$ which vanish along the boundary of Ω. More generally, $W^{k,p}_0(\Omega)$ is a space of functions whose derivatives $D^\alpha u$ vanish along $\partial\Omega$, for $|\alpha| \leq k-1$.

Definition 3.2 *In the special case where $p = 2$, we define the* **Hilbert-Sobolev** *space $H^k(\Omega) \doteq W^{k,2}(\Omega)$. The space $H^k(\Omega)$ is endowed with the inner product*

$$\langle u, v\rangle_{H^k} \doteq \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u \, D^\alpha v \, dx \,. \tag{3.3}$$

Similarly, we define $H^k_0(\Omega) \doteq W^{k,2}_0(\Omega)$.

Theorem 3.1 (basic properties of Sobolev spaces).

(i) Each Sobolev space $W^{k,p}(\Omega)$ is a Banach space.

(ii) The space $W^{k,p}_0(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$. Hence it is a Banach space, with the same norm.

(iii) The spaces $H^k(\Omega)$ and $H^k_0(\Omega)$ are Hilbert spaces.

Proof. 1. Let $u, v \in W^{k,p}(\Omega)$. For $|\alpha| \leq k$, call $D^\alpha u$, $D^\alpha v$ their weak derivatives. Then, for any $\lambda, \mu \in I\!R$, the linear combination $\lambda u + \mu v$ is a locally integrable function. One easily checks that its weak derivatives are

$$D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v \,. \tag{3.4}$$

Therefore, $D^\alpha(\lambda u + \mu v) \in \mathbf{L}^p(\Omega)$ for every $|\alpha| \leq k$. This proves that $W^{k,p}(\Omega)$ is a vector space.

2. Next, we check that (3.1) and (3.2) satisfy the conditions (N1)–(N3) defining a norm.

Indeed, for $\lambda \in I\!R$ and $u \in W^{k,p}$ one has

$$\|\lambda u\|_{W^{k,p}} = |\lambda| \, \|u\|_{W^{k,p}} \,,$$

$$\|u\|_{W^{k,p}} \geq \|u\|_{\mathbf{L}^p} \geq 0 \,,$$

with equality holding if and only if $u = 0$.

Moreover, if $u, v \in W^{k,p}(\Omega)$, then for $1 \leq p < \infty$ Minkowski's inequality yields

$$\begin{aligned}
\|u+v\|_{W^{k,p}} &= \left(\sum_{|\alpha|\leq k} \|D^\alpha u + D^\alpha v\|^p_{\mathbf{L}^p} \right)^{1/p} \\
&\leq \left(\sum_{|\alpha|\leq k} \Big(\|D^\alpha u\|_{\mathbf{L}^p} + \|D^\alpha v\|_{\mathbf{L}^p} \Big)^p \right)^{1/p} \\
&\leq \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{\mathbf{L}^p} \right)^{1/p} + \left(\sum_{|\alpha|\leq k} \|D^\alpha v\|^p_{\mathbf{L}^p} \right)^{1/p} \\
&= \|u\|_{W^{k,p}} + \|v\|_{W^{k,p}}.
\end{aligned}$$

In the case $p = \infty$, the above computation is replaced by

$$\|u+v\|_{W^{k,\infty}} = \sum_{|\alpha|\leq k} \|D^\alpha u + D^\alpha v\|_{\mathbf{L}^\infty} \leq \sum_{|\alpha|\leq k} \Big(\|D^\alpha u\|_{\mathbf{L}^\infty} + \|D^\alpha v\|_{\mathbf{L}^\infty} \Big) = \|u\|_{W^{k,\infty}} + \|v\|_{W^{k,\infty}}.$$

3. To conclude the proof of (i), we need to show that the space $W^{k,p}(\Omega)$ is complete, hence it is a Banach space.

Let $(u_n)_{n\geq 1}$ be a Cauchy sequence in $W^{k,p}(\Omega)$. For any multi-index α with $|\alpha| \leq k$, the sequence of weak derivatives $D^\alpha u_n$ is Cauchy in $\mathbf{L}^p(\Omega)$. Since the space $\mathbf{L}^p(\Omega)$ is complete, there exist functions u and u_α, such that

$$\|u_n - u\|_{\mathbf{L}^p} \to 0, \qquad \|D^\alpha u_n - u_\alpha\|_{\mathbf{L}^p} \to 0 \qquad \text{for all } |\alpha| \leq k. \tag{3.5}$$

By Lemma 1.3, the limit function u_α is precisely the weak derivative $D^\alpha u$. Since this holds for every multi-index α with $|\alpha| \leq k$, the convergence $u_n \to u$ holds in $W^{k,p}(\Omega)$. This completes the proof of (i).

4. The fact that $W^{k,p}_0(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$ follows immediately from the definition. The fact that (3.3) is an inner product is also clear. □

Example 7. Let $\Omega =]a,b[$ be an open interval. By Corollary 2.2, each element of the space $W^{1,p}(]a,b[)$ coincides a.e. with an absolutely continuous function $f :]a,b[\mapsto I\!R$ having derivative $f' \in \mathbf{L}^p(]a,b[)$.

Example 8. Let $\Omega = B(0,1) \subset I\!R^n$ be the open ball centered at the origin with radius one. Fix $\gamma > 0$ and consider the radially symmetric function

$$u(x) \doteq |x|^{-\gamma} = \left(\sum_{i=1}^n x_i^2 \right)^{-\gamma/2} \qquad 0 < |x| < 1.$$

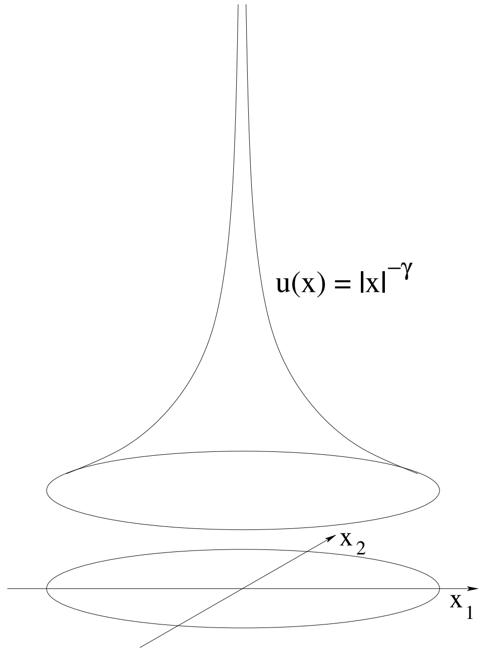

Figure 4: For certain values of p, n a function $u \in W^{1,p}(I\!R^n)$ may not be continuous, or bounded.

Observe that $u \in \mathcal{C}^1(\Omega \setminus \{0\})$. Outside the origin, its partial derivatives are computed as

$$u_{x_i} \;=\; -\frac{\gamma}{2}\left(\sum_{i=1}^n x_i^2\right)^{-(\gamma/2)-1} 2x_i \;=\; \frac{-\gamma x_i}{|x|^{\gamma+2}}. \tag{3.6}$$

Hence, the gradient $\nabla u = (u_{x_1}, \ldots, u_{x_n})$ has norm

$$|\nabla u(x)| \;=\; \left(\sum_{i=1}^n |u_{x_i}(x)|^2\right)^{1/2} \;=\; \frac{\gamma}{|x|^{\gamma+1}}\,.$$

On the open set $\Omega \setminus \{0\}$, the function u clearly admits weak derivatives of all orders, and these coincide with the classical ones.

We wish to understand in which cases the formula (3.6) defines the weak derivatives of u on the entire domain Ω. This means

$$\int_\Omega u_{x_i}\,\phi\,dx \;=\; -\int_\Omega u\,\phi_{x_i}\,dx$$

for every smooth function $\phi \in \mathcal{C}_c^\infty(\Omega)$ whose support is a compact subset of Ω (and not only for those functions ϕ whose support is a compact subset of $\Omega \setminus \{0\}$).

Observe that, for any $\varepsilon > 0$, one has

$$\int_{\varepsilon<|x|<1} (u\phi)_{x_i}\,dx \;=\; \int_{|x|=\varepsilon} u(x)\phi(x)\,\nu_i(x)\,dS$$

where dS is the $(n-1)$-dimensional measure on the surface of the ball $B(0,\varepsilon)$, and $\nu_i(x) = -x_i/|x|$, so that $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the unit normal pointing toward the interior of the ball

$B(0,\varepsilon)$. Since ϕ is a bounded continuous function, we have

$$\int_{|x|=\varepsilon} |u(x)\phi(x)\,\nu_i(x)|\,dS \;\leq\; \|\phi\|_{\mathbf{L}^\infty}\,\varepsilon^{-\gamma}\,\sigma_n\varepsilon^{n-1} \;\to\; 0 \qquad\qquad \text{as} \;\; \varepsilon\to 0$$

provided that $n-1>\gamma$. In other words, if $\gamma<n-1$, then

$$\int_\Omega \frac{-\gamma x_i}{|x|^{\gamma+2}}\,\phi(x)\,dx \;=\; \int_\Omega \frac{1}{|x|^\gamma}\,\phi_{x_i}(x)\,dx \qquad\qquad \text{for every test function} \;\; \phi\in\mathcal{C}^\infty_c(\Omega)\,,$$

and the locally integrable function u_{x_i} defined at (3.6) is in fact the weak derivative of u on the whole domain Ω.

Observe that

$$\int_\Omega \left(\frac{1}{|x|^{\gamma+1}}\right)^p dx \;=\; \int_{x\in I\!R^n,\ |x|<1} |x|^{-p(\gamma+1)}\,dx \;=\; \sigma_n\int_0^1 r^{n-1}r^{-p(\gamma+1)}\,dr \;<\; \infty$$

if and only if $n-1-p(\gamma+1)>-1$, i.e. $\gamma<\frac{n-p}{p}$.

The previous computations show that, if $0<\gamma<\frac{n-p}{p}$, then $u\in W^{1,p}(\Omega)$.

Notice that u is absolutely continuous (in fact, smooth) on a.e. line parallel to one of the coordinate axes. However, there is no way to change u on a set of measure zero, so that it becomes continuous on the whole domain Ω.

An important question in the theory of Sobolev spaces is whether one can estimate the norm of a function in terms of the norm of its first derivatives. The following result provides an elementary estimate in this direction. It is valid for domains Ω which are contained in a slab, say

$$\Omega \;\subseteq\; \{x=(x_1,x_2,\ldots,x_n)\,;\quad a<x_1<b\}\,. \tag{3.7}$$

Theorem 3.2 (Poincare's inequality - I). *Let $\Omega\subset I\!R^n$ be an open set which satisfies (3.7) for some $a,b\in I\!R$. Then, every $u\in H^1_0(\Omega)$ satisfies*

$$\|u\|_{\mathbf{L}^2(\Omega)} \;\leq\; 2(b-a)\;\|D_{x_1}u\|_{\mathbf{L}^2(\Omega)}\,. \tag{3.8}$$

Proof. 1. Assume first that $u\in\mathcal{C}^\infty_c(\Omega)$. We extend u to the whole space $I\!R^n$ by setting $u(x)=0$ for $x\notin\Omega$. Using the variables $x=(x_1,x')$ with $x'=(x_2,\ldots,x_n)$, we compute

$$u^2(x_1,x') \;=\; \int_a^{x_1} 2uu_{x_1}(t,x')\,dt\,.$$

An integration by parts yields

$$\begin{aligned}
\|u\|^2_{\mathbf{L}^2} \;&=\; \int_{I\!R^n} u^2(x)\,dx \;=\; \int_{I\!R^{n-1}}\int_a^b 1\cdot\left(\int_a^{x_1} 2uu_{x_1}(x_1,x')\,dt\right)dx_1\,dx'\\
&=\; \int_{I\!R^{n-1}}\int_a^b (b-x_1)\,2uu_{x_1}(t,x')\,dx_1\,dx' \;\leq\; 2(b-a)\int_{I\!R^n}|u|\,|u_{x_1}|\,dx\\
&\leq\; 2(b-a)\|u\|_{\mathbf{L}^2}\|u_{x_1}\|_{\mathbf{L}^2}\,.
\end{aligned}$$

Dividing both sides by $\|u\|_{\mathbf{L}^2}$ we obtain (3.8), for every $u \in \mathcal{C}_c^\infty(\Omega)$.

2. Now consider any $u \in H_0^1(\Omega)$. By assumption there exists a sequence of functions $u_n \in \mathcal{C}_c^\infty(\Omega)$ such that $\|u_n - u\|_{H^1} \to 0$. By the previous step, this implies

$$\|u\|_{\mathbf{L}^2} \;=\; \lim_{n\to\infty} \|u_n\|_{\mathbf{L}^2} \;\leq\; \lim_{n\to\infty} 2(b-a)\|D_{x_1}u_n\|_{\mathbf{L}^2} \;=\; 2(b-a)\,\|D_{x_1}u\|_{\mathbf{L}^2}\,.$$

□

To proceed in the analysis of Sobolev spaces, we need to derive some more properties of weak derivatives.

Theorem 3.3 (properties of weak derivatives). *Let $\Omega \subseteq I\!\!R^n$ be an open set, $p \in [1,\infty]$, and $|\alpha| \leq k$. If $u, v \in W^{k,p}(\Omega)$, then*

(i) The restriction of u to any open set $\widetilde{\Omega} \subset \Omega$ is in the space $W^{k,p}(\widetilde{\Omega})$.

(ii) $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$.

(iii) If $\eta \in \mathcal{C}^k(\Omega)$, then the product satisfies $\eta\, u \in W^{k,p}(\Omega)$. Moreover there exists a constant C depending on Ω and on $\|\eta\|_{\mathcal{C}^k}$ but not on u, such that

$$\|\eta u\|_{W^{k,p}(\Omega)} \;\leq\; C\,\|u\|_{W^{k,p}(\Omega)}\,. \tag{3.9}$$

(iv) Let $\Omega' \subseteq I\!\!R^n$ be an open set and let $\varphi : \Omega' \mapsto \Omega$ be a $\mathcal{C}^k$ diffeomorphism whose Jacobian matrix has a uniformly bounded inverse. Then the composition satisfies $u \circ \varphi \in W^{k,p}(\Omega')$. Moreover there exists a constant C, depending on Ω' and on $\|\varphi\|_{\mathcal{C}^k}$ but not on u, such that

$$\|u \circ \varphi\|_{W^{k,p}(\Omega')} \;\leq\; C\,\|u\|_{W^{k,p}(\Omega)}\,. \tag{3.10}$$

Proof. The statement (i) is an obvious consequence of the definitions, while (ii) follows from Lemma 1.2.

To prove (iii), by assumption all derivatives of η are bounded, namely

$$\|D^\beta \eta\|_{\mathbf{L}^\infty} \;\leq\; \|\eta\|_{\mathcal{C}^k} \qquad\qquad \text{for all } \; |\beta| \leq k\,.$$

Hence the bound (3.9) follows from the representation formula (2.7).

We prove (iv) by induction on k. By assumption, the $n \times n$ Jacobian matrix $(D_{x_i}\varphi_j)_{i,j=1,\ldots,n}$ has a uniformly bounded inverse. Hence the case $k = 0$ is clear.

Next, assume that the result is true for $k = 0, 1, \ldots, m-1$. If $u \in W^{m,p}(\Omega)$, we have

$$\|D_{x_i}(u \circ \varphi)\|_{W^{m-1,p}(\Omega')} \;\leq\; C'\,\|\nabla u\|_{W^{m-1,p}(\Omega)}\,\|\varphi\|_{\mathcal{C}^m(\Omega')} \;\leq\; C\,\|u\|_{W^{m,p}(\Omega)}\,,$$

showing that the result is true also for $k = m$. By induction, this achieves the proof. □

4 Approximations and extensions of Sobolev functions

Theorem 4.1 (approximation with smooth functions). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $u \in W^{k,p}(\Omega)$ with $1 \leq p < \infty$. Then there exists a sequence of functions $u_k \in \mathcal{C}^\infty(\Omega)$ such that $\|u_k - u\|_{W^{k,p}(\Omega)} \to 0$ as $n \to \infty$.*

Proof. 1. Let $\varepsilon > 0$ be given. Consider the following locally finite open covering of the set Ω, shown in fig. 2:

$$V_1 \doteq \Big\{x \in \Omega\,;\ \ d(x, \partial\Omega) > \frac{1}{2}\Big\}, \qquad V_j \doteq \Big\{x \in \Omega\,;\ \ \frac{1}{j+1} \ <\ d(x, \partial\Omega) < \frac{1}{j-1}\Big\} \quad j = 2, 3, \ldots$$

Let $\eta_1, \eta_2, \ldots$ be a smooth partition of unity subordinate to the above covering. By Theorem 7.3, for every $j \geq 0$ the product $\eta_j u$ is in $W^{k,p}(\Omega)$. By construction, it has support contained in V_j.

2. Consider the mollifications $J_\epsilon * (\eta_j u)$. By Lemma 7.3, for every $|\alpha| \leq k$ we have

$$D^\alpha(J_\epsilon * (\eta_j u)) \ =\ J_\epsilon * (D^\alpha(\eta_j u)) \ \to\ D^\alpha(\eta_j u)$$

as $\epsilon \to 0$. Since each η_j has compact support, here the convergence takes place in $\mathbf{L}^p(\Omega)$. Therefore, for each $j \geq 0$ we can find $0 < \epsilon_j < 2^{-j}$ small enough so that

$$\|\eta_j u - J_{\epsilon_j} * (\eta_j u)\|_{W^{k,p}(\Omega)} \ \leq\ \varepsilon\, 2^{-j}.$$

3. Consider the function

$$U \ \doteq\ \sum_{j=1}^{\infty} J_{\varepsilon_j} * (\eta_j u)$$

Notice that the above series may not converge in $W^{k,p}$. However, it is certainly pointwise convergent because every compact set $K \subset \Omega$ intersects finitely many of the sets V_j. Restricted to K, the above sum contains only finitely many non-zero terms. Since each term is smooth, this implies $U \in \mathcal{C}^\infty(\Omega)$.

4. Consider the subdomains

$$\Omega_{1/n} \ \doteq\ \Big\{x \in \Omega\,;\ \ d(x, \partial\Omega) > \frac{1}{n}\Big\}.$$

Recalling that $\sum_j \eta_j(x) \equiv 1$, for every $n \geq 1$ we find

$$\|U - u\|_{W^{k,p}(\Omega_{1/n})} \leq \ \sum_{j=1}^{n+2} \|\eta_j u - J_{\varepsilon_j} * (\eta_j u)\|_{W^{k,p}(\Omega_{1/n})} \ \leq\ \sum_{j=1}^{n+2} \varepsilon\, 2^{-j} \ \leq\ \varepsilon.$$

This yields

$$\|U - u\|_{W^{k,p}(\Omega)} \ =\ \sup_{n \geq 1}\ \|U - u\|_{W^{k,p}(\Omega_{1/n})} \ \leq\ \varepsilon\,.$$

Since $\varepsilon > 0$ was arbitrary, this proves that the set of $\mathcal{C}^\infty$ function is dense on $W^{k,p}(\Omega)$. □

Using the above approximation result, we obtain a first regularity theorem for Sobolev functions (see fig. 5).

Theorem 4.2 (relation between weak and strong derivatives). *Let $u \in W^{1,1}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open set having the form*

$$\Omega \;=\; \Big\{x = (x, x')\,;\quad x' \doteq (x_2, \ldots, x_n) \in \Omega'\,,\quad \alpha(x') < x_1 < \beta(x')\Big\} \tag{4.1}$$

(possibly with $\alpha \equiv -\infty$ or $\beta \equiv +\infty$). Then there exists a function $\tilde{u}$ with $\tilde{u}(x) = u(x)$ for a.e. $x \in \Omega$, such that the following holds. For a.e. $x' = (x_2, \ldots, x_n) \in \Omega' \subset \mathbb{R}^{n-1}$ (w.r.t. the $(n-1)$-dimensional Lebesgue measure), the map

$$x_1 \mapsto \tilde{u}(x_1, x')$$

is absolutely continuous. Its derivative coincides a.e. with the weak derivative $D_{x_1} u$.

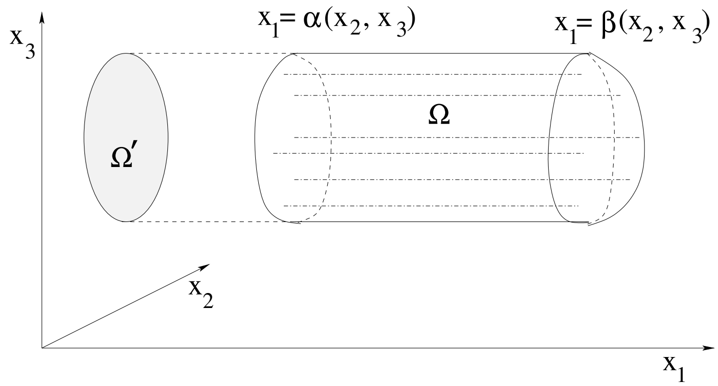

Figure 5: The domain Ω at (4.1). If u has a weak derivative $D_{x_1} u \in \mathbf{L}^1(\Omega)$, then (by possibly changing its values on a set of measure zero), the function u is absolutely continuous on almost every segment parallel to the x_1-axis, and its partial derivative $\partial u/\partial x_1$ coincides a.e. with the weak derivative.

Proof. 1. By the previous theorem, there exists a sequence of functions $u_k \in \mathcal{C}^\infty(\Omega)$ such that

$$\|u_k - u\|_{W^{1,1}} \;<\; 2^{-k}\,. \tag{4.2}$$

We claim that this implies the pointwise convergence

$$u_k(x) \to u(x), \qquad\qquad D_{x_1} u_k(x) \to D_{x_1} u(x) \qquad\qquad \text{for a.e. } x \in \Omega\,.$$

Indeed, consider the functions

$$f(x) \;\doteq\; |u_1(x)| + \sum_{k=1}^\infty |u_{k+1}(x) - u_k(x)|\,, \qquad g(x) \;\doteq\; |D_{x_1} u_1(x)| + \sum_{k=1}^\infty \Big|D_{x_1} u_{k+1}(x) - D_{x_1} u_k(x)\Big|\,. \tag{4.3}$$

By (4.2), there holds

$$\|u_k - u_{k+1}\|_{W^{1,1}} \;\leq\; 2^{1-k},$$

hence $f, g \in \mathbf{L}^1(\Omega)$ and the series in (4.3) are absolutely convergent for a.e. $x \in \Omega$. Therefore, they converge pointwise almost everywhere. Moreover, we have the bounds

$$|u_k(x)| \leq f(x), \qquad |D_{x_1} u_k(x)| \leq g(x) \qquad \text{for all } n \geq 1, \quad x \in \Omega. \tag{4.4}$$

2. Since $f, g \in \mathbf{L}^1(\Omega)$, by Fubini's theorem there exists a null set $\mathcal{N} \subset \Omega'$ (w.r.t. the $n-1$ dimensional Lebesgue measure) such that, for every $x' \in \Omega' \setminus \mathcal{N}$ one has

$$\int_{\alpha(x')}^{\beta(x')} f(x_1, x')\, dx_1 < \infty, \qquad \int_{\alpha(x')}^{\beta(x')} g(x_1, x')\, dx_1 < \infty. \tag{4.5}$$

Fix such a point $x' \in \Omega' \setminus \mathcal{N}$. Choose a point $y_1 \in]\alpha(x'),\ \beta(x')[$ where the pointwise convergence $u_k(y_1, x') \to u(y_1, x')$ holds. For every $\alpha(x') < x_1 < \beta(x')$, since u_k is smooth we have

$$u_k(x_1, x') = u_k(y_1, x') + \int_{y_1}^{x_1} D_{x_1} u_k(s, x')\, ds. \tag{4.6}$$

We now let $n \to \infty$ in (4.6). By (4.4) and (4.5), the functions $D_{x_1} u_k(\cdot, x')$ are all bounded by the integrable function $g(\cdot, x') \in \mathbf{L}^1$. By the dominated convergence theorem, the right hand side of (4.6) thus converges to

$$\tilde{u}(x_1, x') \doteq u(y_1, x') + \int_{y_1}^{x_1} D_{x_1} u(s, x')\, ds. \tag{4.7}$$

Clearly, the right hand side of (4.7) is an absolutely continuous function of the scalar variable x_1. On the other hand, the left hand side satisfies

$$\tilde{u}(x_1, x') \doteq \lim_{n \to \infty} u_k(x_1, x') = u(x_1, x') \qquad \text{for a.e. } x_1 \in [\alpha(x'),\ \beta(x')].$$

This achieves the proof. □

Remark 4.1 *(i) It is clear that a similar result holds for any other derivative $D_{x_i} u$, with $i = 1, 2, \ldots, n$.*

(ii) If $u \in W^{k,p}(\widetilde{\Omega})$ and $\Omega \subset \widetilde{\Omega}$, then the restriction of u to Ω lies in the space $W^{k,p}(\Omega)$. Even if the open set $\widetilde{\Omega}$ has a complicated topology, the result of Theorem 4.2 can be applied to any cylindrical subdomain $\Omega \subset \widetilde{\Omega}$ admitting the representation (4.1).

(iii) If $\Omega \subset \mathbb{R}^n$ is a bounded open set and $u \in W^{k,p}(\Omega)$, then $u \in W^{k,q}(\Omega)$ for every $q \in [1, p]$.

The next theorem shows that, given a bounded open domain $\Omega \subset \mathbb{R}^n$ with $\mathcal{C}^1$ boundary, each function $u \in W^{1,p}(\Omega)$ can be extended to a function $Eu \in W^{1,p}(\mathbb{R}^n)$.

Theorem 4.3 (extension operators). *Let $\Omega \subset\subset \widetilde{\Omega} \subset \mathbb{R}^n$ be open sets, such that the closure of Ω is a compact subset of $\widetilde{\Omega}$. Moreover, assume that $\partial\Omega \in \mathcal{C}^1$. Then there exists a bounded linear operator $E : W^{1,p}(\Omega) \mapsto W^{1,p}(\mathbb{R}^n)$ and a constant C such that*

(i) $Eu(x) = u(x)$ for a.e. $x \in \Omega$,

(ii) $Eu(x) = 0$ for $x \notin \widetilde{\Omega}$,

(iii) One has the bound $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}$.

Proof. **1.** We first prove that the same result holds in the case where the domain is a half space: $\Omega = \{x = (x_1, x_2, \ldots, x_n)\,;\quad x_1 > 0\}$, and $\widetilde{\Omega} = I\!R^n$. In this case, any function $u \in W^{1,p}(\Omega)$ can be extended to the whole space $I\!R^n$ by reflection, i.e. by setting

$$(E^\sharp u)(x_1, x_2, \ldots, x_n) \;\doteq\; u\,(\,|x_1|,\, x_2,\, x_3, \ldots,\, x_n)\,. \tag{4.8}$$

By Theorem 4.2, for every $i \in \{1, \ldots, n\}$ the function u is absolutely continuous along a.e. line parallel to the x_i-axis. Hence the same is true of the extension $E^\sharp u$. A straightforward computation involving integration by parts shows that the first order weak derivatives of $E^\sharp u$ exist on the entire space $I\!R^n$ and satisfy

$$\left\{\begin{array}{lcll} D_{x_1} E^\sharp u(-x_1, x_2, \ldots, x_n) & = & -\,D_{x_1} u(x_1, x_2, \ldots, x_n), & \\ D_{x_j} E^\sharp u(-x_1, x_2, \ldots, x_n) & = & D_{x_j} u(x_1, x_2, \ldots, x_n) & (j = 2, \ldots, n), \end{array}\right.$$

for all $x_1 > 0$, $x_2, \ldots, x_n \in I\!R$. The extension operator $E^\sharp : W^{1,p}(\Omega) \mapsto W^{1,p}(I\!R^n)$ defined at (4.8) is clearly linear and bounded, because

$$\|E^\sharp u\|_{W^{1,p}(I\!R^n)} \;\leq\; 2\|u\|_{W^{1,p}(\Omega)}\,.$$

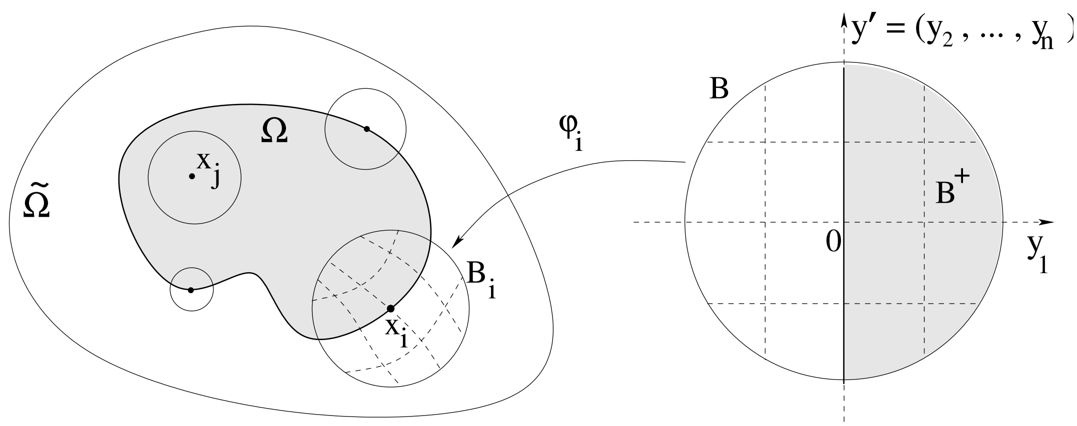

Figure 6: The open covering of the set Ω. For every ball $B_i = B(x_i,\, r_i)$ there is a $\mathcal{C}^1$ bijection φ_i mapping the open unit ball $B \subset I\!R^n$ onto B_i. For those balls B_i having center on the boundary Ω, the intersection $B_i \cap \Omega$ is mapped into $B^+ = B \cap \{y_1 > 0\}$.

2. To handle the general case, we use a partition of unity. For every $x \in \overline{\Omega}$ (the closure of Ω), choose a radius $r_x > 0$ such that the open ball $B(x, r_x)$ centered at x with radius $r_x 0$ satisfies the following conditions

- If $x \in \Omega$, then $B(x, r_x) \subset \Omega$.

- If $x \in \partial\Omega$, then $B(x, r_x) \subset \widetilde{\Omega}$. Moreover, calling $B \doteq B(0,1)$ the open unit ball in $I\!R^n$, there exists a $\mathcal{C}^1$ bijection $\varphi^x : B \mapsto B(x, r_x)$, whose inverse is also $\mathcal{C}^1$, which maps the half ball

$$B^+ \;\doteq\; \left\{y = (y_1, y_2, \ldots, y_n)\,;\quad \sum_{i=1}^n y_i^2 < 1\,,\quad y_1 > 0\right\}$$

onto the set $B(x, r^x) \cap \Omega$.

Choosing $r_x > 0$ sufficiently small, the existence of such a bijection follows from the assumption that Ω has a $\mathcal{C}^1$ boundary.

Since Ω is bounded, its closure $\overline{\Omega}$ is compact. Hence it can be covered with finitely many balls $B_i = B(x_i, r_i)$, $i = 1, \ldots, N$. Let $\varphi_i : B \mapsto B_i$ be the corresponding bijections. Recall that φ_i maps B^+ onto $B_i \cap \Omega$.

3. Let $\eta_1, \ldots, \eta_N$ be a smooth partition of unity subordinated to the above covering. For every $x \in \Omega$ we thus have

$$u(x) \;=\; \sum_{i=1}^{N} \eta_i(x) u(x)\,. \tag{4.9}$$

We split the set of indices as

$$\{1, 2, \ldots, N\} \;=\; \mathcal{I} \cup \mathcal{J},$$

where $\mathcal{I}$ contains the indices with $x_i \in \Omega$ while $\mathcal{J}$ contains the indices with $x_i \in \partial\Omega$.

For every $i \in \mathcal{J}$, we have $\eta_i u \in W^{1,p}(B_i \cap \Omega)$. Hence by Theorem 3.3 (iv), one has $(\eta_i u) \circ \varphi_i \in W^{1,p}(B^+)$. Applying the extension operator $E^\sharp$ defined at (4.8) one obtains

$$E^\sharp\Big((\eta_i u) \circ \varphi_i\Big) \;\in\; W^{1,p}(B^+)\,, \qquad\qquad E^\sharp\Big((\eta_i u) \circ \varphi_i\Big) \circ \varphi_i^{-1} \;\in\; W^{1,p}(B_i)\,.$$

Summing together all these extensions, we define

$$Eu \;\doteq\; \sum_{i \in \mathcal{I}} \eta_i u + \sum_{i \in \mathcal{J}} E^\sharp\Big((\eta_i u) \circ \varphi_i\Big) \circ \varphi_i^{-1}.$$

It is now clear that the extension operator E satisfies all requirements. Indeed, (i) follows from (4.9). The property (ii) stems from the fact that, for every $u \in W^{1,p}(\Omega)$, the support of Eu is contained in $\cup_{i=1}^{N} B_i \subseteq \widetilde{\Omega}$. Finally, since E is defined as the sum of finitely many bounded linear operators, the bound (iii) holds, for some constant C. □

For any open set $\Omega \subseteq I\!R^n$, in Theorem 4.1 we approximated a function $u \in W^{k,p}(\Omega)$ with $\mathcal{C}^\infty$ functions u_k defined on the same open set Ω. In principle, these approximating functions u_k might have wild behavior near the boundary of Ω. Assuming that the boundary $\partial\Omega$ is $\mathcal{C}^1$, using the extension theorem, we now show that u can be approximated in $W^{1,p}$ by smooth functions which are defined on the entire space $I\!R^n$.

Corollary 4.1 *Let $\Omega \subset I\!R^n$ be a bounded open set with $\mathcal{C}^1$ boundary. Given any $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$, there exists a sequence of smooth functions $u_k \in \mathcal{C}_c^\infty(I\!R^n)$ such that the restrictions of u_k to Ω satisfy*

$$\lim_{k \to \infty} \|u_k - u\|_{W^{1,p}(\Omega)} \;=\; 0\,. \tag{4.10}$$

Moreover,

$$\|u_k\|_{W^{1,p}(I\!R^n)} \;\leq\; C\,\|u\|_{W^{1,p}(\Omega)}\,, \tag{4.11}$$

for some constant C depending on p and Ω but not on u.

Proof. Let $\widetilde{\Omega} = B(\Omega, 1)$ be the open neighborhood of radius one around the set Ω. By Theorem 4.3 the function u admits an extension $Eu \in W^{1,p}(I\!R^n)$ which vanishes $\widetilde{\Omega}$. Then the mollifications $u_k = J_{1/k} * Eu \in \mathcal{C}_c^\infty(I\!R^n)$ satisfy all requirements (4.10)-(4.11). Indeed,

$$\|u_k\|_{W^{1,p}(I\!R^n)} \;\leq\; \|Eu\|_{W^{1,p}(I\!R^n)} \;\leq\; C\,\|u\|_{W^{1,p}(\Omega)}\,, \tag{4.12}$$

$$\lim_{k\to\infty} \|u_k - u\|_{W^{1,p}(\Omega)} \;\leq\; \lim_{k\to\infty} \|u_k - Eu\|_{W^{1,p}(I\!R^n)} \;=\; 0.$$

□

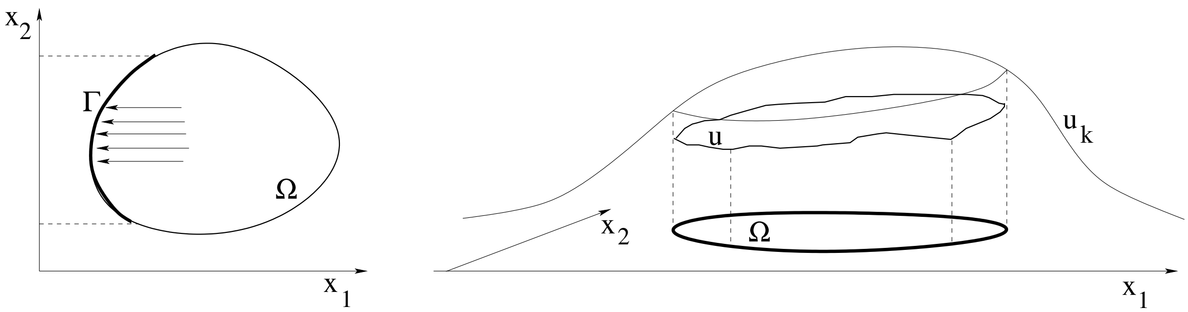

Figure 7: Making sense of the trace of a function $u \in W^{1,p}(\Omega)$. Left: the thick curve Γ is a portion of the boundary which admits the representation $\{x_1 = \alpha(x_2)\}$. Since u is absolutely continuous along almost all lines parallel to the x_1-axis, along Γ one can define the trace of of u by taking pointwise limits along horizontal lines. Right: the standard way to construct a trace. Given a sequence of smooth functions $u_k \in \mathcal{C}^\infty(I\!R^n)$ whose restrictions $u_k\big|_\Omega$ converge to u in $W^{1,p}(\Omega)$, one defines the trace as $Tu \doteq \lim_{k\to\infty} u_k\big|_{\partial\Omega}$. This limit is well defined in $\mathbf{L}^1(\partial\Omega)$ and does not depend on approximating sequence.

5 Traces

Let $\Omega \subset I\!R^n$ be a bounded open set with $\mathcal{C}^1$ boundary. Call $\mathbf{L}^p(\partial\Omega)$ the corresponding space of functions $v : \partial\Omega \mapsto I\!R$, with norm

$$\|v\|_{\mathbf{L}^p(\partial\Omega)} \;\doteq\; \left(\int_{\partial\Omega} |v|^p dS\right)^{1/p} \tag{5.1}$$

where dS denotes the $(n-1)$-dimensional surface measure on $\partial\Omega$.

Given a function $u \in W^{1,p}(\Omega)$, we seek to define the "boundary values" of u. Before giving a precise theorem in this direction, let us consider various possible approaches to this problem.

(i) If u is uniformly continuous on the open set Ω, one can simply extend u by continuity to the closure $\overline{\Omega}$. For any boundary point $x \in \partial\Omega$, we can thus define the *Trace* of u at x by setting

$$Tu(x) \;\doteq\; \lim_{y\to x\,,\; y\in\Omega} u(y)\,. \tag{5.2}$$

By uniform continuity, this limit is well defined and yields a continuous function $Tu : \partial\Omega \mapsto I\!R$. In general, however, a function $u \in W^{1,p}(\Omega)$ may well be discontinuous. Hence the limit in (5.2) may not exist.

(ii) Consider a portion of the boundary $\partial\Omega$ that can be written in the form (see fig. 7, left)

$$\{x = (x_1, \ldots, x_n) \doteq (x_1, x')\,; \quad x_1 = \alpha(x')\}.$$

By Theorem 4.2, the function u is absolutely continuous along a.e. line parallel to the x_1-axis. Therefore the pointwise limit

$$v(\alpha(x'), x') \;=\; \lim_{x_1 \to \alpha(x'),\ (x_1, x') \in \Omega} u(x_1, x')$$

exists for a.e. x'.

This provides is a good way to think about the trace of a function. With this approach, however, the properties of these pointwise limits are not easy to derive. Also, it is not immediately clear whether by taking the limit along the x_1 direction or along some other direction one always obtains the same boundary values.

(iii) The standard approach, which we shall adopt in the sequel, relies on the approximation of u with smooth functions u_k defined on the entire space $I\!R^n$ (see fig. 7). By Corollary 2.2 there exists a sequence of smooth functions $u_k \in \mathcal{C}^\infty(I\!R^n)$ such that their restrictions to Ω approach u in $W^{1,p}(\Omega)$. A more careful analysis shows that the restrictions of these smooth functions to $\partial\Omega$ form a Cauchy sequence in $\mathbf{L}^p(\partial\Omega)$. Hence it determines a unique limit $v = Tu \in \mathbf{L}^p(\partial\Omega)$. This limit, that does not depend on the approximating sequence $(u_k)_{k\geq 1}$ is called the *trace* of u on $\partial\Omega$.

Theorem 5.1 (trace operator). *Let $\Omega \subset I\!R^n$ be a bounded open set with $\mathcal{C}^1$ boundary, and let $1 \leq p < \infty$. Then there exists a bounded linear operator $T : W^{1,p}(\Omega) \mapsto \mathbf{L}^p(\partial\Omega)$ with the following properties.*

(i) If $u \in W^{1,p}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, then

$$Tu \;=\; u|_{\partial\Omega}\,. \tag{5.3}$$

(ii) There exists a constant C depending only on p and on the set Ω such that

$$\|Tu\|_{\mathbf{L}^p(\partial\Omega)} \;\leq\; C\,\|u\|_{W^{1,p}(\Omega)}\,. \tag{5.4}$$

Proof. Given any function $u \in \mathcal{C}^\infty(I\!R^n)$, we claim that its restrictions to Ω and to $\partial\Omega$ satisfy

$$\|u\|_{\mathbf{L}^p(\partial\Omega)} \;\leq\; C\,\|u\|_{W^{1,p}(\Omega)}\,, \tag{5.5}$$

for some constant C depending only on p and on the set Ω. This key inequality will be proved in the next two steps.

1. Let $u \in \mathcal{C}^\infty(I\!R^n)$ be a smooth function which vanishes outside the unit ball B. Consider the $(n-1)$-dimensional disc $\Gamma = \{(x_1, \ldots, x_n)\,;\ \ \sum_i x_i^2 < 1\,,\ \ x_1 = 0\,,\}$. Setting $x' = (x_2, \ldots, x_n)$ and $B^+ \doteq B \cap \{x_1 > 0\}$, we claim that

$$\int_\Gamma |u(x')|^p\,dx' \;\leq\; C_1 \int_{B^+} \Big(|u(x)|^p + |\nabla u(x)|^p\Big)\,dx \tag{5.6}$$

for some constant C_1 depending only on p. Indeed, since u vanishes outside B, using Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ with $q = \frac{p}{p-1}$ we find

$$\begin{aligned}\int_\Gamma |u(0,x')|^p\,dx' &= -\int_\Gamma \left(\int_0^\infty \frac{d}{dx_1}|u(x_1,x')|^p\,dx_1\right)dx'\\ &\leq \int_\Gamma \left(\int_0^\infty p|u|^{p-1}\,|u_{x_1}|\,dx_1\right)dx' \;\leq\; C_1\int_{B^+}\Big(|u|^p + |u_{x_1}|^p\Big)\,dx\,.\end{aligned}$$

2. As in the proof of Theorem 4.3 (see fig. 6), we can cover the compact set $\overline{\Omega}$ with finitely many balls $B_1,\ldots,B_N$ having the following properties. For every $i \in \{1,\ldots,N\}$ there exists a $\mathcal{C}^1$ bijection $\varphi_i : B \mapsto B_i$ with $\mathcal{C}^1$ inverse, such that either $B_i \subseteq \Omega$, or

$$\varphi_i(B^+) = (B_i \cap \Omega)$$

Let $\eta_1,\ldots,\eta_N$ be a smooth partition of unity subordinated to this covering. Call $\mathcal{J} \subseteq \{1,\ldots,N\}$ the set of indices for which $\varphi_i(B)$ intersects the boundary of Ω. If $u \in \mathcal{C}^\infty(I\!R^n)$, we now have the estimate

$$\begin{aligned}\|u\|^p_{\mathbf{L}^p(\partial\Omega)} &\leq \sum_i \|u\eta_i\|^p_{\mathbf{L}^p(\partial\Omega)} \;\leq\; \sum_i C_2\cdot\|(\eta_i u)\circ\varphi_i\|^p_{\mathbf{L}^p(\Gamma)}\\ &\leq\; C_3\sum_{i\in\mathcal{J}}\int_{B_+}\Big(|(\eta_i u)\circ\varphi_i|^p + \Big|\nabla((\eta_i u)\circ\varphi_i)\Big|^p\Big)\,dy\\ &\leq\; C_4\sum_{i\in\mathcal{J}}\int_{B_i\cap\Omega}\Big(|\eta_i u|^p + |\nabla(\eta_i u)|^p\Big)\,dx \;\leq\; C_5\int_\Omega\Big(|u|^p + |\nabla u|^p\Big)\,dx\,.\end{aligned}$$

for suitable constants $C_2,\ldots,C_5$. Notice that the third inequality is a consequence of (5.6). To obtain the other inequalities, we used the fact that the functions η_i, φ_i, and φ_i^{-1} are bounded with uniformly bounded derivatives. The above arguments establish the a priori bound (5.5).

3. Next consider any function $u \in W^{1,p}(\Omega)$. By Corollary 4.1 there exists a sequence of functions $u_m \in \mathcal{C}^\infty(I\!R^n)$ such that $\|u_m - u\|_{W^{1,p}(\Omega)} \to 0$. By the previous step, for any $n, m \geq 1$ we have

$$\limsup_{m,n\to\infty}\|u_m - u_n\|_{\mathbf{L}^p(\partial\Omega)} \;\leq\; C\limsup_{m,n\to\infty}\|u_m - u_n\|_{W^{1,p}(\Omega)} \;=\; 0. \tag{5.7}$$

Hence the restrictions $u_k|_{\partial\Omega}$ provide a Cauchy sequence in $\mathbf{L}^p(\partial\Omega)$. The limit of this sequence is a function $Tu \in \mathbf{L}^p(\partial\Omega)$, which we call the *Trace* of u. By (5.7), this limit does not depend on the choice of the approximating sequence.

4. It remains to check that the operator $u \mapsto Tu$ constructed in the previous step satisfies the required properties (i)–(ii).

If $u \in W^{1,p}(\Omega) \cap \mathcal{C}^0(\Omega)$, then its extension $Eu \in W^{1,p}(I\!R^n)$ constructed in Theorem 4.3 is continuous on the whole space $I\!R^n$. The sequence of mollifications $u_k \doteq J_{1/k} * (Eu)$ converge to Eu uniformly on compact sets. In particular, they converge to u uniformly on $\partial\Omega$. This proves (i). By the definition of trace, the bound (ii) is an immediate consequence of (5.5). □

Remark 5.1 Let $u \mapsto Eu$ be an extension operator, constructed in Theorem 4.3. Then the trace operator can be defined as

$$Tu \;=\; \lim_{\varepsilon\to 0} (J_\varepsilon * Eu)\Big|_{\partial\Omega} \tag{5.8}$$

where J_ε denotes a mollifier and the limit takes place in $\mathbf{L}^p(\partial\Omega)$.

6 Embedding theorems

In one space dimension, a function $u : I\!\!R \mapsto I\!\!R$ which admits a weak derivative $Du \in \mathbf{L}^1(I\!\!R)$ is absolutely continuous (after changing its values on a set of measure zero). On the other hand, if $\Omega \subseteq I\!\!R^n$ with $n \geq 2$, there exist functions $u \in W^{1,p}(\Omega)$ which are not continuous, and not even bounded. This is indeed the case of the function $u(x) = |x|^{-\gamma}$, for $0 < \gamma < \frac{n-p}{p}$.

In several applications to PDEs or to the Calculus of Variations, it is important to understand the degree of regularity enjoyed by functions $u \in W^{k,p}(I\!\!R^n)$. We shall prove two basic results in this direction.

1. (Morrey). If $p > n$, then every function $u \in W^{1,p}(I\!\!R^n)$ is Hölder continuous (after a modification on a set of measure zero).

2. (Gagliardo-Nirenberg). If $p < n$, then every function $u \in W^{1,p}(I\!\!R^n)$ lies in the space $\mathbf{L}^{p^*}(I\!\!R^n)$, with the larger exponent $p^* = p + \frac{p^2}{n-p}$.

In both cases, the result can be stated as an embedding theorem: after a modification on a set of measure zero, each function $u \in W^{1,p}(\Omega)$ also lies in some other Banach space X. Typically $X = C^{0,\gamma}$, or $X = \mathbf{L}^q$ for some $q > p$. The basic approach is as follows:

I - Prove an a priori inequality valid for all smooth functions. Given any function $u \in \mathcal{C}^\infty \cap W^{k,p}(\Omega)$, one proves that u also lies in another Banach space X, and there exists a constant C depending on k, p, Ω but not on u, such that

$$\|u\|_X \;\leq\; C\,\|u\|_{W^{k,p}} \qquad\qquad \text{for all} \;\; u \in \mathcal{C}^\infty \cap W^{k,p}(\Omega)\,. \tag{6.1}$$

II - Extend the embedding to the entire space, by continuity. Since $\mathcal{C}^\infty$ is dense in $W^{k,p}$, for every $u \in W^{k,p}(\Omega)$ we can find a sequence of functions $u_n \in \mathcal{C}^\infty$ such that $\|u - u_n\|_{W^{k,p}} \to 0$. By (6.1),

$$\limsup_{m,n\to\infty} \|u_m - u_n\|_X \;\leq\; \limsup_{m,n\to\infty} C\,\|u_m - u_n\|_{W^{k,p}} \;=\; 0\,.$$

Therefore the functions u_n form a Cauchy sequence also in the space X. By completeness, $u_n \to \tilde u$ for some $\tilde u \in X$. Observing that $\tilde u(x) = u(x)$ for a.e. $x \in \Omega$, we conclude that, up to a modification on a set $\mathcal{N} \subset \Omega$ of measure zero, each function $u \in W^{k,p}(\Omega)$ lies also in the space X.

6.1 Morrey's inequality

In this section we prove that, if $u \in W^{1,p}(I\!R^n)$, where the exponent p is higher than the dimension n of the space, then u coincides a.e. with a Hölder continuous function.

Theorem 6.1 (Morrey's inequality). *Assume $n < p < \infty$ and set $\gamma \doteq 1 - \frac{n}{p} > 0$. Then there exists a constant C, depending only on p and n, such that*

$$\|u\|_{\mathcal{C}^{0,\gamma}(I\!R^n)} \leq C\,\|u\|_{W^{1,p}(I\!R^n)} \tag{6.2}$$

for every $u \in \mathcal{C}^1(I\!R^n) \cap W^{1,p}(I\!R^n)$.

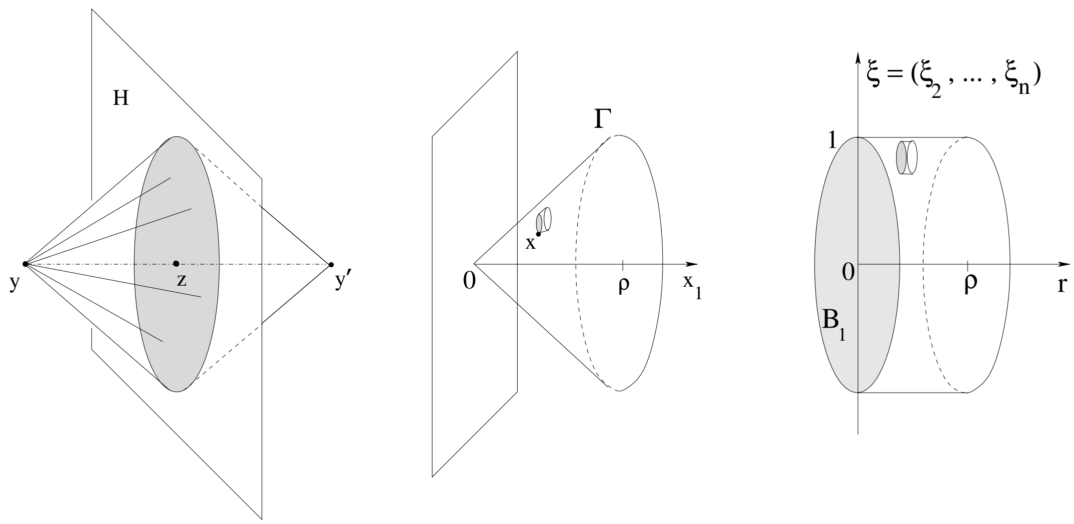

Figure 8: Proving Morrey's inequality. Left: the values $u(y)$ and $u(y')$ are compared with the average value of u on the $(n-1)$-dimensional ball centered at the mid-point z. Center and right: a point x in the cone Γ is described in terms of the coordinates $(r,\xi) \in [0,\rho] \times B_1$.

Proof. 0. Before giving the actual proof, we outline the underlying idea. From an integral estimate on the gradient of the function u, say

$$\int_{I\!R^n} |\nabla u(x)|^p \, dx \leq C_0, \tag{6.3}$$

we seek a pointwise estimate of the form

$$|u(y) - u(y')| \leq C_1 |y - y'|^\gamma \qquad \text{for all } y, y' \in I\!R^n. \tag{6.4}$$

To achieve (6.4), a natural attempt is to write

$$|u(y) - u(y')| = \left| \int_0^1 \left[\frac{d}{d\theta} u\left(\theta y + (1-\theta) y'\right) \right] d\theta \right| \leq \int_0^1 \left| \nabla u(\theta y + (1-\theta) y') \right| |y - y'| \, d\theta\,. \tag{6.5}$$

However, the integral on the right hand side of (6.5) only involves values of ∇u over the segment joining y with y'. If the dimension of the space is $n > 1$, this segment has zero measure. Hence the integral in (6.5) can be arbitrarily large, even if the integral in (6.3) is small. To address this difficulty, we shall compare both values $u(y)$, $u(y')$ with the average value u_B of the function u over an $(n-1)$-dimensional ball centered at the midpoint $z = \frac{y+y'}{2}$, as shown in Figure 8, left. Notice that the difference $|u(y) - u_B|$ can be estimated by an integral of $|\nabla u|$ ranging over a cone of dimension n. In this way the bound (6.3) can thus be brought into play.

1. We now begin the proof, with a preliminary computation. On $I\!R^n$, consider the cone

$$\Gamma \doteq \left\{ x = (x_1, x_2, \ldots, x_n) \, ; \quad \sum_{j=2}^{n} x_j^2 \le x_1^2 \, , \quad 0 < x_1 < \rho \right\}$$

and the function

$$\psi(x) = \frac{1}{x_1^{n-1}} \, . \tag{6.6}$$

Let $q = \frac{p}{p-1}$ be the conjugate exponent of p, so that $\frac{1}{p} + \frac{1}{q} = 1$. We compute

$$\|\psi\|^q_{\mathbf{L}^q(\Gamma)} = \int_\Gamma \left(\frac{1}{x_1^{n-1}} \right)^q dx = \int_0^\rho c_{n-1} x_1^{n-1} \left(\frac{1}{x_1^{n-1}} \right)^q dx_1 = c_{n-1} \int_0^\rho s^{(n-1)(1-q)} \, ds \, ,$$

where the constant c_{n-1} gives the volume of the unit ball in $I\!R^{n-1}$. Therefore, $\psi \in \mathbf{L}^q(\Gamma)$ if and only if $n < p$. In this case,

$$\|\psi\|_{\mathbf{L}^q(\Gamma)} = \left(c_{n-1} \int_0^\rho s^{-\frac{n-1}{p-1}} \, ds \right)^{\frac{1}{q}} = c \left(\rho^{\frac{p-n}{p-1}} \right)^{\frac{p-1}{p}} = c \, \rho^{\frac{p-n}{p}} \, , \tag{6.7}$$

for some constant c depending only on n and p.

2. Consider any two distinct points $y, y' \in I\!R^n$. Let $\rho \doteq \frac{1}{2}|y - y'|$. The hyperplane passing through the midpoint $z \doteq \frac{y+y'}{2}$ and perpendicular to the vector $y - y'$ has equation

$$H = \left\{ x \in I\!R^n \, ; \quad \langle x - z \, , \; y - y' \rangle = 0 \right\} .$$

Inside H, consider the $(n-1)$-dimensional ball centered at z with radius ρ,

$$B_\rho \doteq \left\{ x \in H \, ; \quad |x - z| < \rho \right\} .$$

Calling u_A the average value of u on the ball B_ρ, the difference $|u(y) - u(y')|$ will be estimated as

$$|u(y) - u(y')| \le |u(y) - u_A| + |u_A - u(y')| \, . \tag{6.8}$$

3. By a translation and a rotation of coordinates, we can assume

$$y = (0, \ldots, 0) \in I\!R^n, \qquad B_\rho = \left\{ x = (x_1, x_2, \ldots, x_n) \, ; \quad x_1 = \rho \, , \quad \sum_{i=2}^{n} x_i^2 \le \rho^2 \right\} .$$

To compute the average value u_A, let B_1 be the unit ball in $I\!R^{n-1}$, and let c_{n-1} be its $(n-1)$-dimensional measure. Points in the cone Γ will be described using an alternative system of coordinates. To $(r,\xi) = (r, \xi_2, \ldots, \xi_n) \in [0,\rho] \times B_1$ we associate the point $x(r,\xi) \in \Gamma$, defined by

$$(x_1, x_2, \ldots, x_n) = (r, r\xi) = (r, r\xi_2, \ldots, r\xi_n). \tag{6.9}$$

Define $U(r,\xi) = u(r, r\xi)$, and observe that $U(0,\xi) = u(0)$ for every ξ. Therefore

$$U(\rho,\xi) = U(0,\xi) + \int_0^\rho \left[\frac{\partial}{\partial r} U(r,\xi)\right] dr\,,$$

$$u_B - u(0) = \frac{1}{c_{n-1}} \int_{B_1} \left(\int_0^\rho \left[\frac{\partial}{\partial r} U(r,\xi)\right] dr \right) d\xi\,. \tag{6.10}$$

We now change variables, transforming the integral (6.10) over $[0,\rho] \times B_1$ into an integral over the cone Γ. Computing the Jacobian matrix of the transformation (6.9) we find that its determinant is r^{n-1}, hence

$$dx_1\, dx_2 \ \cdots \ dx_n = r^{n-1} dr\, d\xi_2 \ \cdots \ d\xi_n\,.$$

Moreover, since $|\xi| \leq 1$, the directional derivative of u in the direction of the vector $(1, \xi_2, \ldots, \xi_n)$ is estimated by

$$\left|\frac{\partial}{\partial r} U(r,\xi)\right| = \left|u_{x_1} + \sum_{i=2}^n \xi_i u_{x_i}\right| \leq 2|\nabla u(r,\xi)|\,. \tag{6.11}$$

Using (6.11) in (6.10), and the estimate (6.7) on the $\mathbf{L}^q$ norm of the function $\psi(x) \doteq x_1^{1-n}$, we obtain

$$\begin{aligned} \Big|u_A - u(0)\Big| &\leq \frac{2}{c_{n-1}} \int_\Gamma \frac{1}{x_1^{n-1}} |\nabla u(x)|\, dx \\ &\leq \frac{2}{c_{n-1}} \|\psi\|_{\mathbf{L}^q(\Gamma)} \|\nabla u\|_{\mathbf{L}^p(\Gamma)} \qquad \left(q = \frac{p}{p-1}\right) \\ &\leq C\, \rho^{\frac{p-n}{p}} \|u\|_{W^{1,p}(\mathbb{R}^n)} \end{aligned} \tag{6.12}$$

for some constant C. Notice that the last two steps follow from Hölder's inequality and (6.7).

4. Using (6.12) to estimate each term on the right hand side of (6.8), and recalling that $\rho = \frac{1}{2}|y - y'|$, we conclude

$$|u(y) - u(y')| \leq 2C \left(\frac{|y-y'|}{2}\right)^{\frac{p-n}{p}} \|u\|_{W^{1,p}(\mathbb{R}^n)}. \tag{6.13}$$

This shows that u is Hölder continuous with exponent $\gamma = \frac{p-n}{p}$.

5. To estimate $\sup_y |u(y)|$ we observe that, by (6.13), for some constant C_1 one has

$$|u(y)| \leq |u(x)| + C_1 \|u\|_{W^{1,p}(\mathbb{R}^n)} \qquad \text{for all } x \in B(y,1)\,.$$

Taking the average of the right hand side over the ball centered at y with radius one we obtain

$$|u(y)| \leq \fint_{B(y,1)} |u(x)|\, dx + C_1 \|u\|_{W^{1,p}(\mathbb{R}^n)} \leq C_2 \|u\|_{\mathbf{L}^p(\mathbb{R}^n)} + C_1 \|u\|_{W^{1,p}(\mathbb{R}^n)}. \tag{6.14}$$

6. Together, (6.13)-(6.14) yield

$$\|u\|_{\mathcal{C}^{0,\gamma}(\mathbb{R}^n)} \doteq \sup_y |u(y)| + \sup_{y\neq y'} \frac{|u(y)-u(y')|}{|y-y'|^\gamma} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

for some constant C depending only on p and n. □

Since $\mathcal{C}^\infty$ is dense in $W^{1,p}$, Morrey's inequality yields

Corollary 6.1 (embedding). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^1$ boundary. Assume $n < p < \infty$ and set $\gamma \doteq 1 - \frac{n}{p} > 0$. Then every function $f \in W^{1,p}(\Omega)$ coincides a.e. with a function $\tilde f \in \mathcal{C}^{0,\gamma}(\Omega)$. Moreover, there exists a constant C such that*

$$\|\tilde f\|_{\mathcal{C}^{0,\gamma}} \leq C\|f\|_{W^{1,p}} \qquad \textit{for all } f \in W^{1,p}(\Omega). \tag{6.15}$$

Proof. 1. Let $\widetilde\Omega \doteq \{x \in \mathbb{R}^n\,;\ \ d(x,\Omega) < 1\}$ be the open neighborhood of radius one around the set Ω. By Theorem 4.3 there exists a bounded extension operator E, which extends each function $f \in W^{1,p}(\Omega)$ to a function $Ef \in W^{1,p}(\mathbb{R}^n)$ with support contained inside $\widetilde\Omega$.

2. Since $\mathcal{C}^1(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$, we can find a sequence of functions $g_n \in \mathcal{C}^1(\mathbb{R}^n)$ converging to Ef in $W^{1,p}(\mathbb{R}^n)$. By Morrey's inequality

$$\limsup_{m,n\to\infty} \|g_m - g_n\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \limsup_{m,n\to\infty} \|g_m - g_n\|_{W^{1,p}(\mathbb{R}^n)} = 0.$$

This proves that the sequence $(g_n)_{n\geq 1}$ is a Cauchy sequence also in the space $\mathcal{C}^{0,\gamma}$. Therefore it converges to a limit function $g \in C^{0,\gamma}(\mathbb{R}^n)$, uniformly for $x \in \mathbb{R}^n$.

3. Since $g_n \to Ef$ in $W^{1,p}(\mathbb{R}^n)$, we also have $g(x) = (Ef)(x)$ for a.e. $x \in \mathbb{R}^n$. In particular, $g(x) = f(x)$ for a.e. $x \in \Omega$. Since the extension operator E is bounded, from the bound (6.2) we deduce (6.15). □

6.2 The Gagliardo-Nirenberg inequality

Next, we study the case $1 \leq p < n$. We define the **Sobolev conjugate** of p as

$$p^* \doteq \frac{np}{n-p} > p. \tag{6.16}$$

Notice that p^* depends not only on p but also on the dimension n of the space. Indeed,

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{6.17}$$

As a preliminary, we describe a useful application of the generalized Hölder's inequality. Let $n-1$ non-negative functions $g_1, g_2, \ldots, g_{n-1} \in \mathbf{L}^1(\Omega)$ be given. Since $g_i^{\frac{1}{n-1}} \in \mathbf{L}^{n-1}(\Omega)$ for each i, using the generalized Hölder inequality one obtains

$$\int_\Omega g_1^{\frac{1}{n-1}}\ g_2^{\frac{1}{n-1}} \cdots g_{n-1}^{\frac{1}{n-1}}\ ds \leq \prod_{i=1}^{n-1} \|g_i^{\frac{1}{n-1}}\|_{\mathbf{L}^{n-1}} = \prod_{i=1}^{n-1} \|g_i\|_{\mathbf{L}^1}^{\frac{1}{n-1}}. \tag{6.18}$$

Theorem 6.2 (Gagliardo-Nirenberg inequality). *Assume $1 \leq p < n$. Then there exists a constant C, depending only on p and n, such that*

$$\|f\|_{\mathbf{L}^{p^*}(\mathbb{R}^n)} \;\leq\; C\,\|\nabla f\|_{\mathbf{L}^p(\mathbb{R}^n)} \qquad \textit{for all}\;\; f \in C^1_c(\mathbb{R}^n)\,. \tag{6.19}$$

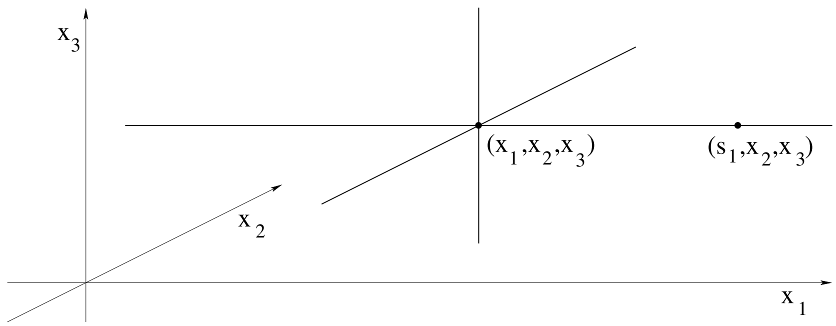

Figure 9: Proving the Gagliardo-Nirenberg inequality. The integral $\int_{-\infty}^{\infty} |D_{x_1} f(s_1, x_2, x_3)|\, ds_1$ depends on x_2, x_3 but not on x_1. Similarly, the integral $\int_{-\infty}^{\infty} |D_{x_2} f(x_1, s_2, x_3)|\, ds_2$ depends on x_1, x_3 but not on x_2.

Proof. 1. For each $i \in \{1, \ldots, n\}$ and every point $x = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n$, since f has compact support we can write

$$f(x_1, \ldots, x_i, \ldots, x_n) \;=\; \int_{-\infty}^{x_i} D_{x_i} f(x_1, \ldots, s_i, \ldots, x_n)\, ds_i\,.$$

In turn, this yields

$$|f(x_1, \ldots, x_n)| \;\leq\; \int_{-\infty}^{\infty} \Big| D_{x_i} f(x_1, \ldots, s_i, \ldots, x_n)\Big|\, ds_i \qquad 1 \leq i \leq n\,,$$

$$|f(x)|^{\frac{n}{n-1}} \;\leq\; \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |D_{x_i} f(x_1, \ldots, s_i, \ldots, x_n)|\, ds_i \right)^{\frac{1}{n-1}}. \tag{6.20}$$

We now integrate (6.20) w.r.t. x_1. Observe that the first factor on the right hand side does not depend on x_1. This factor behaves like a constant and can be taken out of the integral. The product of the remaining $n-1$ factors is handled using (6.18). This yields

$$\begin{aligned} \int_{-\infty}^{\infty} |f|^{\frac{n}{n-1}}\, dx_1 \;&\leq\; \left(\int_{-\infty}^{\infty} |D_{x_1} f|\, ds_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |D_{x_i} f|\, ds_i \right)^{\frac{1}{n-1}} dx_1 \\ &\leq\; \left(\int_{-\infty}^{\infty} |D_{x_1} f|\, ds_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_{x_i} f|\, ds_i\, dx_1 \right)^{\frac{1}{n-1}}. \end{aligned} \tag{6.21}$$

Notice that the second inequality was obtained by applying the generalized Hölder inequality to the $n-1$ functions $g_i = \int_{-\infty}^{\infty} |D_{x_i} f| ds_i$, $i = 2, \ldots, n$.

We now integrate both sides of (6.21) w.r.t. x_2. Observe that one of the factors appearing in the product on the right hand side of (6.21) does not depend on the variable x_2 (namely, the one involving integration w.r.t. s_2). This factor behaves like a constant and can be taken out of the integral. The product of the remaining $n-1$ factors is again estimated using Hölder's inequality. This yields

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|^{\frac{n}{n-1}}\,dx_1\,dx_2 \ &\leq\ \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|D_{x_1}f|\,dx_1dx_2\right)^{\frac{1}{n-1}}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|D_{x_2}f|\,dx_1dx_2\right)^{\frac{1}{n-1}}\\ &\qquad\times\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|D_{x_i}f|\,ds_i\,dx_1dx_2\right)^{\frac{1}{n-1}}.\end{aligned}\tag{6.22}$$

Proceeding in the same way, after n integrations we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|f|^{\frac{n}{n-1}}\,dx_1\cdots dx_n \ &\leq\ \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|D_{x_i}f|\,dx_1\cdots dx_n\right)^{\frac{1}{n-1}}\\ &\leq\ \left(\int_{I\!R^n}|\nabla f|\,dx\right)^{\frac{n}{n-1}}.\end{aligned}\tag{6.23}$$

This already implies

$$\|f\|_{\mathbf{L}^{n/(n-1)}} \ =\ \left(\int_{I\!R^n}|f|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \ \leq\ \int_{I\!R^n}|\nabla f|\,dx\,,\tag{6.24}$$

proving the theorem in the case where $p=1$ and $p^*=\frac{n}{n-1}$.

2. To cover the general case where $1<p<n$, we apply (6.24) to the function

$$g\ \doteq\ |f|^{\beta}\qquad\text{with}\qquad \beta\ \doteq\ \frac{p(n-1)}{n-p}\,.\tag{6.25}$$

Using the standard Hölder's inequality one obtains

$$\begin{aligned}\left(\int_{I\!R^n}|f|^{\frac{\beta n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \ &\leq\ \int_{I\!R^n}\beta|f|^{\beta-1}|\nabla f|\,dx\\ &\leq\ \beta\left(\int_{I\!R^n}|f|^{\frac{(\beta-1)p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left(\int_{I\!R^n}|\nabla f|^p\,dx\right)^{\frac{1}{p}}.\end{aligned}\tag{6.26}$$

Our choice of β in (6.25) yields

$$\frac{(\beta-1)p}{p-1}\ =\ \frac{\beta n}{n-1}\ =\ \frac{np}{n-p}\ =\ p^*.$$

Therefore, from (6.26) it follows

$$\left(\int_{I\!R^n}|f|^{p^*}\,dx\right)^{\frac{n-1}{n}}\ \leq\ \beta\left(\int_{I\!R^n}|f|^{p^*}\,dx\right)^{\frac{p-1}{p}}\left(\int_{I\!R^n}|\nabla f|^p\,dx\right)^{\frac{1}{p}}.$$

Observing that $\frac{n-1}{n}-\frac{p-1}{p}=\frac{n-p}{np}=\frac{1}{p^*}$, we conclude

$$\left(\int_{I\!R^n}|f|^{p^*}\,dx\right)^{\frac{1}{p^*}}\ \leq\ C\left(\int_{I\!R^n}|\nabla f|^p\,dx\right)^{\frac{1}{p}}.$$

□

If the domain $\Omega \subset I\!R^n$ is bounded, then $\mathbf{L}^q(\Omega) \subseteq \mathbf{L}^{p^*}(\Omega)$ for every $q \in [1, p^*]$. Using the Gagliardo-Nirenberg inequality we obtain

Corollary 6.2 (embedding). *Let $\Omega \subset I\!R^n$ be a bounded open domain with $\mathcal{C}^1$ boundary, and assume $1 \leq p < n$. Then, for every $q \in [1, p^*]$ with $p^* \doteq \frac{np}{n-p}$, there exists a constant C such that*

$$\|f\|_{\mathbf{L}^q(\Omega)} \leq C\,\|f\|_{W^{1,p}(\Omega)} \qquad \textit{for all } \ f \in W^{1,p}(\Omega)\,. \tag{6.27}$$

Proof. Let $\widetilde{\Omega} \doteq \{x \in I\!R^n\,;\ \ d(x, \Omega) < 1\}$ be the open neighborhood of radius one around the set Ω. By Theorem 4.3 there exists a bounded extension operator $E : W^{1,p}(\Omega) \mapsto W^{1,p}(I\!R^n)$, with the property that Ef is supported inside $\widetilde{\Omega}$, for every $f \in W^{1,p}(\Omega)$. Applying the Gagliardo-Nirenberg inequality to Ef, for suitable constants C_1, C_2, C_3 we obtain

$$\|f\|_{\mathbf{L}^q(\Omega)} \leq C_1\,\|f\|_{\mathbf{L}^{p^*}(\Omega)} \leq C_2\,\|Ef\|_{\mathbf{L}^{p^*}(I\!R^n)} \leq C_3\|f\|_{W^{1,p}(\Omega)}\,.$$

6.3 High order Sobolev estimates

Let $\Omega \subset I\!R^n$ be a bounded open set with $\mathcal{C}^1$ boundary, and let $u \in W^{k,p}(\Omega)$. The number

$$k - \frac{n}{p}$$

will be called the **net smoothness** of u. As in Fig. 10, let m be the integer part and let $0 \leq \gamma < 1$ be the fractional part of this number, so that

$$k - \frac{n}{p} = m + \gamma\,. \tag{6.28}$$

In the following, we say that a Banach space X is **continuously embedded** in a Banach space Y if $X \subseteq Y$ and there exists a constant C such that

$$\|u\|_Y \leq C\,\|u\|_X \qquad \text{for all } \ u \in X\,.$$

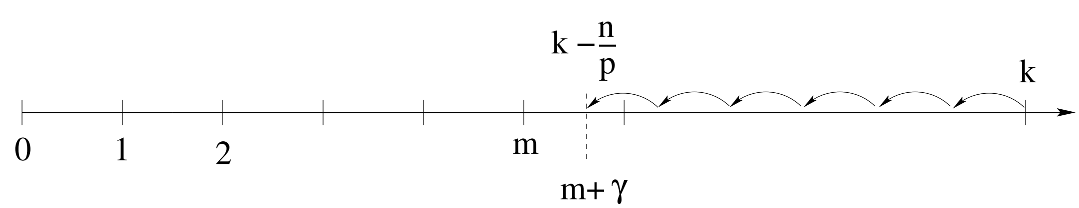

Figure 10: Computing the "net smoothness" of a function $f \in W^{k,p} \subset \mathcal{C}^{m,\gamma}$.

Theorem 6.3 (general Sobolev embeddings). *Let $\Omega \subset I\!R^n$ be a bounded open set with $\mathcal{C}^1$ boundary, and consider the space $W^{k,p}(\Omega)$. Let m, γ be as in (6.28). Then the following continuous embeddings hold.*

(i) If $k - \frac{n}{p} < 0$ *then* $W^{k,p}(\Omega) \subseteq \mathbf{L}^q(\Omega)$, *with* $\frac{1}{q} = \frac{1}{p} - \frac{k}{n} = \frac{1}{n}\left(\frac{n}{p} - k\right)$.

(ii) If $k - \frac{n}{p} = 0$, *then* $W^{k,p}(\Omega) \subseteq \mathbf{L}^q(\Omega)$ *for every* $1 \le q < \infty$.

(iii) If $m \ge 0$ *and* $\gamma > 0$, *then* $W^{k,p}(\Omega) \subseteq \mathcal{C}^{m,\gamma}(\Omega)$.

(iv) If $m \ge 1$ *and* $\gamma = 0$, *then for every* $0 \le \gamma' < 1$ *one has* $W^{k,p}(\Omega) \subseteq \mathcal{C}^{m-1,\gamma'}(\Omega)$.

Remark 6.1 Functions in a Sobolev space are only defined up to a set of measure zero. More precisely, by saying that $W^{k,p}(\Omega) \subseteq \mathcal{C}^{m,\gamma}(\Omega)$ we mean the following. For every $u \in W^{k,p}(\Omega)$ there exists a function $\tilde{u} \in \mathcal{C}^{m,\gamma}(\Omega)$ such that $\tilde{u}(x) = u(x)$ for a.e. $x \in \Omega$. Moreover, there exists a constant C, depending on k, p, m, γ but not on u, such that

$$\|u\|_{\mathcal{C}^{m,\gamma}(\Omega)} \le C\,\|\tilde{u}\|_{W^{k,p}(\Omega)}.$$

Proof of the theorem. 1. We start by proving (i). Assume $k - \frac{n}{p} < 0$ and let $u \in W^{k,p}(\Omega)$. Since $D^\alpha u \in W^{1,p}(\Omega)$ for every $|\alpha| \le k-1$, the Gagliardo-Nirenberg inequality yields

$$\|D^\alpha u\|_{L^{p^*}(\Omega)} \le C\,\|u\|_{W^{k,p}(\Omega)} \qquad |\alpha| \le k-1.$$

Therefore $u \in W^{k-1,p^*}(\Omega)$, where p^* is the Sobolev conjugate of p.

This argument can be iterated. Set $p_1 = p^*$, $p_2 \doteq p_1^*$, $\ldots$, $p_j \doteq p_{j-1}^*$. By (6.17) this means

$$\frac{1}{p_1} = \frac{1}{p} - \frac{1}{n}, \qquad \ldots \qquad \frac{1}{p_j} = \frac{1}{p} - \frac{j}{n},$$

provided that $jp < n$. Using the Gagliardo-Nirenberg inequality several times, we obtain

$$W^{k,p}(\Omega) \subseteq W^{k-1,p_1}(\Omega) \subseteq W^{k-2,p_2}(\Omega) \subseteq \cdots \subseteq W^{k-j,p_j}(\Omega). \tag{6.29}$$

After k steps we find that $u \in W^{0,p_k}(\Omega) = \mathbf{L}^{p_k}(\Omega)$, with $\frac{1}{p_k} = \frac{1}{p} - \frac{k}{n} = \frac{1}{q}$. Hence $p_k = q$ and (i) is proved.

2. In the special case $kp = n$, repeating the above argument, after $k-1$ steps we find

$$\frac{1}{p_{k-1}} = \frac{1}{p} - \frac{k-1}{n} = \frac{1}{n}.$$

Therefore $p_{k-1} = n$ and

$$W^{k,p}(\Omega) \subset W^{1,n}(\Omega) \subseteq W^{1,n-\varepsilon}(\Omega)$$

for every $\varepsilon > 0$. Using the Gagliardo-Nirenberg inequality once again, we obtain

$$u \in W^{1,n-\varepsilon}(\Omega) \subseteq \mathbf{L}^q(\Omega) \qquad q = \frac{n(n-\varepsilon)}{n-(n-\varepsilon)} = \frac{n^2 - \varepsilon n}{\varepsilon}.$$

Since $\varepsilon > 0$ was arbitrary, this proves (ii).

3. To prove (iii), assume that $m \geq 0$ and $\gamma > 0$ and let $u \in W^{k,p}(\Omega)$. We use the inequalities (6.29), choosing j to be the smallest integer such that $p_j > n$. We thus have

$$\frac{1}{p} - \frac{j}{n} = \frac{1}{p_j} < \frac{1}{n} < \frac{1}{p} - \frac{j-1}{n}, \qquad u \in W^{k-j,p_j}(\Omega).$$

Hence, for every multi-index α with $|\alpha| \leq k - j - 1$, Morrey's inequality yields

$$D^\alpha u \in W^{1,p_j}(\Omega) \subseteq \mathcal{C}^{0,\gamma}(\Omega) \qquad \text{with } \gamma = 1 - \frac{n}{p_j} = 1 - \frac{n}{p} + j.$$

Since α was any multi-index with length $\leq k - j - 1$, the above implies

$$u \in \mathcal{C}^{k-j-1,\gamma}(\Omega).$$

To conclude the proof of (iii), it suffices to check that

$$k - \frac{n}{p} = (k - j - 1) + \left(1 - \frac{n}{p} + j\right),$$

so that $m = k - j - 1$ is the integer part of the number $k - \frac{n}{p}$, while γ is its fractional part.

4. To prove (iv), assume that $m \geq 1$ and $j \doteq \frac{n}{p}$ is an integer. Let $u \in W^{k,p}(\Omega)$, and fix any multi-index with $|\alpha| \leq j - 1$. Using the Gagliardo-Nirenberg inequality as in step **2**, we obtain

$$D^\alpha u \in W^{k-j,p}(\Omega) \subseteq W^{1,q}(\Omega)$$

for every $1 < q < \infty$. Hence, by Morrey's inequality

$$D^\alpha u \in W^{1,q}(\Omega) \subseteq \mathcal{C}^{0,1-\frac{n}{q}}(\Omega).$$

Since q can be chosen arbitrarily large, this proves (iv). □

Example 9. Let Ω be the open unit ball in $I\!R^5$, and assume $u \in W^{4,2}(\Omega)$. Applying two times the Nirenberg-Gagliardo inequality and then Morrey's inequality, we obtain

$$u \in W^{4,2}(\Omega) \subset W^{3,\frac{10}{3}}(\Omega) \subset W^{2,10}(\Omega) \subset C^{1,\frac{1}{2}}(\Omega).$$

Observe that the *net smoothness* of u is $k - \frac{n}{p} = 4 - \frac{5}{2} = 1 + \frac{1}{2}$.

7 Compact embeddings

Let $\Omega \subset I\!R^n$ be a bounded open set with $\mathcal{C}^1$ boundary. In this section we study the embedding $W^{1,p}(\Omega) \subset \mathbf{L}^q(\Omega)$ in greater detail and show that, when $\frac{1}{q} > \frac{1}{p} - \frac{1}{n}$, this embedding is compact. Namely, from any sequence $(u_m)_{m\geq 1}$ which is bounded in $W^{1,p}$ one can extract a subsequence which converges in $\mathbf{L}^q$.

As a preliminary we observe that, if $p > n$, then every function $u \in W^{1,p}(\Omega)$ is Hölder continuous. In particular, if $(u_m)_{m\geq 1}$ is a bounded sequence in $W^{1,p}(\Omega)$ then the functions u_m are equicontinuous and uniformly bounded. By Ascoli's compactness theorem we can extract a subsequence $(u_{m_j})_{j\geq 1}$ which converges to a continuous function u uniformly on Ω. Since Ω is bounded, this implies $\|u_{m_j} - u\|_{\mathbf{L}^q(\Omega)} \to 0$ for every $q \in [1, \infty]$. This already shows that the embedding $W^{1,p}(\Omega) \subset \mathbf{L}^q(\Omega)$ is compact whenever $p > n$ and $1 \leq q \leq \infty$.

In the remainder of this section we thus focus on the case $p < n$. By the Gagliardo-Nirenberg inequality, the space $W^{1,p}(\Omega)$ is continuously embedded in $\mathbf{L}^{p^*}(\Omega)$, where $p^* = \frac{np}{n-p}$. In turn, since Ω is bounded, for every $1 \leq q \leq p^*$ we have the continuous embedding $\mathbf{L}^{p^*}(\Omega) \subseteq \mathbf{L}^q(\Omega)$.

Theorem 7.1 (Rellich-Kondrachov compactness theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^1$ boundary. Assume $1 \leq p < n$. Then for each $1 \leq q < p^* \doteq \frac{np}{n-p}$ one has the compact embedding*

$$W^{1,p}(\Omega) \ \subset\subset \ \mathbf{L}^q(\Omega).$$

Proof. 1. Let $(u_m)_{m\geq 1}$ be a bounded sequence in $W^{1,p}(\Omega)$. Using Theorem 4.3 on the extension of Sobolev functions, we can assume that all functions u_m are defined on the entire space $\mathbb{R}^n$ and vanish outside a compact set K:

$$Supp(u_m) \ \subseteq \ K \ \subset \ \widetilde{\Omega} \ \doteq \ B(\Omega, 1)\,. \tag{7.1}$$

Here the right hand side denotes the open neighborhood of radius one around the set Ω.

Since $q < p^*$ and $\widetilde{\Omega}$ is bounded, we have

$$\|u_m\|_{\mathbf{L}^q(\mathbb{R}^n)} \ = \ \|u_m\|_{\mathbf{L}^q(\widetilde{\Omega})} \ \leq \ C\,\|u_m\|_{\mathbf{L}^{p^*}(\widetilde{\Omega})} \ \leq \ C'\,\|u_m\|_{W^{1,p}(\widetilde{\Omega})}$$

for some constants C, C'. Hence the sequence u_m is uniformly bounded in $\mathbf{L}^q(\mathbb{R}^n)$.

2. Consider the mollified functions $u_m^\varepsilon \doteq J_\varepsilon * u_m$. By (7.1) we can assume that all these functions are supported inside $\widetilde{\Omega}$. We claim that

$$\|u_m^\varepsilon - u_m\|_{\mathbf{L}^q(\widetilde{\Omega})} \ \to \ 0 \qquad as \quad \varepsilon \to 0, \qquad uniformly\ w.r.t.\ \ m. \tag{7.2}$$

Indeed, if u_m is smooth, then (performing the changes of variable $y' = \varepsilon y$ and $z = x - \varepsilon t y$)

$$\begin{aligned}
u_m^\varepsilon(x) - u_m(x) \ &= \ \int_{|y'|<\varepsilon} J_\varepsilon(y')\,[u_m(x-y') - u_m(x)]\,dy' \\
&= \ \int_{|y|<1} J(y)[u_m(x-\varepsilon y) - u_m(x)]\,dy \\
&= \ \int_{|y|<1} J(y)\left(\int_0^1 \frac{d}{dt}\Big(u_m(x-\varepsilon t y)\Big)\,dt\right)\,dy \\
&= \ -\,\varepsilon\int_{|y|<1} J(y)\left(\int_0^1 \nabla u_m(x-\varepsilon t y)\cdot y\ dt\right)\,dy\,.
\end{aligned}$$

In turn, this yields

$$\int_{\widetilde{\Omega}} |u_m^\varepsilon(x) - u_m(x)|\,dx \;\leq\; \varepsilon \int_{\widetilde{\Omega}} \int_{|y|\leq 1} J(y) \left(\int_0^1 |\nabla u_m(x-\varepsilon t y)|\,dt \right) dy\,dx$$

$$\leq\; \varepsilon \int_{\widetilde{\Omega}} |\nabla u_m(z)|\,dz.$$

By approximating u_m in $W^{1,p}$ with a sequence of smooth functions, we see that the same estimate remains valid for all functions $u_m \in W^{1,p}(\widetilde{\Omega})$. We have thus shown that

$$\|u_m^\varepsilon - u_m\|_{\mathbf{L}^1(\widetilde{\Omega})} \;\leq\; \varepsilon \|\nabla u_m\|_{\mathbf{L}^1(\widetilde{\Omega})} \;\leq\; \varepsilon C \|u_m\|_{W^{1,p}(\widetilde{\Omega})}, \tag{7.3}$$

for some constant C. Since the norms $\|u_m\|_{W^{1,p}}$ satisfy a uniform bound independent of m, this already proves our claim (7.2) in the case $q = 1$.

3. To prove (7.2) also for $1 < q < p^*$, we now use the interpolation inequality for $\mathbf{L}^p$ norms. Choose $0 < \theta < 1$ such that

$$\frac{1}{q} \;=\; \theta \cdot 1 + (1-\theta) \cdot \frac{1}{p^*}\,.$$

Then

$$\|u_m^\varepsilon - u_m\|_{\mathbf{L}^q(\widetilde{\Omega})} \;\leq \|u_m^\varepsilon - u_m\|^{\theta}_{\mathbf{L}^1(\widetilde{\Omega})} \cdot \|u_m^\varepsilon - u_m\|^{1-\theta}_{\mathbf{L}^{p^*}(\widetilde{\Omega})} \;\leq\; C_0\,\varepsilon^\theta. \tag{7.4}$$

for some constant C_0 independent of m. Indeed, in the above expression, the $\mathbf{L}^1$ norm is bounded by (7.3), while the $\mathbf{L}^{p^*}$ norm is bounded by a constant, because of the Gagliardo-Nirenberg inequality.

4. Fix any $\delta > 0$, and choose $\varepsilon > 0$ small enough so that (7.4) yields

$$\|u_m^\varepsilon - u_m\|_{\mathbf{L}^q(\widetilde{\Omega})} \;\leq\; C_0\,\varepsilon^\theta \;\leq\; \frac{\delta}{2} \qquad\qquad \text{for all}\;\; m \geq 1\,.$$

Recalling that $u_m^\varepsilon = J_\varepsilon * u_m$, we have

$$\|u_m^\varepsilon\|_{\mathbf{L}^\infty} \;\leq\; \|J_\varepsilon\|_{\mathbf{L}^\infty}\,\|u_m\|_{\mathbf{L}^1} \;\leq\; C_1,$$

$$\|\nabla u_m^\varepsilon\|_{\mathbf{L}^\infty} \;\leq\; \|\nabla J_\varepsilon\|_{\mathbf{L}^\infty}\,\|u_m\|_{\mathbf{L}^1} \;\leq\; C_2,$$

where C_1, C_2 are constants depending on ε but not on m. The above inequalities show that, for each fixed $\varepsilon > 0$, the sequence $(u_m^\varepsilon)_{m\geq 1}$ is uniformly bounded and equicontinuous. By Ascoli's compactness theorem, there exists a subsequence $(u_{m_j}^\varepsilon)$ which converges uniformly on $\widetilde{\Omega}$ to some continuous function u^ε. We now have

$$\begin{aligned} &\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{\mathbf{L}^q} \\ &\quad\leq\; \limsup_{j,k\to\infty} \Big(\|u_{m_j} - u_{m_j}^\varepsilon\|_{\mathbf{L}^q} + \|u_{m_j}^\varepsilon - u^\varepsilon\|_{\mathbf{L}^q} + \|u^\varepsilon - u_{m_k}^\varepsilon\|_{\mathbf{L}^q} + \|u_{m_k}^\varepsilon - u_{m_k}\|_{\mathbf{L}^q} \Big) \\ &\quad\leq\; \tfrac{\delta}{2} + 0 + 0 + \tfrac{\delta}{2}\,. \end{aligned} \tag{7.5}$$

5. The proof is now concluded by a standard diagonalization argument. By the previous step we can find an infinite set of indices $I_1 \subset I\!\!N$ such that the subsequence $(u_m)_{m\in I_1}$ satifies

$$\limsup_{\ell,m\to\infty,\ \ell,m\in I_1} \|u_\ell - u_m\|_{\mathbf{L}^q} \ \leq\ 2^{-1}.$$

By induction on $j = 1, 2, \ldots$, after I_{j-1} has been constructed, we choose an infinite set of indices $I_j \subset I_{j-1} \subset I\!\!N$ such that the subsequence $(u_m)_{m\in I_j}$ satisfies

$$\limsup_{\ell,m\to\infty,\ \ell,m\in I_j} \|u_\ell - u_m\|_{\mathbf{L}^q} \ \leq\ 2^{-j}.$$

After the subsets I_j have been constructed for all $j \geq 1$, again by induction on j we choose a sequence of integers $m_1 < m_2 < m_3 < \cdots$ such that $m_j \in I_j$ for every j. The subsequence $(u_{m_j})_{j\geq 1}$ satisfies

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{\mathbf{L}^q} \ =\ 0\,.$$

Therefore this subsequence is Cauchy, and converges to some limit $u \in \mathbf{L}^q$. □

As a first application of the compact embedding theorem, we now prove and estimate on the difference between a function u and its average value on the domain Ω.

Theorem 7.2 (Poincare's inequality - II). *Let $\Omega \subset I\!\!R^n$ be a bounded, connected open set with $\mathcal{C}^1$ boundary, and let $p \in [1, \infty]$. Then there exists a constant C depending only on p and Ω such that*

$$\left\| u - \fint_\Omega u\,dx \right\|_{\mathbf{L}^p(\Omega)} \ \leq\ C\,\|\nabla u\|_{\mathbf{L}^p(\Omega)}\,, \tag{7.6}$$

for every $u \in W^{1,p}(\Omega)$.

Proof. If the conclusion were false, one could find a sequence of functions $u_k \in W^{1,p}(\Omega)$ with

$$\left\| u_k - \fint_\Omega u_k\,dx \right\|_{\mathbf{L}^p(\Omega)} \ >\ k\,\|\nabla u_k\|_{\mathbf{L}^p(\Omega)}$$

for every $k = 1, 2, \ldots$ Then the renormalized functions

$$v_k \ \doteq\ \frac{u_k - \fint_\Omega u_k\,dx}{\left\| u_k - \fint_\Omega u_k\,dx \right\|_{\mathbf{L}^p(\Omega)}}$$

satisfy

$$\fint_\Omega v_k\,dx = 0\,, \qquad \|v_k\|_{\mathbf{L}^p(\Omega)} \ =\ 1, \qquad \|Dv_k\|_{\mathbf{L}^p(\Omega)} \ <\ \frac{1}{k} \qquad\qquad k = 1, 2, \ldots \tag{7.7}$$

Since the sequence $(v_k)_{k\geq 1}$ is bounded in $W^{1,p}(\Omega)$, if $p < \infty$ we can use the Rellich-Kondrachov compactness theorem and find a subsequence that converges in $\mathbf{L}^p(\Omega)$ to some function v. If $p > n$, then by (6.15) the functions v_k are uniformly bounded and Hölder continuous. Using Ascoli's compactness theorem we can thus find a subsequence that converges in $\mathbf{L}^\infty(\Omega)$ to some function v.

By (7.7), the sequence of weak gradients also converges, namely $\nabla v_k \to 0$ in $\mathbf{L}^p(\Omega)$. By Lemma 1.3, the zero function is the weak gradient of the limit function v.

We now have

$$\fint_\Omega v\,dx \;=\; \lim_{k\to\infty} \fint_\Omega v_k dx \;=\; 0.$$

Moreover, since $\nabla v = 0 \in \mathbf{L}^p(\Omega)$, By Corollary 2.1 the function v must be constant on the connected set Ω, hence $v(x) = 0$ for a.e. $x \in \Omega$. But this is in contradiction with

$$\|v\|_{\mathbf{L}^p(\Omega)} \;=\; \lim_{k\to\infty} \|v_k\|_{\mathbf{L}^p(\Omega)} \;=\; 1.$$

□

8 Differentiability properties

By Morrey's inequality, if $\Omega \subset I\!R^n$ and $w \in W^{1,p}(\Omega)$ with $p > n$, then w coincides a.e. with a Hölder continuous function. Indeed, after a modification on a set of measure zero, we have

$$|w(x) - w(y)| \;\leq\; C|x-y|^{1-\frac{n}{p}} \left(\int_{B(x,\,|y-x|)} |\nabla w(z)|^p\,dz\right)^{1/p}. \tag{8.1}$$

This by itself does not imply that u should be differentiable in a classical sense. Indeed, there exist Hölder continuous functions that are nowhere differentiable. However, for functions in a Sobolev space a much stronger differentiability result holds.

Theorem 8.1 (almost everywhere differentiability). *Let $\Omega \subseteq I\!R^n$ and let $u \in W^{1,p}_{loc}(\Omega)$ for some $p > n$. Then u is differentiable at a.e. point $x \in \Omega$, and its gradient equals its weak gradient.*

Proof. Let $u \in W^{1,p}_{loc}(\Omega)$. Since the weak derivatives are in $\mathbf{L}^p_{loc}$, the same is true of the weak gradient $\nabla u \doteq (D_{x_1} u\,,\ \ldots, D_{x_n} u)$. By the Lebesgue differentiation theorem for a.e. $x \in \Omega$ we have

$$\fint_{B(x,r)} |\nabla u(x) - \nabla u(z)|^p\,dz \;\to\; 0 \qquad \text{as} \quad r \to 0\,. \tag{8.2}$$

Fix a point x for which (8.2) holds, and define

$$w(y) \;\doteq\; u(y) - u(x) - \nabla u(x)\cdot(y-x). \tag{8.3}$$

Observing that $w \in W^{1,p}_{loc}(\Omega)$, we can apply the estimate (8.1) and obtain

$$\begin{aligned}|w(y) - w(x)| \;&=\; |w(y)| \;=\; |u(y) - u(x) - \nabla u(x)\cdot(y-x)|\\ &\leq\; C\,|y-x|^{1-\frac{n}{p}} \left(\int_{B(x,\,|y-x|)} |\nabla u(x) - \nabla u(z)|^p\,dz\right)^{1/p}\\ &\leq C'\,|y-x| \left(\fint_{B(x,\,|y-x|)} |\nabla u(x) - \nabla u(z)|^p\,dz\right)^{1/p}\end{aligned}$$

for suitable constants C, C'. Therefore

$$\frac{|w(y) - w(x)|}{|y - x|} \to 0 \qquad \text{as } |y - x| \to 0 .$$

By the definition of w at (8.3), this means that u is differentiable at x in the strong sense, and its gradient coincides with its weak gradient. □

9 Problems

1. Determine which of the following functionals defines a distribution on $\Omega \subseteq I\!R$.

(i) $\Lambda(\phi) = \sum_{k=1}^{\infty} k!\, D^k \phi(k)$, with $\Omega = I\!R$.

(ii) $\Lambda(\phi) = \sum_{k=1}^{\infty} 2^{-k}\, D^k \phi(1/k)$, with $\Omega = I\!R$.

(iii) $\Lambda(\phi) = \sum_{k=1}^{\infty} \frac{\phi(1/k)}{k}$, with $\Omega = I\!R$.

(iv) $\Lambda(\phi) = \int_0^{\infty} \frac{\phi(x)}{x^2}\, dx$, with $\Omega =]0, \infty[$.

2. Give a direct proof that, if $f \in W^{1,p}(\,]a,b[\,)$ for some $a < b$ and $1 < p < \infty$, then, by possibly changing f on a set of measure zero, one has

$$|f(x) - f(y)| \leq C\, |x - y|^{1 - \frac{1}{p}} \qquad \text{for all } x, y \in \,]a, b[\,.$$

Compute the best possible constant C.

3. Consider the open square

$$Q \doteq \{(x_1, x_2)\,;\ \ 0 < x_1 < 1,\ \ 0 < x_2 < 1\} \subset I\!R^2 .$$

Let $f \in W^{1,1}(Q)$ be a function whose weak derivative satisfies $D_{x_1} f(x) = 0$ for a.e. $x \in Q$. Prove that there exists a function $g \in \mathbf{L}^1([0,1])$ such that

$$f(x_1, x_2) = g(x_2) \qquad \text{for a.e. } (x_1, x_2) \in Q .$$

4. Let $\Omega \subseteq I\!R^n$ be an open set and assume $f \in \mathbf{L}^1_{loc}(\Omega)$. Let $g = D_{x_1} f$ be the weak derivative of f w.r.t. x_1. If f is $\mathcal{C}^1$ restricted to an open subset $\Omega' \subseteq \Omega$, prove that g coincides with the partial derivative $\partial f / \partial x_1$ at a.e. point $x \in \Omega'$.

5. (i) Prove that, if $u \in W^{1,\infty}(\Omega)$ for some open, convex set $\Omega \subseteq I\!\!R^n$, then u coincides a.e. with a Lipschitz continuous function.

(ii) Show that there exists a (non-convex), connected open set $\Omega \subset I\!\!R^n$ and a function $u \in W^{1,\infty}(\Omega)$ that does not coincide a.e. with a Lipschitz continuous function.

6. Let $\Omega = B(0,1)$ be the open unit ball in $I\!\!R^n$, with $n \geq 2$. Prove that the unbounded function $\varphi(x) = \log\log\left(1 + \frac{1}{|x|}\right)$ is in $W^{1,n}(\Omega)$.

7. Let $\Omega =]0,1[$. Consider the linear map $T : \mathcal{C}^1([0,1]) \mapsto I\!\!R$ defined by $Tf = f(0)$. Show that this map can be continuously extended, in a unique way, to a linear functional $T : W^{1,1}(\Omega) \mapsto I\!\!R$.

8. Let $\Omega \subset I\!\!R^n$ be a bounded open set, with smooth boundary. For every continuous function $f : \overline{\Omega} \mapsto I\!\!R$, define the trace

$$T : \mathcal{C}^0(\Omega) \mapsto \mathcal{C}^0(\partial\Omega)$$

by letting Tf be the restriction of f to the boundary $\partial\Omega$. Show that the operator T cannot be continuously extended as a map from $\mathbf{L}^p(\Omega)$ into $\mathbf{L}^p(\Omega)$, for any $1 \leq p < \infty$. In other words, a generic function $f \in \mathbf{L}^p(\Omega)$ does not have trace on the boundary $\partial\Omega$.

9. To construct the trace of a function $u : W^{1,p}(\Omega)$, consider the following approach.

(i) Using Theorem 4.3, extend u to a function $Eu \in W^{1,p}(I\!\!R^n)$ defined on the entire space $I\!\!R^n$, including the boundary of Ω.

(ii) Define the trace of u as the restriction of Eu to the boundary of Ω, namely $Tu \doteq Eu|_{\partial\Omega}$.

Explain why this approach is fundamentally flawed.

10. Let $V \subset I\!\!R^n$ be a subspace of dimension m and let $V^\perp$ be the perpendicular subspace, of dimension $n - m$. Let $u \in W^{1,p}(I\!\!R^n)$ with $m < p < n$. Show that, after a modification on a set of measure zero, the following holds.

(i) For a.e. $y \in V^\perp$ (w.r.t. the $n-m$ dimensional measure), the restriction of u to the affine subspace $y + V$ is Hölder continuous with exponent $\gamma = 1 - \frac{m}{p}$.

(ii) The pointwise value $u(y)$ is well defined for a.e. $y \in V^\perp$. Moreover

$$\int_{V^\perp} |u(y)|\, dy \;\leq\; C \cdot \|u\|_{W^{1,p}(I\!\!R^n)}$$

for some constant C depending on m, n, p but not on u.

11. Determine for which values of $p \geq 1$ a generic function $f \in W^{1,p}(I\!R^3)$ admits a trace along the x_1-axis. In other words, set $\Gamma \doteq \{(t,0,0)\,;\quad t \in I\!R\} \subset I\!R^3$ and consider the map $T : \mathcal{C}^1_c(I\!R^3) \mapsto \mathbf{L}^p(\Gamma)$, where $Tf = f_{|\Gamma}$ is the restriction of f to Γ. Find values of p such that this map admits a continuous extension $T : W^{1,p}(I\!R^3) \mapsto \mathbf{L}^p(\Gamma)$.

12. When $k = 0$, by definition $W^{0,p}(\Omega) = \mathbf{L}^p(\Omega)$. If $1 \leq p < \infty$ prove that $W^{0,p}_0(\Omega) = \mathbf{L}^p(\Omega)$ as well. What is $W^{0,\infty}_0(\Omega)$?

13. Let $\varphi : I\!R \mapsto [0,1]$ be a smooth function such that

$$\varphi(r) \;=\; \begin{cases} 1 & \text{if} \quad r \leq 0\,, \\ 0 & \text{if} \quad r \geq 1\,. \end{cases}$$

Given any $f \in W^{k,p}(I\!R^n)$, prove that the functions $f_k(x) \;\doteq\; f(x)\,\varphi(|x| - k)$ converge to f in $W^{k,p}(I\!R^n)$, for every $k \geq 0$ and $1 \leq p < \infty$. As a consequence, show that $W^{k,p}_0(I\!R^n) = W^{k,p}(I\!R^n)$.

14. Let $I\!R_+ \doteq \{x \in I\!R\,;\quad x > 0\}$ and assume $u \in W^{2,p}(I\!R_+)$. Define the symmetric extension of u by setting $Eu(x) \;\doteq\; u(|x|)$. Prove that $Eu \in W^{1,p}(I\!R)$ but $Eu \notin W^{2,p}(I\!R)$, in general.

15. Let $u \in \mathcal{C}^1_c(I\!R^n)$ and fix $p, q \in [1, \infty[$. For a given $\lambda > 0$, consider the rescaled function $u_\lambda(x) \doteq u(\lambda x)$.

(i) Show that there exists an exponent α, depending on n, p, such that

$$\|u_\lambda\|_{\mathbf{L}^p(I\!R^n)} \;=\; \lambda^\alpha \|u\|_{\mathbf{L}^p(I\!R^n)}\,.$$

(ii) Show that there exists an exponent β, depending on n, q, such that

$$\|\nabla u_\lambda\|_{\mathbf{L}^q(I\!R^n)} \;=\; \lambda^\beta \|\nabla u\|_{\mathbf{L}^q(I\!R^n)}\,.$$

(iii) Determine for which values of n, p, q one has $\alpha = \beta$. Compare with (6.16).

16. Let $\Omega \subseteq I\!R^n$ be a bounded open set, with $\mathcal{C}^1$ boundary. Let $(u_m)_{m\geq 1}$ be a sequence of functions which are uniformly bounded in $H^1(\Omega)$. Assuming that $\|u_m - u\|_{\mathbf{L}^2} \to 0$, prove that $u \in H^1(\Omega)$ and

$$\|u\|_{H^1} \;\leq\; \liminf_{m\to\infty} \|u_m\|_{H^1}\,.$$

17. Let $\Omega \doteq \{(x_1, x_2)\,;\quad x_1^2 + x_2^2 < 1\}$ be the open unit disc in $I\!R^2$, and let $\Omega_0 \doteq \Omega \setminus \{(0,0)\}$ be the unit disc minus the origin. Consider the function $f(x) \doteq 1 - |x|$. Prove that

$$\begin{cases} f \;\in\; W^{1,p}_0(\Omega) & \text{for} \;\; 1 \leq p < \infty\,, \\ f \;\in\; W^{1,p}_0(\Omega_0) & \text{for} \;\; 1 \leq p \leq 2\,, \\ f \;\notin\; W^{1,p}_0(\Omega_0) & \text{for} \;\; 2 < p \leq \infty\,. \end{cases}$$

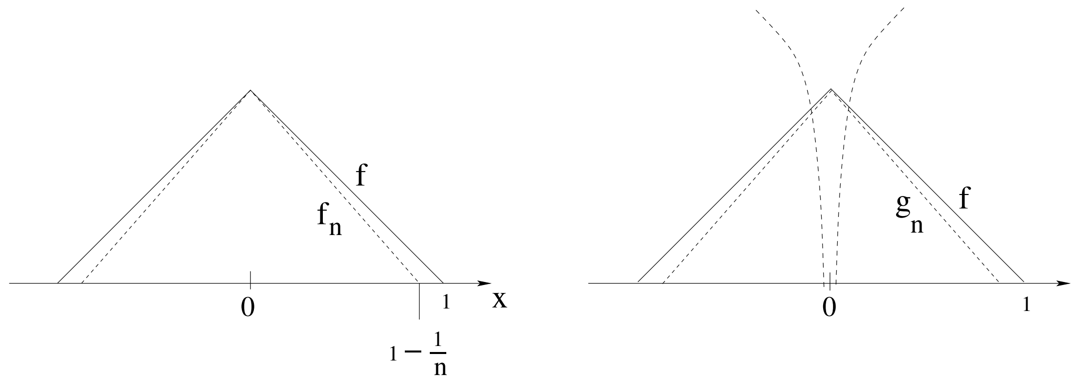

Figure 11: Left: the function f can be approximated in $W^{1,p}$ with functions f_n having compact support in Ω. Right: the function f can be approximated in $W^{1,2}$ with functions g_n having compact support in Ω_0.

18. Let $\Omega = \{(x_1, x_2)\,;\ \ 0 < x_1 < 1,\ 0 < x_2 < 1\}$ be the open unit square in $I\!R^2$.

(i) If $u \in H^1(\Omega)$ satisfies

$$\text{meas}\Big(\{x \in \Omega\,;\ u(x) = 0\}\Big) \ > \ 0\,, \qquad\qquad \nabla u(x) = 0 \quad \text{for a.e. } x \in \Omega$$

prove that $u(x) = 0$ for a.e. $x \in \Omega$.

(ii) For every $\alpha > 0$, prove that there exists a constant C_α with the following property. If $u \in H^1(\Omega)$ is a function such that $\text{meas}(\{x \in \Omega\,;\ u(x) = 0\}) \geq \alpha$, then

$$\|u\|_{\mathbf{L}^2(\Omega)} \leq C_\alpha \, \|\nabla u\|_{\mathbf{L}^2(\Omega)}\,. \tag{9.4}$$

19. Let $(u_n)_{n\geq 1}$ be a sequence of functions in the Hilbert space $H^2(I\!R^3) \doteq W^{2,2}(I\!R^3)$. Assume that

$$\lim_{n\to\infty} u_n(x) = u(x) \quad \text{for all} \;\; x \in I\!R^3, \qquad\qquad \|u_n\|_{H^2} \leq M \qquad \text{for all} \;\; n.$$

Prove that the limit function u coincides a.e. with a continuous function.

20.

(i) Find two functions $f, g \in \mathbf{L}^1_{loc}(I\!R^n)$ such that the product $f \cdot g$ is not locally integrable.

(ii) Show that, if $f, g \in \mathbf{L}^1_{loc}(I\!R)$ are both weakly differentiable, then the product $f \cdot g$ is also weakly differentiable and satisfies the usual product rule: $D_x(fg) = (D_x f) \cdot g + f \cdot (D_x g)$.

(iii) Find two functions $f, g \in \mathbf{L}^1_{loc}(I\!R^n)$ (with $n \geq 2$) with the following properties. For every $i = 1, \ldots, n$ the first order weak derivatives $D_{x_i} f$, $D_{x_i} g$ are well defined. However, the product $f \cdot g$ does not have any weak derivative (on the entire space $I\!R^n$).