MASARYKOVA UNIVERZITA
PŘÍRODOVĚDECKÁ FAKULTA
ÚSTAV MATEMATIKY A STATISTIKY

Coalitional games

Master's Thesis

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The aim of this thesis is to discuss various solution concepts of coalitional games and demonstrate them on specific examples. In the practical part a game with economic or financial background will be introduced and solved.

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Game theory - introduction

Game theory is one of the disciplines of applied mathematics. It studies and analyses strategic decision making and its aim is to provide a mathematical model of conflict situations or cooperation among agents that would help us describe the setting of a given problem and find the best game strategies for all agents.

First game-theoretic ideas occurred in the 18th century as the discussion on game theory topic was recorded in a letter written by Charles Waldegrave in 1713.

Three main game representations

Normal form - In the normal-form representation of a game, each player simultaneously chooses a strategy, and the combination of strategies chosen by the players determines a payoff for each player. A game in normal form $G = ((X_i)_{i \in N}, (u_i)_{i \in N})$ consists of:

- a finite set $N = \{1, ..., n\}$, $n \in \mathbb{N}$, $n \geq 2$ – (the set of **players**)
- for each player $i \in N$ a nonempty set X_i – (the set of **actions** available to a player i)
- for each player $i \in N$ a function $u_i : \prod_{j \in N} X_j \to \mathbb{R}$ – (the **payoff function** of a player i)

A game in normal form can be expressed in several ways, i.e. by payoff matrix. This is, for instance, the payoff matrix of the famous *Prisoner's dilemma*:

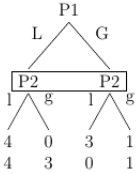
P2

P1		C	NC
	C	1, 1	10, 0
	NC	0, 10	5, 5

Extensive form - The extensive game is a game in which players move after they learn the actions of other players and which consists of:

- Set of players, N
- Player's payoffs as a function of outcomes, $v_i(\cdot)_{i\in N}$
- Order of moves
- Action of players when they can move
- The knowledge that players have when they can move
- Probability distribution over exogenous events

The extensive game can be captured by a tree diagram:



Characteristic function - Games in a form of characteristic function are most typically coalitional games to which my thesis is devoted. Such a game consists of:

- a finite set $N = \{1, ..., n\}$, $n \in \mathbb{N}$, $n \geq 2$ – (the set of **players**)
  Let 2^N be a set of all subsets of the set of players N. Then the elements of the set 2^N are called **coalitions.**
- a mapping $v : 2^N \to \mathbb{R}$ satisfying following conditions:
  - $v(\emptyset) = 0$
  - for arbitrary $S, T \subseteq N : S \cap T = \emptyset : v(S \cap T) \geq v(S) + v(T)$ – (a **characteristic function**). The number $v(S)$ is called a **payoff of the coalition** S.

Coalitional game

A coalitional game is a game in which players can cooperate - before choosing a strategy they have a chance to negotiate and make agreements about what strategies they will play to increase their payoff. Generally, players can but do not have to collaborate and they do so only in case it is beneficial for them.

Two fundamental problems:

- Coalition Structure Generation
- Payoff Distribution

Solution concepts

- Core
- Shapley value
- N-M stable set
- Bargaining set
- Kernel
- Nucleolus
- ε core

Core

It is a set of such payoff distributions from which no group of players is motivated to deviate as they would not be better off.

Let S be a non-empty coalition, $\emptyset \neq S \subseteq N$, and $\mathbf{x}, \mathbf{y}$ are some payoff distributions. Distribution $\mathbf{x}$ *dominates for a coalition S* distribution $\mathbf{y}$, denoted $x \succ_S y$, if:

- $\forall i \in S : x_i > y_i$
- $\sum_{i \in S} x_i \leq v(S)$.

Distribution $\mathbf{x}$ *dominates* distribution $\mathbf{y}$, denoted $x \succ y$, if there exists $\emptyset \neq S \subseteq N$ such that $x \succ_S y$. The ***core*** is the set of all undominated payoffs and it is denoted by $C(v)$. From collective and individual rationality it follows:

$$C(v) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \forall S \subseteq N : \sum_{i \in S} x_i \geq v(S), \sum_{i \in N} x_i = v(N) \right\}.$$

Example

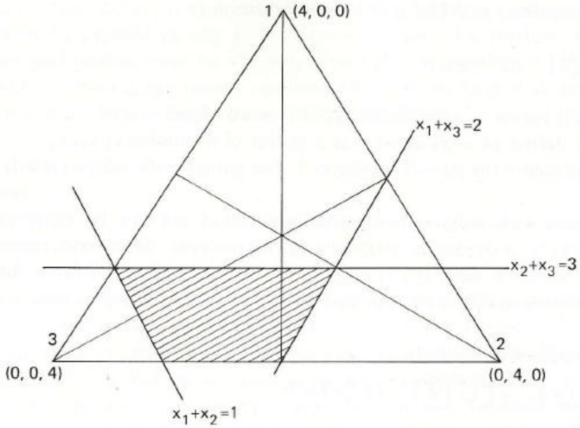
- Consider the following game of three players in a form of characteristic function:

$$v(\overline{1}) = v(\overline{2}) = v(\overline{3}) = 0,$$
$$v(\overline{12}) = 1, v(\overline{13}) = 2, v(\overline{23}) = 3,$$
$$v(\overline{123}) = 4.$$

- The core of this game is a set of distributions $\mathbf{x} = (x_1, x_2, x_3)$ satisfying the following conditions:

$$C(v) = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \forall S \subseteq N : x_1 + x_2 + x_3 = 4 \quad (1)$$
$$x_1 + x_2 \geq 1 \quad (2)$$
$$x_1 + x_3 \geq 2 \quad (3)$$
$$x_2 + x_3 \geq 3 \quad (4)$$
$$x_1, x_2, x_3 \geq 0. \quad (5)$$

Geometric representation



The particular integer solution can be for instance vectors $(1, 1, 2)$, $(0, 2, 2)$, $(1, 0, 3)$...

Shapley value

- Fair division
- Shapley payoff vector $\psi = (\psi_1, \ldots, \psi_n)$
- Unique
- ϑ - a set of all games with respect to the given set of players
- We denote the game using her characteristic function v
- Player i is called *dummy* if:

$$i \notin S \subseteq N : v(S \cup \{i\}) \leq v(S) + v(\{i\}).$$

- Players i and j are *interchangeable* if:

$$i, j \notin S \subseteq N : v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S).$$

Shapley value

Four axioms:

- **Efficiency**: The payoff distribution has to be efficient, thus there is no waste from dividing the payoff among players, i.e. $\sum_{i=1}^{n} x_i = v(N)$.
- **A dummy player gets nothing**: The player i, who is dummy in the game v, obtains as much as he would get in case of not joining the coalition, i.e. $\psi_i(v) = v(\{i\})$.
- **Symmetry**: In case players i and j are interchangeable in the game v they get paid the same amount, i.e. $\psi_i(v) = \psi_j(v)$.
- **Additivity:** For every two games v and w it holds that $\psi_i(v + w) = \psi_i(v) + \psi_i(w)$, where $v + w$ is a game defined as $(v + w)(S) = v(S) + w(S)$ for $\forall S \subseteq N$.

Shapley value

There exists a unique function $\psi : \vartheta \to \mathbb{R}^n$ satisfying all four axioms. The ith component of the Shapley vector reffering to the payoff of player i is given as follows:

$$\psi_i(v) = \sum_{i \in S \subseteq N} \frac{(|S| - 1)!(n - |S|)!}{n!} \left[v(S) - v(S - \{i\}) \right].$$

Example

$$v(\overline{1}) = v(\overline{2}) = v(\overline{3}) = 0,$$
$$v(\overline{12}) = 1, v(\overline{13}) = 2, v(\overline{23}) = 3,$$
$$v(\overline{123}) = 4$$

- It is not necessary to include singleton coalitions, the contribution of each player to such a coalition is zero - it cancels out.
- Player 1 belongs to coalitions $\{1,2\}, \{1,3\}, \{1,2,3\}$.

$$\psi_1(v) = \frac{(2-1)!(3-2)!}{3!} \cdot 1 + \frac{(2-1)!(3-2)!}{3!} \cdot 2 + \\ + \frac{(3-1)!(3-3)!}{3!} \cdot 1 = \frac{5}{6}$$

Example

- Player 2 belongs to coalitions $\{1,2\},\{2,3\},\{1,2,3\}$.

$$\psi_2(v) = \frac{(2-1)!(3-2)!}{3!}\cdot 1 + \frac{(2-1)!(3-2)!}{3!}\cdot 3 + \\ + \frac{(3-1)!(3-3)!}{3!}\cdot 2 = \frac{8}{6}$$

- Player 3 belongs to coalitions $\{1,3\},\{2,3\},\{1,2,3\}$.

$$\psi_3(v) = \frac{(2-1)!(3-2)!}{3!}\cdot 2 + \frac{(2-1)!(3-2)!}{3!}\cdot 3 + \\ + \frac{(3-1)!(3-3)!}{3!}\cdot 3 = \frac{11}{6}$$

Thus Shapley vector is $\psi = \left(\frac{5}{6}, \frac{8}{6}, \frac{11}{6}\right)$. It can be observed that $\frac{5}{6}+\frac{8}{6}+\frac{11}{6} = 4 = v(N)$, therefore, this distribution is effective.

N-M stable set

- Johnu von Neumann and Oskar Morgenstern
- Coalition S dissatisfied with the the current division of $v(N)$ can credibly object by suggesting a stable division x of $v(N)$ that is better for all the members of S.
- For every outcome that is not stable some coalition has a credible objection.
- No coalition has a credible objection to any stable outcome.
- $\langle N, v \rangle$ is a coalitional game
- X is a set of all payoff distribution of this game
- Distributin x is *an objection* of the coalition S to the distribution y if x dominates for the coalition S y, $x \succ_S y$, i.e. $x_i > y_i$ for $\forall i \in S$ and $x(S) \leq v(S)$.
- The core of the game $\langle N, v \rangle$ is the set of all distributions to which there exists no objection, i.e. $C(v) = \{y \in X | \nexists S, x : x \succ_S y\}$.

N-M stable set

- A subset Y of the set of all distributions X of the game $\langle N, v\rangle$ is ***a stable set***, or ***N-M solution***, if it satisfies the condition of *internal* and *external stability*:
  - *Internal stability:* If $y \in Y$, then for no $z \in Y$ does there exist a coalition S for which $x \succ_S y$.
  - *External stability:* If $z \in X \setminus Y$, then there exists $y \in Y$ such that $y \succ_S z$ for some coalition S.

Nucleolus

- *The excess* is the key term:

$$e(S,x) = v(S) - \sum_{i \in S} x_i.$$

It is calculated for every coalition which gives us the whole vector of excesses $E(x)$ of a distribution x where $E_l(x) = e(S_l, x)$ for all $l = 1, 2, \ldots, 2^{|N|} - 1$, the components of which are sorted in non-decreasing order.

- ***The nucleolus*** of a game is then the set of efficient distributions for which the vector of excesses $E(x)$ is lexicographically minimal. Thus let $X = \left\{ x : \sum_{j=1}^{n} x_j = v(N) \right\}$ is the set of all effective allocations. We say that $\nu \in X$ is ***a nucleolus*** if for every $x \in X$ it holds that $O(\nu) \preceq_L O(x)$, where $\preceq_L$ is a lexicographical ordering.

Solving for the nucleolus using linear programming

Our aim is to find a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ that minimizes the maximum of the excesses $e(\mathbf{x}, S)$ over all coalitions S subject to $\sum_{i=1}^{n} x_i = v(N)$ which is, in fact, the problem of minimizing the largest excess (i.e. a linear function) subject to a system of linear constraints. Thus, it can be solved using tools of linear programming.

Solving the introduced optimization problem will not necessarily determine a unique solution. If the initial solution is not unique, you must fix the values of excesses that cannot be further lowered and then solve the new problem to minimize the maximum of the rest of the allocations. Continue in this manner until a unique optimal solution is obtained.

Example

Consider three IT firms A, B and C. New software, the cost of which is \$ 1,200,000, has been developed and launched. Each firm evaluated its utility from buying the software with respect to its cost and announced its willingness to pay. Firm A is willing to pay \$ 1,000,000 for the software, firm B \$ 800,000 and firm C \$ 500,000. The software is transferable, therefore, firms can split the cost by buying it together. Determine the *nucleolus* of the game.

Example - characteristic function

The value of each coalition is given by the total amount of savings. It is obvious that values of each singleton equals to 0 as none of the firms is willing to pay the whole amount of \$ 1,200,000 and buy the software on its own. (From now on all amounts will be expressed in hundreds of thousands dollars.) If firms A and B agree to buy it together, they, as a coalition, save $(10+8)-12=\$6$. Similarly, if firm A decides to buy the software together with firm C, their coalition saves \$3. For the coalition of firms B and C, the saving would be worth \$1 and in case all three firms decided to split the bill, they would save \$11. In particular,

$$v(\emptyset) = v(A) = v(B) = v(C) = 0,$$
$$v(AB) = 6, v(AC) = 3, v(BC) = 1,$$
$$v(ABC) = 11.$$

Example - the primal linear program

For this game, the primal linear program is as follows:
max α s.t.

$$
\begin{aligned}
x_1 + \alpha &\leq 0 \\
x_2 + \alpha &\leq 0 \\
x_3 + \alpha &\leq 0 \\
x_1 + x_2 + \alpha &\leq 6 \\
x_1 + x_3 + \alpha &\leq 3 \\
x_2 + x_3 + \alpha &\leq 1 \\
x_1 + x_2 + x_3 &= 11 \\
x_1, x_2, x_3 &\leq 0
\end{aligned}
$$

Example - the dual linear program

The corresponding dual linear program is:
max $0y_1 + 0y_2 + 0y_3 + 6y_{12} + 3y_{13} + y_{23} + 11a_1 - 11a_2 + 0S_1 + 0S_2 + 0S_3$ s.t.

$$\begin{aligned} y_1 + 0y_2 + 0y_3 + y_{12} + y_{13} + 0y_{23} + a_1 - a_2 + S_1 &= 0 \\ 0y_1 + y_2 + 0y_3 + y_{12} + 0y_{13} + y_{23} + a_1 - a_2 + S_2 &= 0 \\ 0y_1 + 0y_2 + y_3 + 0y_{12} + y_{13} + y_{23} + a_1 - a_2 + S_3 &= 0 \\ y_1 + y_2 + y_3 + y_{12} + y_{13} + y_{23} + 0a_1 - 0a_2 &= 1 \\ y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, S_1, S_2, S_3 &\leq 0 \end{aligned}$$

y_{123} is a free variable, however, all variables must be non-negative, therefore, y_{123} is expressed as a difference of two non-negative variables $a_1 - a_2, a_1, a_2 \geq 0$

Example - solution

Running the dual linear program we obtain its optimal solution. By the Duality Theorem the optimal solution for the primal LP exists and values of both objective functions are equal.

Plugging the acquired values in we get a concrete solution for x_1, however, only a limitation for x_2 and x_3. Therefore, we need to update the LP and solve it again.

The new initial feasible basis for the new LP has to be found. To do so we use the eight step procedure described in [3].

The particular values of the solution have not been found yet, I am currently working on it.

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