The Continuous Transition from Micro- to Macro-Mechanics

(*Die Naturwissenschaften*, 28, pp. 664-666, 1926)

BUILDING on ideas of de Broglie [1] and Einstein,[2] I have tried to show [3] that the usual differential equations of mechanics, which attempt to define the co-ordinates of a mechanical system as functions of the time, are no longer applicable for "small" systems; instead there must be introduced a certain *partial* differential equation, which defines a variable ψ ("wave function") as a function of the co-ordinates and the time. As in the differential equation of a vibrating string or of any other vibrating system, ψ is given as a superposition of pure time harmonic (*i.e.* "sinusoidal") vibrations, the frequencies of which agree exactly with the spectroscopic "term frequencies" of the micro-mechanical system. For example, in the case of the linear Planck oscillator [4] where the energy function is

$$\frac{m}{2}\left(\frac{dq}{dt}\right)^2 + 2\pi^2\nu_0^2 m q^2, \tag{1}$$

when we put, instead of the displacement q, the dimensionless variable

$$x = q \cdot 2\pi\sqrt{\frac{m\nu_0}{h}}, \tag{2}$$

we get ψ as the superposition of the following proper vibrations: [5]

$$\begin{cases} \psi_n = e^{-\frac{x^2}{2}} H_n(x) e^{2\pi i \nu_n t} \\ \left(\nu_n = \dfrac{2n+1}{2}\nu_0; \quad n = 0,\ 1,\ 2,\ 3\ \ldots\right). \end{cases} \tag{3}$$

The H_n's are the polynomials [6] named after Hermite. If they are

[1] L. de Broglie, *Ann. de Physique* (10), 3, p. 22, 1925 (Thèses, Paris, 1924).
[2] A. Einstein, *Berlin Ber.* 1925, p. 9 *et seq.*
[3] *Ann. d. Physik*; the essays here collected.
[4] *i.e.* a particle of mass m which, moving in a straight line, is attracted towards a fixed point in it, with a force proportional to its displacement q from this point; according to the usual mechanics, such a particle executes sine vibrations of frequency ν_0.
[5] i means $\sqrt{-1}$. On the right-hand side the real part is to be taken, as usual.
[6] Cf. Courant-Hilbert, *Methoden der mathematischen Physik*, I. chap. ii. § 10, 4, p. 76 (Berlin, Springer, 1924).

multiplied by $e^{-\frac{x^2}{2}}$ and the "normalising factor" $(2^n n!)^{-\frac{1}{2}}$ they are called Hermite's orthogonal functions. They represent therefore the amplitudes of the proper vibrations.

The first five are represented in Fig. 1. The similarity between this and the well-known picture of the vibrations of a string is very great.

At first sight it appears very strange to try to describe a process, which we previously regarded as belonging to particle mechanics, by a system of such proper vibrations. For this chosen simple case, I would like to demonstrate here *in concreto* the transition to macroscopic mechanics, by showing that a *group* of proper vibrations of *high* order-number n ("quantum number") and of relatively small order-number differences ("quantum number differences") may represent a "particle", which is executing the "motion", expected

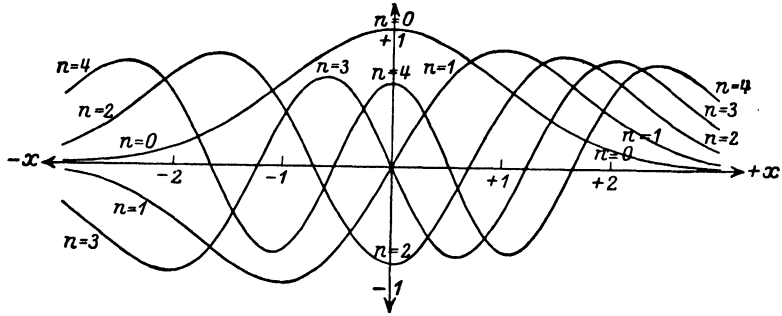

FIG. 1.—The first five proper vibrations of the Planck oscillator according to undulatory mechanics. Outside of the region $-3 \leq x \leq +3$ represented here, all five functions approach the x-axis in monotonic fashion.

from the usual mechanics, *i.e.* oscillating with the frequency ν_0. I choose a number $A \gg 1$ (*i.e.* great compared with 1) and form the following aggregate of proper vibrations:

$$\psi = \sum_{n=0}^{\infty} \left(\frac{A}{2}\right)^n \frac{\psi_n}{n!} = e^{\pi i \nu_0 t} \sum_{n=0}^{\infty} \left(\frac{A}{2} e^{2\pi i \nu_0 t}\right)^n \frac{1}{n!} e^{-\frac{x^2}{2}} H_n(x). \tag{4}$$

Thus the *normalised* proper vibrations (see above) are taken with the coefficients

$$\frac{A^n}{\sqrt{2^n n!}}, \tag{5}$$

and this, as is easily seen,[1] results in the singling out of a relatively small group in the neighbourhood of the n-value given by

$$n = \frac{A^2}{2}. \tag{6}$$

[1] $z^n/n!$ has, *as function of n, for large values of z*, a single extremely high and relatively very sharp maximum at $n = z$. By taking square roots and with $z = A^2/2$, we get the series of numbers (5).

The summation of the series (4) is made possible by the following identity [1] in x and s :

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} e^{-\frac{x^2}{2}} H_n(x) = e^{-s^2+2sx-\frac{x^2}{2}}. \tag{7}$$

Thus

$$\psi = e^{\pi i \nu_0 t - \frac{A^2}{4} e^{4\pi i \nu_0 t} + Axe^{2\pi i \nu_0 t} - \frac{x^2}{2}}. \tag{8}$$

Now we take, as is provided for, the real part of the right-hand side and after a short calculation obtain

$$\psi = e^{\frac{A^2}{4} - \frac{1}{2}(x - A\cos 2\pi\nu_0 t)^2} \cos\left[\pi\nu_0 t + (A \sin 2\pi\nu_0 t) \cdot \left(x - \frac{A}{2}\cos 2\pi\nu_0 t\right)\right]. \tag{9}$$

This is the *final result*, in which the first factor is our first interest. It represents a relatively tall and narrow " hump ", of the form of a " Gaussian error-curve ", which at a given moment lies in the neighbourhood of the position

$$x = A \cos 2\pi\nu_0 t. \tag{10}$$

The breadth of the hump is of the order of magnitude unity and therefore very small compared with A, by hypothesis. According to (10), the hump oscillates under exactly the same law as would operate in the usual mechanics for a particle having (1) as its energy function. The amplitude in terms of x is A, and thus in terms of q is

$$a = \frac{A}{2\pi}\sqrt{\frac{h}{m\nu_0}}. \tag{11}$$

Ordinary mechanics gives for the *energy* of a particle of mass m, which oscillates with this amplitude and with frequency ν_0,

$$2\pi^2 a^2 \nu_0^2 m = \frac{A^2}{2} h\nu_0, \tag{12}$$

i.e. from (6) exactly $nh\nu_0$, where n is the average quantum number of the selected group. The " correspondence " is thus complete in this respect also.

The *second* factor in (9) is in general a function whose absolute value is small compared with unity, and which varies very rapidly with x and also t. It ploughs many deep and narrow furrows in the profile of the first factor, and makes a *wave group* out of it, which is represented—schematically only—in Fig. 2. The x-scale of Fig. 2 is naturally much smaller than that of Fig. 1 ; Fig. 2 requires to be magnified five times before being directly compared with Fig. 1. A more exact consideration of the second factor of (9) discloses the following interesting details, which cannot be seen in Fig. 2, which only represents *one* stage. The *number* and *breadth* of the " furrows " or " wavelets " within the particle vary with the time. The wavelets are most *numerous* and *narrowest* when passing through the centre $x = 0$; they become *completely smoothed out* at the turning points $x = \pm A$, because

[1] Courant-Hilbert, *loc. cit.* eqn. (58).

there, by (10), $\cos 2\pi\nu_0 t = \pm 1$ and thus $\sin 2\pi\nu_0 t$ becomes equal to zero, so that the second factor of (9) is absolutely independent of x. The entire extension of the wave group (" density of the particle ") remains, however, always the same. The variability of the " corrugation " is to be conceived as depending on the *velocity*, and, as such, is completely intelligible from all general aspects of undulatory mechanics—but I do not wish to discuss this further at present.

Our wave group always remains compact, and does *not* spread out into larger regions as time goes on, as we were accustomed to make it do, for example, in optics. It is admitted that this does not mean much in one dimension, and that a hump on a string will behave quite similarly. But it is easily seen that, by multiplying together two or three expressions like (4), written in x, in y, and in z respectively, we can represent also the *plane* and the *spatial* oscillator respectively, *i.e.* a plane or spatial wave group which moves round a harmonic ellipse.[1] Also such a wave group will remain compact, in contrast, *e.g.*, to a

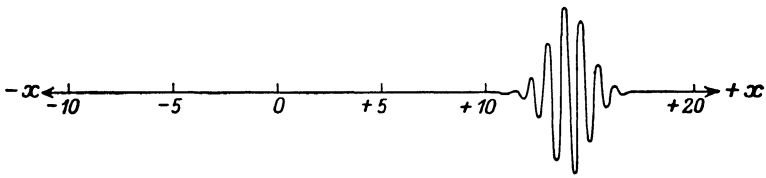

FIG. 2.—Oscillating wave group as the representation of a particle in wave mechanics.

wave packet in classical optics, which is dissipated in the course of time. The distinction may originate in the fact that our group is built up out of separate *discrete* harmonic components, and not out of a *continuum* of such.

I wish to mention, in conclusion, that a general additive constant, C, let us say, which should be added to all the ν_n's in (3), (and corresponds to the " rest-energy " of the particle) does not alter the essentials. It only affects the square bracket in (9), adding $2\pi Ct$ thereto. Hence the oscillations *within* the wave group become very much quicker with respect to the *time*, while the oscillation of the group as a whole, given by (10), and its " corrugation ", remain quite unaffected.

We can definitely foresee that, in a similar way, wave groups can be constructed which move round highly quantised Kepler ellipses and are the representation by wave mechanics of the hydrogen electron. But the technical difficulties in the calculation are greater than in the especially simple case which we have treated here.

[1] We may point out, in passing, the interesting fact that the quantum levels of the *plane* oscillator are *integral*, but for the *spatial* oscillator they again become " half-integral ". Similarly for the rotator. This half-integralness, which is spectroscopically so significant, is thus connected with the " oddness " of the number of the dimensions of space.