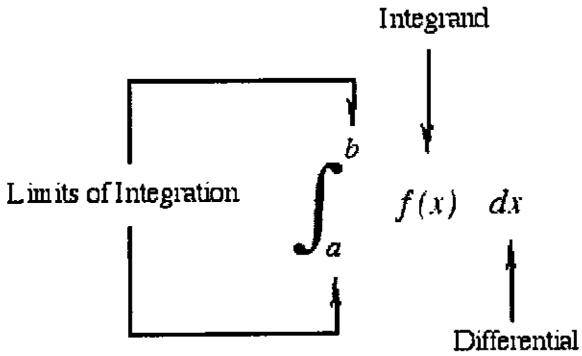
DEFINITE INTEGRALS

http://ocw.mit.edu/courses/mathematics/18-01-single-variable-calculus-fall-2006/video-lectures/embed18/



Listen to and watch the video and decide whether the statements are true or false.

a) Integration is the only part of calculus.

b) The professor starts with the geometric perspective because he will use it in the whole course.

c) There are many opinions about what an integral is.

d) The professor wants to find out how large the area above the curve is.

e) The x axis is normally the bottom.

f) x, a, and b are the limits of the function.

g) We don´t know where indefinite integrals start and end.

h) Rectangles in the example are not real rectangles.

i) The second step is to multiply the areas of rectangles.

j) The limit will be taken as the rectangles get infinitely small.

Transcript - Lecture 18

Listen again and try to correct wrong words.

PROFESSOR: So we're going on to the third unit here. So we're **going** started with Unit 3. And this is our intro to integration. It's basically the second half of calculus after differentiation. Today what I'll talk about is what are known as definite integrals. Actually, it looks like, are we missing a **bund** of overhead lights? Is there a reason for that? Hmm. Let's see. Ahh. Alright. OK, that's a little brighter now. Alright. So the idea of definite integrals can be **tackled** in a number of ways. But I will be consistent with the rest of the presentation in the course. We're going to start with the geometric point of view. And the geometric point of view is, the problem we want to solve us to find the area **below** a curve. The other point of view that one can take, and we'll mention that at the end of this lecture, is the idea of a **corresponding** sum. So keep that in mind that there's a lot going on here. And there are many different interpretations of what the integral is.

Now, so let's draw a picture here. I'll start at a place a and end at a place b. And I have some curve here. And what I have in mind is to find this area here. And, of course, in order to do that, I need more information than just where we start and where we end. I also need the bottom and the top. By **convection**, the bottom is the x axis and the top is the curve that we've specified, which is y = f(x). And we have a notation for this, which is the notation using calculus for this as opposed to some geometric notation. And that's the following **exercise**. It's called an integral, but now it's going to have what are **called** as limits on it. It will start at a and end at b. And we write in the function f(x) dx. So this is what's known as a definite integral. And it's interpreted geometrically as the area under the curve. The only difference between this **sample** of symbols and what we had before with indefinite integrals is that before we didn't specify where it started and where it ended.

Now, in order to understand what to do with this **thing**, I'm going to just describe very abstractly what we do. And then carry out one example in detail. So, to **calculate** this area, we're going to follow initially three steps. First of all, we're going to divide into rectangles. And unfortunately, because it's impossible to divide a curvy **space** into rectangles, we're going to cheat. So they're only quote-**nonquote** rectangles. They're almost rectangles. And the second thing we're going to do is to add up the areas. And the third thing we're going to do is to rectify this problem that we didn't actually **bit** the answer on the nose. That we were missing some pieces or were choosing some extra bits. And the way we'll rectify that is by taking the limit as the rectangles get thin. Infinitesimally thin, very thin.

UNIT 5

FOCUS A

RIEMANN SUMS

In the text, L^1 will refer to integration over the unit interval, and all functions that occur will be defined on the real line and will have period 1: $f(x) = f(x+1)$ for all x.

Some time in 1962, during one of the Department's weekly post-colloquium parties, we sat around on the floor of Marvin Knopp's apartment in University Houses, drinking, when Anatole Beck, who was teaching Lebesque integration, made the following

CONJECTURE. *The Riemann sums of f, namely, the averages*

$$(M_n f)(x) = \frac{1}{n}\sum_{k=1}^{n} f\left(x + \frac{k}{n}\right)$$

converge to $\int_0^1 f(t)dt$ *at almost every x.*

I objected immediately and loudly that this couldn't possibly be true. We argued for a while and then made a *big* bet: $ 20 to be spent on a party by the winner. If I produced a counter-example within a year, I won. If not, I lost.

Next morning I remembered the most elementary theorem on diophantine approximation: to every irrational x correspond infinitely many rational numbers $r = p/q$ (p and q are integers) such that $|x - r| < q^{-2}$. Using this, it is easy to see that if $1/2 < \alpha < 1$ and $f(x) = |x|^{-\alpha}$ on [−1/2, 1/2] then f is in L^1, but

$$\limsup_{n\to\infty} (M_n f)(x) = +\infty$$

at every irrational x.

So I had my counter-example. But Anatole insisted that he had only been talking about *bounded* functions. Since no one who had listened to us could or would remember exactly what was said, and since I wasn't quite sure myself, I accepted the challenge to find a bounded counter-example. Thinking about it off and on, it took me almost a year to produce one, but the effort was definitely worth while.

I knew the following theorem of Jessen which gives a positive result for certain subsequences:

If $\{n_i\}$ *is an increasing sequence of positive integers in which each* n_i *is a divisor of* n_{i+1} *then, for every f in* L^1,

$$\lim_{i\to\infty} (M_{n_i} f)(x) = \int_0^1 f(t)dt$$

at almost every x.

(When $n_i = a^i$ for some $a > 1$, this is a special case of Birkhoff's ergodic theorem. Jessen's theorem is the only gap theorem I know in which divisibility plays a role.) To win my bet I therefore had to avoid divisibility. Here is a simplified version of what I finally came up with.

THEOREM. *If* $\{p_i\}$ *is an increasing sequence of primes, and* $\in > 0$, *then there is a measurable f such that* $0 \leq f \leq 1$, $\int_0^1 f(t)dt < \in$, *but*

$$\limsup_{i \to \infty} (M_{p_i} f)(x) \geq \frac{1}{2}$$

at every x.

The proof actually produces such an *f* which is the characteristic function of an open set.

An interesting fact (a consequence of Dirichlet's theorem about primes in arithmetic progressions) is that there exist sequences $\{n_i\}$ which satisfy Jessen's hypothesis whereas the shifted sequences $\{1 + n_i\}$ consist entirely of primes. What matters here is thus not the rapidity with which a sequence $\{n_i\}$ tends to ∞, but rather its arithmetic structure. This unexpected delicate number-theoretic aspect of the behaviour of $\{M_n f\}$ pleased me much more than just winning the bet.

Notice:

- *phrasal verbs:* to think *about.* – to have ideas or opinions in one's mind
  to think *of* – to consider; to remember
  to come *up with* (sth.) – to find; to produce (a solution, an answer)
  Phrasal verbs are very common in informal style.
- Contractions (can't, isn't ...) are not used in formal specialised texts.
- (to think about sth.) *off and on* – sometimes, occasionally (*čes.* tu a tam)
- it is *worth while* – deserving attention, time and effort etc. (*čes.* stojí to za to)

Exercises

1. Is the text written in a formal or informal style? Why?

2. Complete the following sentences with the appropriate forms of the verbs from the box below:

see	make	pay	know	say	find	apply	forget	be	give

a) Subspace *M* of L^2 is .... to be translation invariant.
b) Elementary proofs may be .... in Chapter VII.
c) The result extends to L^2, as can be .... from Theorem 9.13.
d) An extremely short proofs by P. J. Cohen is also ....
e) The functional analysis proof is ...... to many analysts and has probably .... independently discovered several times in recent years.
f) Attention is ..... to the closeness of the approximation.
g) Lebesgue's theory of integration is ...... to problems about holomorphic functions.
h) Part (b) was proved by Lebesgue but seems to have been ....
i) On earlier attempts that were ..... towards the construction of a satisfactory theory of integration ....

3. Put the following into the passive, mentioning the agent where necessary:

a) We use this room only on special occasions.
b) Someone switched on a light and opened the door.

c) The librarian said that they were starting a new system because students were not returning books.
d) People must not leave their luggage in the hall.
e) Members may keep books for three weeks. After that they must return them.
f) Someone has already told him to report for duty at six.
g) No one can do anything unless someone gives us more information.
h) The organisers will exhibit the documents till the end of the month.
i) Who wrote the article?
j) He expected us to solve the problem.
k) They showed her the easiest way to do it.
l) The author has written a special edition for mathematicians.
m) Did the idea interest you?
n) Students are doing a lot of the work.

4. ***Read the texts and put the verb in brackets into the active or passive. Use the tense that is shown at the beginning of each passage:***

a) Present Progressive

Chaos on M25 set to continue

The chaos on London's orbital motorway, the M25, looks set to continue for a few weeks. Important repairs (1)........................... (carry out) between junctions 18 and 19 at the moment, and traffic (2)...................... (divert) along other routes. At the M40 intersection, a bridge (3) .... (widen) and this (4) .... (cause) long delays and tailbacks of up to 5 miles. Motorists (5) .... (advise) to avoid the area if at all possible.

b) Present Perfect

Forest fires sweep across France

In the last few days, uncontrollable forest fires (1) .... (sweep) across the South of France. Thousands of square miles of trees (2) .... (burn) and hundreds of homes (3) .... (destroy). So far no one (4) .... (kill) but scores of people (5) .... (injure) trying to fight the blaze. The fires are close to many of the popular holiday resorts and tourists (6) .... (warn) not to travel to the affected areas. A number of tour operators in England (7) .... (cancel) holidays to the region and alternative destinations (8) .... (arrange).

c) Past Progressive

Bomb suspect arrested

Patrick Mallory, who escaped from prison last year while he (1) .... (transfer) from Belfast, was arrested last night as he (2) .... (try) to leave the country. A police spokesman said he (3) .... (hold) in Horseferry Road police station, and (4) .... (question) by detectives from the Special Branch. The spokesman said that Mallory's home (5) .... (search) and that two other men, who had been staying with Mallory at the time of the arrest, (6) .... (help) police with their enquiries.

5. ***Study these definitions. How do the prefixes change the meaning of a word? What other prefixes with the same meaning do you know? With what kind of words can we use these prefixes?***

expected	supposed	**unexpected**	not supposed
accurate	careful, exact	**inaccurate**	not careful or exact
probable	likely to happen	**improbable**	not likely to happen

Give the opposites of the words in the table below:

VERB	OPPOSITE	ADJECTIVE	OPPOSITE
like	*dislike*	able	*unable*
connect	....	capable	....
dress	....	conscious	....
cover	....	correct	....
believe	....	legal	....
please	....	personal	....
agree	....	polite	....
understand	....	proper	....
dependent	....	regular	....

6. *Read out the following:*

a) $\int \frac{\cos x \, \mathrm{d}x}{1+\sin^2 x} = \arctan \sin x + c$

b) $\int \tan x \, \mathrm{d}x = -\ln |\cos x| + c$

c) $\int_a^b \sin x \, \mathrm{d}x = \cos a - \cos b$

d) $\int \frac{\mathrm{d}x}{(x^2+a^2)} = \frac{1}{2a^2} \cdot \frac{x}{x^2+a^2} + \frac{1}{2a^3} \arctan \frac{x}{a} + c$

e) $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}} \mathrm{d}x = \frac{3}{7} t^4 (4t^3 - 7) + c$

f) $\int_0^\pi \frac{x \sin x}{1+\cos^2 x} \mathrm{d}x = \frac{\pi^2}{4}$

Riemann Sums Qs.

Adapted from Rozšiřující materiály pro výuku anglického jazyka, Křepinská, Houšková, Bubeníková, Matfyzpres 2006

Answer these questions after reading the text.

1) What where the colleagues betting on and what under which conditions?
2) What is diophantine approximation?
3) Why was Anatole not satisfied with the proposed solution?
4) Why is Jessen's theorem unique?
5) What does *ergodic* mean?
6) What was the most important result for the author?