LA - WEEK 6 EUCLIDEAN GEOMETRY

Let $\mathcal{U}$ (usually $\mathbb{R}^n$) be a real vector space. Elements of $\mathcal{U}$ will be called sometimes points.

Affine subspace of $\mathcal{U}$ is a nonemty subset of $\mathcal{U}$ of the form

$$\mathcal{M} = M + V$$

V is called the direction of $\mathcal{M}$ and denoted $Z(\mathcal{M})$

where $M \in \mathcal{U}$ is a point and $V \subseteq \mathcal{U}$ is a vector subspace of $\mathcal{U}$. $\dim \mathcal{M} = \dim V$.

Example 1 $\mathcal{U} = \mathbb{R}^4$

$$\rho : [0,1,2,3] + a(1,1,0,2) + b(2,3,-1,4)$$

is so called parametric description of a 2-dim. affine subspace (plane) given by the point $M = [0,1,2,3]$ and the vector subspace

$$Z(\mathcal{M}) = V = \{a(1,1,0,2) + b(2,3,-1,4) \in \mathbb{R}^4;\ a, b \in \mathbb{R}\}$$

Example 2 $\mathcal{U} = \mathbb{R}^4$. The set of all solutions of the system of linear equations

$$4x_1 - 2x_2 + x_3 - x_4 = 5$$
$$x_1 - x_2 + x_3 + 2x_4 = 3$$

is also a 2-dim affine subspace $\pi = M + V$ where M is a one solution of the system and vector subspace V is the set of the solutions of the homogeneous system

$Z(\mathcal{M}) = V$:

$$4x_1 - 2x_2 + x_3 - x_4 = 0$$
$$x_1 - x_2 + x_3 + 2x_4 = 0$$

Let $\mathcal{U}$ be a real vector space with a scalar product $\langle \, , \, \rangle : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$.

The <u>distance of two points</u> $A, B \in \mathcal{U}$ is the norm of the vector $B - A$.

$$\text{dist}\,(A, B) = \| B - A \| = \sqrt{\langle A-B, A-B \rangle}.$$

The <u>distance of a point A and an affine space</u> $\mathcal{N}$ is defined as

$$\text{dist}\,(A, \mathcal{N}) = \min \{ \text{dist}\,(A, Y) \, ; \, Y \in \mathcal{N} \}$$

We compute this distance using an orthogonal projections.

<u>Theorem 1</u>

(a) The distance of a point A from an affine subspace $\mathcal{N} = B + Z(\mathcal{N})$ is equal to the norm of the orthogonal projection of the vector $A - B$ to the space $Z(\mathcal{N})^{\perp}$:

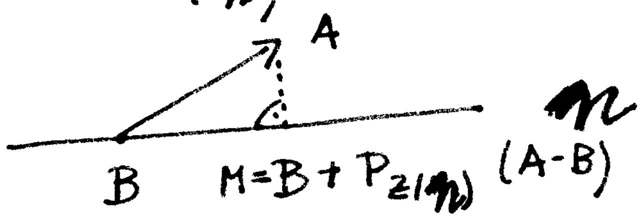
$$\text{dist}(A, B) = \| P_{Z(\mathcal{N})^{\perp}} (A - B) \|$$

(b) The following assertions are equivalent:

(1) $\text{dist}\,(A, \mathcal{N}) = \| A - M \|$ for a point $M \in \mathcal{N}$.

(2) $A - M \perp Z(\mathcal{N})$

(3) $M = B + P_{Z(\mathcal{N})} (A - B)$



Proof of (a) Let $X \in \mathcal{N}$ be arbitrary. Then $X = B + v$, where $v \in Z(\mathcal{N})$. It holds

$$\|A - X\|^2 = \|A - B - v\|^2 = \|\underbrace{(A-B) - P_{Z(\mathcal{N})}(A-B)}_{\substack{P_{Z(\mathcal{N})^\perp}(A-B) \\ \in Z(\mathcal{N})^\perp}} + \underbrace{P_{Z(\mathcal{N})}(A-B) - v}_{\in Z(\mathcal{N})}\|^2$$

$$= \|P_{Z(\mathcal{N})^\perp}(A-B)\|^2 + \|P_{Z(\mathcal{N})}(A-B) - v\|^2$$

$$\geq \|P_{Z(\mathcal{N})^\perp}(A-B)\|^2$$

The equality occurs just for $v = P_{Z(\mathcal{N})}(A-B)$.

Hence

$$\operatorname{dist}(A, \mathcal{N}) = \min_{X \in \mathcal{N}} \|A - X\| = \|P_{Z(\mathcal{N})^\perp}(A-B)\|$$

Proof of (b) omitted. $\square$

The <u>distance of two affine spaces</u> $\mathcal{M}$ and $\mathcal{N}$ is

$$\operatorname{dist}(\mathcal{M}, \mathcal{N}) = \min \{ \operatorname{dist}(X, Y);\ X \in \mathcal{M}, Y \in \mathcal{N} \}$$

It holds: If $\mathcal{M} = A + Z(\mathcal{M})$, $\mathcal{N} = B + Z(\mathcal{N})$, then

$$\boxed{\operatorname{dist}(\mathcal{M}, \mathcal{N}) = \operatorname{dist}(A, B + Z(\mathcal{N}) + Z(\mathcal{M}))}.$$

$$\operatorname{dist}(\mathcal{M}, \mathcal{N}) = \min \{ \operatorname{dist}(A + u, B + v),\ u \in Z(\mathcal{M}), v \in Z(\mathcal{N}) \}$$

$$= \min \{ \|A + u - B - v\|,\ u \in Z(\mathcal{M}), v \in Z(\mathcal{N}) \} =$$

$$= \min \{ \|A - (B + v - u)\|,\ v \in Z(\mathcal{N}), u \in Z(\mathcal{M}) \}$$

$$= \operatorname{dist}(A, B + Z(\mathcal{N}) + Z(\mathcal{M}))$$

Hence we get

THEOREM 2 (a) The distance between $\mathcal{M} = A + Z(\mathcal{M})$ and $\mathcal{N} = B + Z(\mathcal{N})$ is the norm of the orthogonal projection of the vector $A - B$ into $\left(Z(\mathcal{N}) + Z(\mathcal{M})\right)^{\perp}$.

(b) The following acsertions for points $M \in \mathcal{M}$ and $N \in \mathcal{N}$ are equivalent:

(1) $\operatorname{dist}(\mathcal{M}, \mathcal{N}) = \| M - N \|$

(2) $M - N \perp Z(\mathcal{M}) + Z(\mathcal{N})$

(3) $M - N = P_{(Z(\mathcal{M}) + Z(\mathcal{N}))^{\perp}} (A - B)$

Example 3 For $U = \mathbb{R}^4$ compute the distance of a point $A = (x_1, x_2, x_3, x_4)$ from a hyperplane $\mathcal{N} = \{ y \in \mathbb{R}^4, \ a y_1 + b y_2 + c y_3 + d y_4 + e = 0 \}$ where $d \neq 0$.

Solution: $\operatorname{dist}(A, \mathcal{N}) = \| P_{Z(\mathcal{N})^{\perp}} (A - B) \|$ for a point $B \in \mathcal{N}$. Let us choose $B = (0, 0, 0, -\frac{e}{d})$

$Z(\mathcal{N})$: $a y_1 + b y_2 + c y_3 + d y_4 = 0$

$Z(\mathcal{N})^{\perp} = [(a, b, c, d)]$

We compute the orthogonal projection P of the vector $A - B = (x_1, x_2, x_3, x_4 + \frac{e}{d})$ into $Z(\mathcal{N})^{\perp}$

$P(A - B) = \alpha \cdot (a, b, c, d) = \alpha \cdot u$

$$(A-B) - P(A-B) \perp (a,b,c,d) = n$$

$$\langle A-B, n\rangle - \alpha \langle n,n\rangle = 0$$

$$\alpha = \frac{\langle A-B, n\rangle}{\langle n,n\rangle} = \frac{a x_1 + b x_2 + c x_3 + d x_4 + e}{a^2+b^2+c^2+d^2}$$

$$\mathrm{dist}\,(A,\varkappa) = \|\alpha n\| = \ldots\ldots = \frac{|a x_1 + b x_2 + c x_3 + d x_4 + e|}{\sqrt{a^2+b^2+c^2+d^2}}$$

Example 4 In $\mathbb{R}^4$ compute the distance between the line p : $(5,4,4,5) + r(0,0,1,-4)$
and the plane ρ : $(4,1,1,0) + t(1,-1,0,0) + s(2,0,-1,0)$
and find point $M \in p$ and $N \in \rho$ such that

$$\mathrm{disp}(p,\rho) = \|M-N\|.$$

(Result : $\mathrm{dist} = 5$, $(Z(\varkappa) + Z(\rho))^{\perp} = [(2,2,4,1)]$
$M = (5,4,5,1)$, $N = (3,2,1,0)$)

Homework 6 Compute the distance of two planes in $\mathbb{R}^4$

$$\sigma : (4,5,3,2) + t(1,2,2,2) + s(2,0,2,1)$$
$$\tau : (1,-2,1,-3) + r(2,-2,1,2) + p(1,-2,0,1)$$

and find $M \in \sigma$, $N \in \tau$ such that

$$\mathrm{dist}\,(\sigma,\tau) = \|M-N\|.$$

(M and N need not be determined by this property uniquely !)

Angles between affine subspaces

① The angle between two nonzero vectors u and v is $\sphericalangle(u,v) = \alpha \in [0,\pi]$ such that

$$\cos\alpha = \frac{\langle u,v\rangle}{\|u\|\,\|v\|} \in [-1,1]$$

② The angle between two lines $[u]$ and $[v]$, where $u \neq \vec{0}$, $v \neq \vec{0}$ is $\sphericalangle([u],[v]) = \alpha \in [0,\frac{\pi}{2}]$ such that

$$\cos\alpha = \frac{|\langle u,v\rangle|}{\|u\|\,\|v\|} \in [0,1].$$

③ The angle between two vector subspaces V and W such that $V \cap W = \{\vec{0}\}$ is

$$\sphericalangle(V,W) = \min\{\sphericalangle([v],[w]),\ v\in V\setminus\{o\},\ w\in W\setminus\{o\}\}$$

If $V = \{\vec{o}\}$ or $W = \{\vec{o}\}$ then

$$\sphericalangle(V,W) = 0$$

④ The angle between two vector subspaces V and W such that $V \cap W \neq \{\vec{o}\}$ is

$$\sphericalangle(V,W) = \sphericalangle\left(V\cap(V\cap W)^{\perp},\ W\cap(V\cap W)^{\perp}\right)$$

Example for the case ④: Consider two planes in $\mathbb{R}^3$ with intersection a line.

⑤ The angle between two affine subspaces $\mathcal{M}$ and $\mathcal{N}$ is

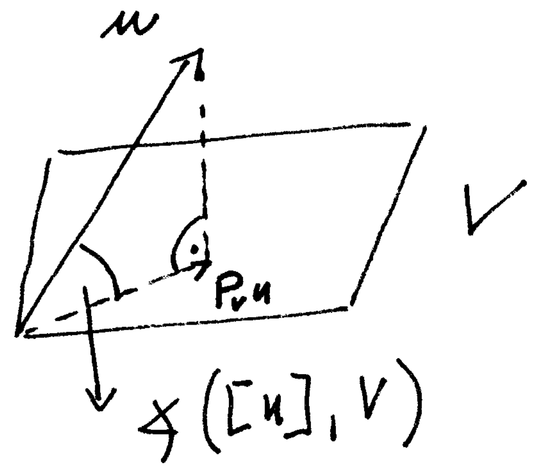
$$\sphericalangle(\mathcal{M},\mathcal{N}) = \sphericalangle(Z(\mathcal{M}), Z(\mathcal{N}))$$

Theorem 3 Let $u \in \mathcal{U} \setminus \{\vec{o}\}$ and V is a subspace of $\mathcal{U}$. Then

$$\cos \sphericalangle ([u], V) = \frac{\| P_V u \|}{\| u \|}$$

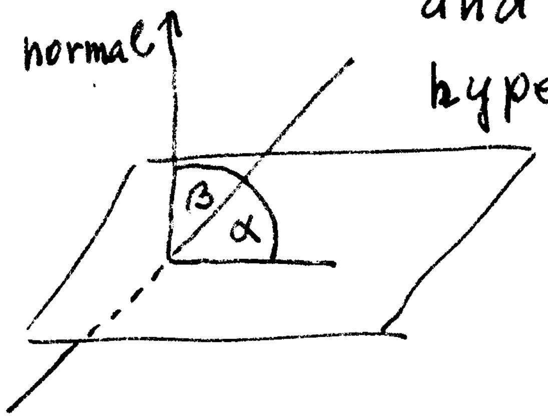
where P_V is ~~an~~ the orthogonal projection on the vector subspace V.

Proof by a picture



Theorem 4 The angle between a line and a hyperplane (hyperplane is an affine subspace of dim $n-1$ in the space of dim n) is

$\frac{\pi}{2}$ - the angle betwee the line and the normal line to the hyperplane



$$\alpha = \frac{\pi}{2} - \beta$$

Example $U = \mathbb{R}^4$ with an orthonormal basis e_1, e_2, e_3, e_4.

$V = [e_1+e_2, e_1+e_2+e_3]$, $W = [e_2+e_4, e_2+e_3+e_4]$

$V \cap W = [e_3]$

$$\measuredangle(V, W) = \measuredangle(V \cap [e_3]^\perp, W \cap [e_3]^\perp) =$$

$$= \measuredangle([e_1+e_2], [e_2+e_4]) = \alpha$$

$$\cos\alpha = \frac{|\langle e_1+e_2, e_2+e_4\rangle|}{\|e_1+e_2\| \, \|e_2+e_4\|} = \frac{1}{2} \qquad \alpha = \frac{\pi}{3}$$

Example Compute the angle between the line $p : (1, 2, 3, 4) + t(-3, 15, 1, -5)$ and the plane $\rho : r(1, -5, -2, 10) + s(1, 8, -2, -16)$

Homework 7 Compute the angle between

$\sigma : t(1, 1, 1, 1) + s(1, -1, 1, -1)$ and

$\tau : r(2, 2, 1, 0) + p(1, -2, 2, 0)$