# Řešitelský seminář, 21. 2. 2017

**Problem 1.** *Let* $A$ *and* $B$ *be* $2 \times 2$ *real matrices such that* $AB = A^2B^2 - (AB)^2$ *and* $\det(B) = 2$*. Evaluate* $\det(A + 2B) - \det(B + 2A)$.

**Problem 2.** *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is a nonconstant function such that* $f(x) \le f(y)$ *whenever* $x \le y$*. Prove that there exist* $a \in \mathbb{R}$ *and* $c > 0$ *such that* $f(a + x) - f(a - x) \ge cx$ *for all* $x \in [0, 1]$.

**Problem 3.** *Prove that for all* $x > 0$, $\sin x > x - \frac{x^3}{6}$.

**Problem 4.** *Which number is larger,* $\pi^3$ *or* $3^\pi$*?*

**Problem 5.** *Let* $A$ *be a linear transformation on* $\mathbb{R}^3$ *whose matrix (relative to the usual basis for* $\mathbb{R}^3$*) is both symmetric and orthogonal. Prove that* $A$ *is either plus or minus the identity, or a rotation by 180° about some axis, or a reflection about some two-dimensional subspace of* $\mathbb{R}^3$.

**Problem 6.** *Let* $n \ge 2$ *be an integer and let* $(K, +, \cdot)$ *be a commutative field with the property:*

$$\underbrace{1 + \cdots + 1}_{m \text{ times}} \ne 0, m = 2, \ldots, n.$$

*Consider a polynomial* $f \in K[x]$ *of degree* $n$ *and* $G$ *a subgroup of the aditive group* $(K, +)$, $G \ne K$*. Prove that there exists* $a \in K$*, such that* $f(a) \notin G$.