Řešitelský seminář, 11. 4. 2017

Problem 1. *Let $f(x)$ be a polynomial of degree at most n such that*

$$f(k) = \frac{n+1-k}{k+1}$$

for $k = 0, 1, \dots, n$. Find $f(n+1)$.

Problem 2. *Let $a > 0$ be a real number. Find the value of the following integral*

$$\int_{-a}^{a} \frac{\cos t}{a^{\frac{1}{t}} + 1} dt$$

Problem 3. *For a non-negative integer n, let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(n)$ is 1000 in binary, thus $f(23) = 8$.*

1. *Compute*

   $$\sum_{j=1}^{n} f^j(j),$$

   where $f^n(k)$ means function f applied n-times on k.

2. *Show*

   $$\sum_{k=1}^{n} \leq \frac{n^2}{4}.$$

   When does the equality hold?

Problem 4.

$$\sum_{1 \leq i < j \leq n} \frac{|S_i \cap S_j|}{|S_i||S_j|} < 1.$$

Prove that there exits pairwise distinct elements $x_1, \dots x_n$ such that x_i is a member of S_i for each index i.

Domácí úloha

Problem 5. *Let $n \geq 2$ and A_1, A_2,$\dots A_{n+1}$ be $n+1$ points in the n-dimensional Euclidean space, not lying on the same hyperplane, and let B be a point strictly inside the convex hull of A_1, $A_2, \dots A_{n+1}$. Prove that $|\angle A_i B A_1| > 90°$.*