INTRODUCTION TO ALGEBRAIC TOPOLOGY

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2. CW-complexes

2.1. Constructive definition of CW-complexes. *CW-complexes* are all the spaces which can be obtained by the following construction:

(1) We start with a discrete space X^0. Single points of X^0 are called 0-dimensional cells.
(2) Suppose that we have already constructed X^{n-1}. For every element α of an index set J_n take a map $f_\alpha : S^{n-1} = \partial D^n_\alpha \to X^{n-1}$ and put
$$X^n = \bigcup_\alpha \left(X^{n-1} \cup_{f_\alpha} D^n_\alpha\right).$$
Interiors of discs D^n_α are called n-dimensional cells and denoted by e^n_α.
(3) We can stop our construction for some n and put $X = X^n$ or we can proceed with n to infinity and put
$$X = \bigcup_{n=0}^{\infty} X^n.$$
In the latter case X is equipped with inductive topology which means that $A \subseteq X$ is closed (open) iff $A \cap X^n$ is closed (open) in X^n for every n.

Example A. The sphere S^n is a CW-complex with one cell e^0 in dimension 0, one cell e^n in dimension n and the constant attaching map $f : S^{n-1} \to e^0$.

Example B. The real projective space $\mathbb{RP}^n$ is the space of 1-dimensional linear subspaces in $\mathbb{R}^{n+1}$. It is homeomorhic to
$$S^n/(v \simeq -v) \cong D^n/(w \simeq -w), \quad \text{for } w \in \partial D^n = S^{n-1}.$$
However, $S^{n-1}/(w \simeq -w) \cong \mathbb{RP}^{n-1}$. So $\mathbb{RP}^n$ arises from $\mathbb{RP}^{n-1}$ by attaching one n-dimensional cell using the projection $f : S^{n-1} \to \mathbb{RP}^{n-1}$. Hence $\mathbb{RP}^n$ is a CW-complex with one cell in every dimension from 0 to n.

We define $\mathbb{RP}^\infty = \bigcup_{n=1}^{\infty} \mathbb{RP}^n$. It is again a CW-complex.

Example C. The complex projective space $\mathbb{CP}^n$ is the space of complex 1-dimensional linear subspaces in $\mathbb{C}^{n+1}$. It is homeomorhic to
$$\begin{aligned} S^{2n+1}/(v \simeq \lambda v) &\cong \{(w, \sqrt{1-|w|^2}) \in \mathbb{C}^{n+1};\ \|w\| \le 1\}/((w,0) \simeq \lambda(w,0),\ \|w\| = 1) \\ &\cong D^{2n}/(w \simeq \lambda w;\ w \in \partial D^{2n}) \end{aligned}$$
for all $\lambda \in \mathbb{C}$, $|\lambda| = 1$. However, $\partial D^{2n}/(w \simeq \lambda w) \cong \mathbb{CP}^{n-1}$. So $\mathbb{CP}^n$ arises from $\mathbb{CP}^{n-1}$ by attaching one $2n$-dimensional cell using the projection $f : S^{2n-1} = \partial D^{2n} \to \mathbb{CP}^{n-1}$. Hence $\mathbb{CP}^n$ is a CW-complex with one cell in every even dimension from 0 to $2n$.

Define $\mathbb{CP}^\infty = \bigcup_{n=1}^\infty \mathbb{CP}^n$. It is again a CW-complex.

2.2. Another definition of CW-complexes. Sometimes it is advantageous to be able to describe CW-complexes by their properties. We carry it out in this paragraph. Then we show that the both definitions of CW-complexes are equivalent.

Definition. A *cell complex* is a Hausdorff topological space X such that

(1) X as a set is a disjoint union of cells e_α
$$X = \bigcup_{\alpha\in J} e_\alpha.$$
(2) For every cell e_α there is a number, called dimension.
$$X^n = \bigcup_{\dim e_\alpha \le n} e_\alpha$$
is the n-skeleton of X.

(3) Cells of dimension 0 are points. For every cell of dimension ≥ 1 there is a characteristic map
$$\varphi_\alpha : (D^n, S^{n-1}) \to (X, X^{n-1})$$
which is a homeomorphism of $\operatorname{int} D^n$ onto e_α.

The *cell subcomplex* Y of a cell complex X is a union $Y = \bigcup_{\alpha\in K} e_\alpha$, $K \subseteq J$, which is a cell complex with the same characterictic maps as the complex X.

A *CW-complex* is a cell complex satisfying the following conditions:

(C) Closure finite property. The closure of every cell belongs to a finite subcomplex, i. e. subcomplex consisting only from a finite number of cells.

(W) Weak topology property. F is closed in X if and only if $F \cap \bar{e}_\alpha$ is closed for every α.

Example. Examples of cell complexes which are not CW-complexes:

(1) S^2 where every point is 0-cell. It does not satisfy property (W).
(2) D^3 with cells $e^3 = \operatorname{int} B^3$, $e^0_x = \{x\}$ for all $x \in S^2$. It does not satisfy (C).
(3) $X = \{1/n;\ n \geq 1\} \cup \{0\} \subset \mathbb{R}$. It does not satisfy (W).
(4) $X = \bigcup_{n=1}^\infty \{x \in \mathbb{R}^2;\ \|x - (1/n, 0)\| = 1/n\} \subset \mathbb{R}^2$. If it were a CW-complex, the set $\{(1/n, 0) \in \mathbb{R}^2;\ n \geq 1\}$ would be closed in X, and consequently in $\mathbb{R}^2$.

2.3. Equivalence of definitions.

Proposition. *The definitions 2.1 and 2.2 of CW-complexes are equivalent.*

Proof. We will show that a space X constructed according to 2.1 satisfies definition 2.2. The proof in the opposite direction is left as an exercise to the reader.

The cells of dimension 0 are points of X^0. The cells of dimension n are interiors of discs D^n_α attached to X^{n-1} with charakteristic maps
$$\varphi_\alpha : (D^n_\alpha, S^{n-1}_\alpha) \to (X^{n-1} \cup_{f_\alpha} D^n_\alpha, X^{n-1})$$

induced by identity on D^n_α. So X is a cell complex. From the construction 2.1 it follows that X satisfies property (W). It remains to prove property (C). We will carry it out by induction.

Let $n = 0$. Then $\overline{e^0_\alpha} = e^0_\alpha$.

Let (C) holds for all cells of dimension $\leq n-1$. $\overline{e^n_\alpha}$ is a compact set (since it is an image of D^n_α). Its boundary ∂e^n_α is compact in X^{n-1}. Consider the set of indices

$$K = \{\beta \in J;\ \partial e^n_\alpha \cap e_\beta \neq \emptyset\}.$$

If we show that K is finite, from the inductive assumption we get that $\bar{e}^n_\alpha$ lies in a finite subcomplex which is a union of finite subcomplexes for $\bar{e}_\beta$, $\beta \in K$.

Choosing one point from every intersection $\partial e^n_\alpha \cap e_\beta$, $\beta \in K$ we form a set A. A is closed since any intersection with a cell is empty or a onepoint set. Simultaneously, it is open, since every its element a forms an open subset (for $A - \{a\}$ is closed). So A is a discrete subset in the compact set ∂e^n_α, consequently, it is finite. □

2.4. Compact sets in CW complexes.

Lemma. *Let X be a CW-complex. Then any compact set $A \subseteq X$ lies in a finite subcomplex, particularly, there is n such that $A \subseteq X^n$.*

Proof. Consider the set of indices

$$K = \{\beta \in J;\ A \cap e_\beta \neq \emptyset\}.$$

Similarly as in 2.3 we will show that K is a finite set. Then $A \subseteq \bigcup_{\beta \in K} \bar{e}_\beta$ and every $\bar{e}_\beta$ lies in a finite subcomplexes. Hence A itself is a subset of a finite subcomplex. □

2.5. Cellular maps. Let X and Y be CW-complexes. A map $f : X \to Y$ is called a *cellular map* if $f(X^n) \subseteq Y^n$ for all n. In Section 5 we will prove that every map $g : X \to Y$ is homotopic to a cellular map $f : X \to Y$. If moreover, g restricted to a subcomplex $A \subset X$ is already cellular, f can be chosen in such a way that $f = g$ on A.

2.6. Spaces homotopy equivalent to CW-complexes. One can show that every open subset of $\mathbb{R}^n$ is a CW-complex. In [Hatcher], Theorem A.11, it is proved that every retract of a CW-complex is homotopy equivalent to a CW-complex. These two facts imply that every compact manifold with or without boundary is homotopy equivalent to a CW-complex. (See [Hatcher], Corollary A.12.)

2.7. CW complexes and HEP. The most important result of this section is the following theorem:

Theorem. *Let A be a subcomplex of a CW-complex X. Then the pair (X, A) has the homotopy extension property.*

Proof. According to the last theorem in Section 1 it is sufficient to prove that $X \times \{0\} \cup A \times I$ is a retract of $X \times I$. We will prove that it is even a deformation retract. There is a retraction $r_n : D^n \times I \to D^n \times \{0\} \cup S^{n-1} \times I$. (See Section 1.) Then $h_n : D^n \times I \times I \to D^n \times I$ defined by

$$h_n(x, s, t) = (1 - t)(x, s) + tr_n(x, s)$$

is a deformation retraction, i.e. a homotopy between id and r_n.

Put $Y^{-1} = A$, $Y^n = X^n \cup A$. Using h_n we can define a deformation retraction $H_n : Y^n \times I \times I \to Y^n \times I$ for the retract $Y^n \times \{0\} \cup Y^{n-1} \times I$ of $Y^n \times I$. Now define the deformation retraction $H : X \times I \times I \to X \times I$ for the retract $X \times \{0\} \cup A \times I$ succesively on the subspaces $X \times \{0\} \times I \cup Y^n \times I \times I$ with values in $X \times \{0\} \cup Y^n \times I$. For $n = 0$ put

$$H(x, s, t) = (x, s) \qquad \text{for } (x, s) \in X \times \{0\} \text{ or } t \in [0, 1/2],$$
$$H(x, s, t) = H_0(x, s, 2(t - 1/2)) \quad \text{for } x \in Y^0 \text{ and } t \in [1/2, 1].$$

Suppose that we have already defined H on $X \times \{0\} \cup Y^{n-1} \times I$. On $X \times \{0\} \cup Y^n \times I$ we put

$$H(x, s, t) = (x, s) \qquad \text{for } (x, s) \in X \times \{0\} \text{ or } t \in [0, 1/2^{n+1}],$$
$$H(x, s, t) = H_n(x, s, 2^{n+1}(t - 1/2^{n+1})) \quad \text{for } x \in Y^n \text{ and } t \in [1/2^{n+1}, 1/2^n],$$
$$H(x, s, t) = H(H(x, s, 1/2^n), t) \quad \text{for } x \in Y^n \text{ and } t \in [1/2^n, 1].$$

$H : X \times I \times I \to X \times I$ is continuous since so are its restrictions on $X \times \{0\} \times I \cup Y^n \times I \times I$ and the space $X \times I \times I$ is a direct limit of the subspaces $X \times \{0\} \times I \cup Y^n \times I \times I$.

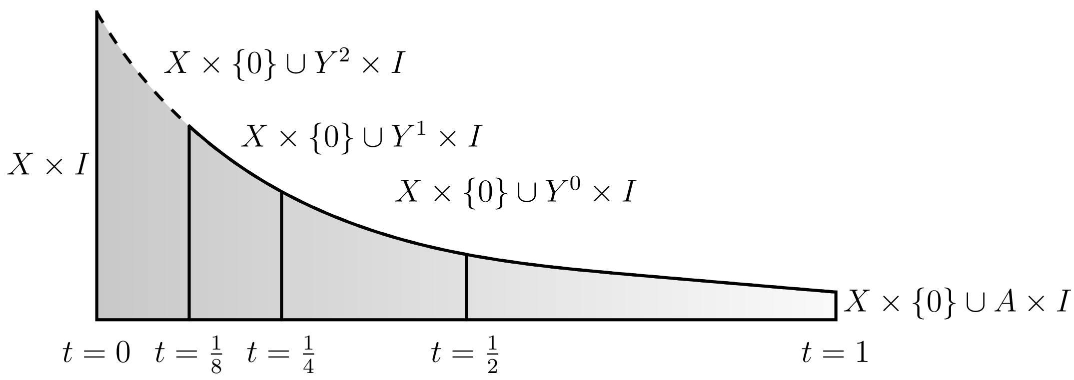

FIGURE 2.1. Image of H depending on t

□

2.8. First criterion for homotopy equivalence.

Proposition. *Suppose that a pair (X, A) has the homotopy extension property and that A is contractible (in A). Then the canonical projection $q : X \to X/A$ is a homotopy equivalence.*

Proof. Since A is contractible, there is a homotopy $h : A \times I \to A$ between id_A and constant map. This homotopy together with $\mathrm{id}_X : X \to X$ can be extended to a homotopy $f : X \times I \to X$. Since $f(A, t) \subseteq A$ for all $t \in I$, there is a homotopy $\tilde{f} : X/A \times I \to X/A$ such that the diagram

$$\begin{array}{ccc} X \times I & \xrightarrow{f} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ X/A \times I & \xrightarrow[\tilde{f}]{} & X/A \end{array}$$

commutes. Define $g : X/A \to X$ by $g([x]) = f(x, 1)$. Then $\mathrm{id}_X \sim g \circ q$ via the homotopy f and $\mathrm{id}_{X/A} \sim q \circ g$ via the homotopy $\tilde{f}$. Hence X is homotopy equivalent to X/A. □

Exercise A. Using the previous criterion show that $S^2/S^0 \sim S^2 \vee S^1$.

Exercise B. Using the previous criterion show that the suspension and the reduced suspension of a CW-complex are homotopy equivalent.

2.9. Second criterion for homotopy equivalence.

Proposition. *Let (X, A) be a pair of CW-complexes and let Y be a space. Suppose that $f, g : A \to Y$ are homotopic maps. Then $X \cup_f Y$ and $X \cup_g Y$ are homotopy equivalent.*

Proof. Let $F : A \times I \to Y$ be a homotopy between f and g. We will show that $X \cup_f Y$ and $X \cup_g Y$ are both deformation retracts of $(X \times I) \cup_F Y$. Consequently, they have to be homotopy equivalent.

We construct a deformation retraction in two steps.

(1) $(X \times \{0\}) \cup_f Y$ is a deformation retract of $(X \times \{0\} \cup A \times I) \cup_F Y$.
(2) $(X \times \{0\} \cup A \times I) \cup_F Y$ is a deformation retract of $(X \times I) \cup_F Y$.

□

Exercise. Let (X, A) be a pair of CW-complexes. Suppose that A is a contractible in X, i. e. there is a homotopy $F : A \to X$ between id_X and const. Using the first criterion show that $X/A \cong X \cup CA/CA \sim X \cup CA$. Using the second criterion prove that $X \cup CA \sim X \vee SA$. Then

$$X/A \sim X \vee SA.$$

Apply it to compute S^n/S^i, $i < n$.

CZ.1.07/2.2.00/28.0041
Centrum interaktivních a multimediálních studijních opor pro inovaci výuky a efektivní učení

evropský sociální fond v ČR
EVROPSKÁ UNIE
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