# INTRODUCTION TO ALGEBRAIC TOPOLOGY

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## 5. Singular cohomology

Cohomology forms a dual notion to homology. To every topological space we assign a graded group $H^*(X)$ equipped with a ring structure given by a product $\cup : H^i(X) \times H^j(X) \to H^{i+j}(X)$. In this section we give basic definitions and properties of singular cohomology groups which are very similar to those of homology groups.

**5.1. Cochain complexes.** A *cochain complex* $(C, \delta)$ is a sequence of Abelian groups (or modules over a ring) and their homomorphisms indexed by integers

$$\dots \xrightarrow{\delta^{n-2}} C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \xrightarrow{\delta^{n+1}} \dots$$

such that

$$\delta^n \delta^{n-1} = 0.$$

$\delta^n$ is called a coboundary operator. A *cochain homomorphism* of cochain complexes $(C, \delta_C)$ and $(D, \delta_D)$ is a sequence of homomorphisms of Abelian groups (or modules over a ring) $f^n : C^n \to D^n$ which commute with the coboundary operators

$$\delta^n_D f_n = f^{n+1} \delta^n_C.$$

**5.2. Cohomology of cochain complexes.** The $n$-th *cohomology group* of a cochain complex $(C, \delta)$ is the group

$$H^n(C) = \frac{\operatorname{Ker} \delta^n}{\operatorname{Im} \delta^{n-1}}.$$

The elements of $\operatorname{Ker} \delta^n = Z^n$ are called *cocycles* of dimension $n$ and the elements of $\operatorname{Im} \delta^{n-1} = B^n$ are called *coboundaries* (of dimension $n$). If a cochain complex is exact, then its cohomology groups are trivial.

The component $f^n$ of the cochain homomorphism $f : (C, \delta_C) \to (D, \delta_D)$ maps cocycles into cocycles and coboundaries into coboundaries. It enables us to define

$$H^n(f) : H^n(C) \to H^n(D)$$

by the prescription $H^n(f)[c] = [f^n(c)]$ where $[c] \in H^n(C)$ and $[f^n(c)] \in H^n(D)$ are classes represented by the elements $c \in Z^n(C)$ and $f^n(c) \in Z^n(D)$, respectively.

**5.3. Long exact sequence in cohomology.** A sequence of cochain homomorphisms

$$\dots \to A \xrightarrow{f} B \xrightarrow{g} C \to \dots$$

is exact if for every $n \in \mathbb{Z}$

$$\dots \to A^n \xrightarrow{f^n} B_n \xrightarrow{g^n} C^n \to \dots$$

is an exact sequence of Abelian groups. Similarly as for homology groups we can prove

**Theorem.** *Let* $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ *be a short exact sequence of cochain complexes. Then there is a so called* connecting homomorphism $\delta^* : H^n(C) \to H^{n+1}(A)$ *such that the sequence*

$$\dots \xrightarrow{\delta^*} H^n(A) \xrightarrow{H^n(i)} H^n(B) \xrightarrow{H^n(j)} H^n(C) \xrightarrow{\delta^*} H^{n+1}(A) \xrightarrow{H^{n+1}(i)} \dots$$

*is exact.*

**5.4. Cochain homotopy.** Let $f, g : C \to D$ be two cochain homomorphisms. We say that they are *cochain homotopic* if there are homomorphisms $s^n : C^n \to D^{n-1}$ such that

$$\delta_D^{n-1} s^n + s^{n+1}\delta_C^n = f^n - g^n \quad \text{for all } n.$$

The relation to be cochain homotopic is an equivalence. The sequence of maps $s^n$ is called a *cochain homotopy.* Similarly as for homology we have

**Theorem.** *If two cochain homomorphism* $f, g : C \to D$ *are cochain homotopic, then*

$$H^n(f) = H^n(g).$$

**5.5. Singular cohomology groups of a pair.** Consider a pair of topological spaces $(X, A)$, an inclusion $i : A \hookrightarrow X$ and an Abelian group $G$. Let

$$C(X, A) = (C_n(X)/C_n(A), \partial_n)$$

be the singular chain complex of the pair $(X, A)$. The *singular cochain complex* $(C(X, A; G), \delta)$ for the pair $(X, A)$ is defined as

$$\begin{aligned} C^n(X, A; G) = \mathrm{Hom}\,(C_n(X, A), G) &\cong \{h \in \mathrm{Hom}(C_n(X), G);\ h|C_n(A) = 0\} \\ &= \mathrm{Ker}\, i^* : \mathrm{Hom}(C_n(X), G) \longrightarrow \mathrm{Hom}(C_n(A), G). \end{aligned}$$

and

$$\delta^n(h) = h \circ \partial_{n+1} \quad \text{for } h \in \mathrm{Hom}(C_n(X, A), G).$$

The $n$-th *cohomology group* of the pair $(X, A)$ with coefficients in the group $G$ is the $n$-th cohomology group of this cochain complex

$$H^n(X, A; G) = H^n(C(X, A; G), \delta).$$

We write $H^n(X; G)$ for $H^n(X, \emptyset; G)$. A map $f : (X, A) \to (Y, B)$ induces the cochain homomorphism $C^n(f) : C^n(Y; G) \to C^n(X; G)$ by

$$C^n(f)(h) = h \circ C_n(f)$$

which restricts to a cochain homomorphism $C^n(Y, B; G) \to C^n(X, A; G)$ since $f(A) \subseteq B$. In cohomology it induces the homomorphism

$$f^* = H^n(f) : H^n(Y, B) \to H^n(X, A).$$

Moreover, $H^n(\mathrm{id}_{(X,A)}) = \mathrm{id}_{H^n(X,A;G)}$ and $H^n(fg) = H^n(g)H^n(f)$. We can conclude that $H^n$ is a contravariant functor (cofunctor) from the category $\mathsf{Top}^2$ into the category $\mathcal{AG}$ of Abelian groups.

**5.6. Long exact sequence for singular cohomology.** Consider inclusions of spaces $i : A \hookrightarrow X$, $i' : B \hookrightarrow Y$ and maps $j : (X, \emptyset) \to (X, A)$, $j' : (Y, \emptyset) \to (Y, B)$ induced by $\mathrm{id}_X$ and $\mathrm{id}_Y$, respectively. Let $f : (X, A) \to (Y, B)$ be a map. Then there are connecting homomorphisms $\delta^*_X$ and $\delta^*_Y$ such that the following diagram

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{\delta^*_X} & H^n(X,A;G) & \xrightarrow{j^*} & H^n(X;G) & \xrightarrow{i^*} & H^n(A;G) & \xrightarrow{\delta^*_X} & H^{n+1}(X,A;G) & \xrightarrow{j^*} & \dots \\
 & & \uparrow f^* & & \uparrow f^* & & \uparrow (f/B)^* & & \uparrow f^* & & \\
\dots & \xrightarrow{\delta^*_Y} & H^n(X,B;G) & \xrightarrow{j'^*} & H^n(Y;G) & \xrightarrow{i'^*} & H^n(B;G) & \xrightarrow{\delta^*_Y} & H^{n+1}(Y,B;G) & \xrightarrow{j'^*} & \dots
\end{array}$$

commutes and its horizontal sequences are exact.

The proof follows from Theorem 5.3 using the fact that

$$0 \to C^n(X, A; G) \xrightarrow{C^n(j)} C^n(X; G) \xrightarrow{C^n(i)} C^n(A; G) \to 0$$

is a short exact sequence of cochain complexes as it follows directly from the definition of $C^n(X, A; G)$.

**Remark A.** Consider the functor $I : \mathsf{Top}^2 \to \mathsf{Top}^2$ which assigns to every pair $(X, A)$ the pair $(A, \emptyset)$. The commutativity of the last square in the diagram above means that $\delta^*$ is a natural transformation of contravariant functors $H^n \circ I$ and $H^{n+1}$ defined on $\mathsf{Top}^2$.

**Remark B.** It is useful to realize how $\delta^* : H^n(A; G) \to H^{n+1}(X, A; G)$ looks like. Every element of $H^n(A; G)$ is represented by a cochain $q \in \mathrm{Hom}(C_n(A); G)$ with a zero coboundary $\delta q \in \mathrm{Hom}(C_{n+1}(A); G)$. Extend $q$ to $Q \in \mathrm{Hom}(C_n(X); G)$ in arbitrary way. Then $\delta Q \in \mathrm{Hom}(C_{n+1}(X), G)$ restricted to $C_{n+1}(A)$ is equal to $\delta q = 0$. Hence it lies in $\mathrm{Hom}(C_{n+1}(X, A); G)$ and from the definition in 5.3 we have

$$\delta^*[q] = [\delta Q].$$

**5.7. Homotopy invariance.** If two maps $f, g : (X, A) \to (Y, B)$ are homotopic, then they induce the same homomorphisms

$$f^* = g^* : H^n(Y, B; G) \to H_n(X, A; G).$$

*Proof.* We already know that the homotopy between $f$ and $g$ induces a chain homotopy $s_*$ between $C_*(f)$ and $C_*(g)$. Then we can define a cochain homotopy between $C^*(f)$ and $C^*(g)$ as

$$s^n(h) = h \circ s_{n-1} \quad \text{for } h \in \mathrm{Hom}(C_n(Y); G)$$

and use Theorem 5.4. □

**Corollary.** *If $X$ and $Y$ are homotopy equivalent spaces, then*

$$H^n(X) \cong H^n(Y).$$

**5.8. Excision Theorem.** Similarly as for singular homology groups there are two equivalent versions of this theorem.

**Theorem A** (Excision Theorem, 1st version)**.** *Consider spaces $C \subseteq A \subseteq X$ and suppose that $\bar{C} \subseteq \operatorname{int} A$. Then the inclusion*

$$i : (X - C, A - C) \hookrightarrow (X, A)$$

*induces the isomorphism*

$$i^* : H^n(X, A; G) \xrightarrow{\cong} H^n(X - C, A - C; G).$$

**Theorem B** (Excision Theorem, 2nd version)**.** *Consider two subspaces $A$ and $B$ of a space $X$. Suppose that $X = \operatorname{int} A \cup \operatorname{int} B$. Then the inclusion*

$$i : (B, A \cap B) \hookrightarrow (X, A)$$

*induces the isomorphism*

$$i^* : H^n(X, A; G) \xrightarrow{\cong} H^n(B, A \cap B; G).$$

The proof of Excision Theorem for singular cohomology follows from the proof of the homology version.

**5.9. Cohomology of finite disjoint union.** Let $X = \coprod_{\alpha=1}^{k} X_\alpha$ be a disjoint union. Then

$$H^n(X; G) = \bigoplus_{\alpha=1}^{k} H^n(X_\alpha).$$

The statement is not generally true for infinite unions.

**5.10. Reduced cohomology groups.** For every space $X \neq \emptyset$ we define the *augmented cochain complex* $(\tilde{C}^*(X; G), \tilde{\delta})$ as follows

$$\tilde{C}^n(X; G) = \operatorname{Hom}(\tilde{C}_n(X); G)$$

with $\tilde{\delta}^n h = h \circ \tilde{\partial}_{n+1}$ for $h \in \operatorname{Hom}(\tilde{C}_n(X); G)$. See 3.14. The *reduced cohomology groups* $\tilde{H}_n(X; G)$ with coefficients in $G$ are the cohomology groups of the augmented cochain complex. From the definition it is clear that

$$\tilde{H}^n(X; G) = H^n(X; G) \quad \text{for } n \neq 0$$

and

$$\tilde{H}^n(*; G) = 0 \quad \text{for all } n.$$

For pairs of spaces we define $\tilde{H}^n(X, A; G) = H^n(X, A; G)$ for all $n$. Then theorems on long exact sequence, homotopy invariance and excision hold for reduced cohomology groups as well.

Considering a space $X$ with base point $*$ and applying the long exact sequence for the pair $(X, *)$, we get that for all $n$

$$\tilde{H}^n(X; G) = \tilde{H}^n(X, *; G) = H^n(X, *; G).$$

Using this equality and the long exact sequence for unreduced cohomology we get that

$$H^0(X;G) \cong H^0(X,*;G) \oplus H^0(*;G) \cong \tilde{H}^0(X) \oplus \mathbb{G}.$$

Analogously as for homology groups we have

**Lemma.** *Let $(X, A)$ be a pair of CW-complexes. Then*

$$\tilde{H}^n(X/A;G) = H^n(X,A;G)$$

*and we have the long exact sequence*

$$\cdots \to \tilde{H}^n(X/A;G) \to \tilde{H}^n(X;G) \to \tilde{H}^n(A;G) \to \tilde{H}^{n+1}(X/A;G) \to \dots$$

**5.11. The long exact sequence of a triple.** Consider a triple $(X, B, A)$, $A \subseteq B \subseteq X$. Denote $i : (B, A) \hookrightarrow (X, A)$ and $j : (X, A) \to (X, B)$ maps induced by the inclusion $B \hookrightarrow X$ and $\mathrm{id}_X$, respectively. Analogously as for homology one can derive the long exact sequence of the triple $(X, B, A)$

$$\dots \xrightarrow{\delta^*} H^n(X,B;G) \xrightarrow{j^*} H^n(X,A;G) \xrightarrow{i^*} H^n(B,A;G) \xrightarrow{\delta^*} H^{n+1}(X,B;G) \xrightarrow{j^*} \dots$$

**5.12. Singular cohomology groups of spheres.** Considering the long exact sequence of the triple $(\Delta^n, \delta\Delta^n, V = \delta\Delta^n - \Delta^{n-1})$: we get that

$$H^i(\Delta^n, \partial\Delta^n; G) = \begin{cases} G & \text{for } i = n, \\ 0 & \text{for } i \neq n. \end{cases}$$

The pair $(D^n, S^{n-1})$ is homeomorphic to $(\Delta^n, \partial\Delta^n)$. Hence it has the same cohomology groups. Using the long exact sequence for this pair we obtain

$$\tilde{H}^i(S^n;G) = H^{i+1}(D^{n+1}, S^n) = \begin{cases} 0 & \text{for } i \neq n, \\ G & \text{for } i = n. \end{cases}$$

**5.13. Mayer-Vietoris exact sequence.** Denote inclusions $A \cap B \hookrightarrow A$, $A \cap B \hookrightarrow B$, $A \hookrightarrow X$, $B \hookrightarrow X$ by $i_A$, $i_B$, $j_A$, $j_B$, respectively. Let $C \hookrightarrow A$, $D \hookrightarrow B$ and suppose that $X = \operatorname{int} A \cup \operatorname{int} B$, $Y = \operatorname{int} C \cup \operatorname{int} D$. Then there is the long exact sequence

$$\dots \xrightarrow{\delta^*} H^n(X,Y;G) \xrightarrow{(j_A^*, j_B^*)} H^n(A,C;G) \oplus H^n(B,D;G)$$
$$\xrightarrow{i_A^* - i_B^*} H_n(A \cap B, C \cap D; G) \xrightarrow{\delta^*} H^{n+1}(X,Y;G) \longrightarrow \dots$$

*Proof.* The coverings $\mathcal{U} = \{A, B\}$ and $\mathcal{V} = \{C, D\}$ satisfy conditions of Lemma 3.12. The sequence of chain complexes

$$0 \longrightarrow C_n(A \cap B, C \cap D) \xrightarrow{i} C_n(A,C) \oplus C_n(B;D) \xrightarrow{j} C_n^{\mathcal{U},\mathcal{V}}(X,Y) \longrightarrow 0$$

where $i(x) = (x, x)$ and $j(x, y) = x - y$ is exact. Applying $\mathrm{Hom}(-, G)$ we get a new short exact sequence of cochain complexes

$$0 \longrightarrow C^n_{\mathcal{U},\mathcal{V}}(X,Y;G) \xrightarrow{j^*} C^n(A,C;G) \oplus C^n(B,D;G) \xrightarrow{i^*} C^n(A \cap B, C \cap D; G) \longrightarrow 0$$

and it induces a long exact sequence. Using Lemma 3.12 we get that $H^n(C_{\mathcal{U},\mathcal{V}}(X,Y;G)) = H^n(X,Y;G)$, which completes the proof. □

**5.14. Computations of cohomology of CW-complexes.** If we know a CW-structure of a space $X$, we can compute its cohomology in the same way as homology. Consider the chain complex from Section 4

$$(H_n(X^n, X^{n-1}), d_n).$$

**Theorem.** *Let $X$ be a CW-complex. The $n$-th cohomology group of the cochain complex*

$$(\mathrm{Hom}(H_n(X^n, X^{n-1}; G), d^n) \quad d^n(h) = h \circ d_n$$

*is isomorphic to the $n$-th singular cohomology group $H^n(X;G)$.*

**Exercise A.** After reading the next section try to prove the theorem above using the results and proofs from Section 4.

**Exercise B.** Compute singular cohomology of real and complex projective spaces with coefficients $\mathbb{Z}$ and $\mathbb{Z}_2$.

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