INTRODUCTION TO ALGEBRAIC TOPOLOGY

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7. Products in cohomology

An internal product in cohomology brings a further algebraic structure. The contravariant functor H^* becomes a cofunctor into graded rings. It enables us to obtain more information on topological spaces and homotopy classes of maps. In this section we will define an internal product – called the cup product and a closely related external product – called the cross product.

7.1. Cup product. Let R be a commutative ring with a unite and let X be a space. For two cochains $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ we define their *cup product* $\varphi \cup \psi \in C^{k+l}(X;R)$

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma/[v_0, v_1, \dots, v_k]) \cdot \psi(\sigma/[v_k, v_{k+1}, \dots, v_{k+l}])$$

for any singular simplex $\sigma : \Delta^{k+l} \to X$. The notation $\sigma/[v_0, v_1, \dots, v_k]$ and $\sigma/[v_k, v_{k+1}, \dots, v_{k+l}]$ stands for σ composed with inclusions of the standard simplices Δ^k and Δ^l into the indicated faces of the standard simplex Δ^{k+l}, respectively. The coboundary operator δ behaves on the cup products of cochains as graded derivation as shown in the following

Lemma.

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi.$$

Proof. For $\sigma \in C_{k+l+1}(X)$ we get

$$\begin{aligned}
(\delta\varphi \cup \psi)(\sigma) + (-1)^k(\varphi \cup \delta\psi)(\sigma) &= \delta\varphi(\sigma/[v_0, v_1, \dots, v_{k+1}])\psi(\sigma/[v_{k+1}, \dots, v_{k+l+1}]) \\
&\quad + (-1)^k\varphi(\sigma[v_0, v_1, \dots, v_k])\delta\psi(\sigma/[v_k, \dots, v_{k+l+1}]) \\
&= \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma/[v_0, \dots, \hat{v}_i, \dots, v_{k+1}])(\psi(\sigma/[v_{k+1}, \dots, v_{k+l+1}])) \\
&\quad + (-1)^k \left(\sum_{j=k}^{k+l+1} (-1)^{j-k} \varphi(\sigma/[v_0, \dots, v_k])\psi(\sigma/[v_k, \dots, \hat{v}_j, \dots, v_{k+l+1}]) \right) \\
&= \sum_{i=0}^{k+l+1} (-1)^i (\varphi \cup \psi)(\sigma/[v_0, \dots, \hat{v}_i, \dots, v_{k+l+1}]) = \delta(\varphi \cup \psi)(\sigma).
\end{aligned}$$

□

Lemma implies that

(1) If φ and ψ are cocycles, then $\varphi \cup \psi$ is a cocycle.
(2) If one of the cochains φ and ψ is a coboundary, then $\varphi \cup \psi$ is a coboundary.

It enables us to define the *cup product*

$$\cup : H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R)$$

by the prescription

$$[\varphi] \cup [\psi] = [\varphi \cup \psi]$$

for cocycles φ and ψ. Since $\cup$ is an R-bilinear map on $H^k(X;R) \times H^l(X;R)$, it can be considered as an R-linear map on the tensor product $H^k(X;R) \otimes_R H^l(X;R)$. Given a pair of spaces (X, A) we can define the cup product as a linear map

$$\begin{aligned} \cup : & H^k(X,A;R) \otimes_R H^l(X;R) \to H^{k+l}(X,A;R), \\ \cup : & H^k(X;R) \otimes_R H^l(X,A;R) \to H^{k+l}(X,A;R), \\ \cup : & H^k(X,A;R) \otimes_R H^l(X,A;R) \to H^{k+l}(X,A;R). \end{aligned}$$

Moreover, if A and B are open in X or A and B are subcomplexes of CW-complex X, one can define

$$\cup : H^k(X,A;R) \otimes_R H^l(X,B;R) \to H^{k+l}(X,A \cup B;R).$$

Exercise. Prove that the previous definitions of cup product for pairs of spaces are correct. For the last case you need Lemma 3.12.

Remark. In the same way as the singular cohomology groups and the cup product have been defined using the singular chain complexes, we can introduce simplicial cohomology groups for Δ-complexes and a cup product in these groups.

7.2. Properties of the cup product are following:

(1) The cup product is associative.
(2) If $X \neq \emptyset$, there is an element $1 \in H^0(X;R)$ such that for all $\alpha \in H^k(X,A;R)$

$$1 \cup \alpha = \alpha \cup 1 = \alpha.$$

(3) For all $\alpha \in H^k(X,A;R)$ and $\beta \in H^l(X,A;R)$

$$\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha,$$

i. e. the cup product is graded commutative.
(4) Naturality of the cup product. For every map $f : (X,A) \to (Y,B)$ and any $\alpha \in H^k(Y,B;R)$, $\beta \in H^l(Y,B;R)$ we have

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta).$$

Remark. Properties (1) – (3) mean that $H^*(X,A;R) = \bigoplus_{i=0}^{\infty} H^i(X,A;R)$ with the cup product is not only a graded group but also a graded ring and that $H^*(X;R)$ is even a graded ring with a unit if $X \neq \emptyset$. Property (4) says that $f : (X,A) \to (Y,B)$ induces a ring homomorphism $f^* : H^*(Y,B;R) \to H^*(X,A;R)$.

Proof. To prove properties (1), (2) and (4) is easy and left to the reader as an exercise. To prove property (3) is more difficult. We refer to [Hatcher], Theorem 3.14, pages 215 – 217 for geometrically motivated proof. Another approach is outlined later in 7.8. □

7.3. Cross product. Consider spaces X and Y and projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. We will define the *cross product* or *external product.* The absolute and relative forms are the linear maps

$$\mu : H^k(X,R) \otimes H^l(Y;R) \to H^{k+l}(X \times Y;R),$$
$$\mu : H^k(X,A;R) \otimes H^l(Y,B;R) \to H^{k+l}(X \times Y, A \times Y \cup X \times B;R)$$

given by

$$\mu(\alpha \otimes \beta) = p_1^*(\alpha) \cup p_2^*(\beta).$$

For the relative form of the cross product we suppose that A and B are open in X and Y, or that A and B are subcomplexes of X and Y, respectively. (See the definition of the cup product.) The name cross product comes from the notation since $\mu(\alpha \otimes \beta)$ is often written as $\alpha \times \beta$.

Exercise. Let $\Delta : X \to X \times X$ be the diagonal $\Delta(x) = (x,x)$. Show that for $\alpha, \beta \in H^*(X;R)$

$$\alpha \cup \beta = \Delta^*\big(\mu(\alpha \otimes \beta)\big).$$

7.4. Tensor product of graded rings. Let $A^* = \bigoplus_{n=0}^{\infty} A^n$ and $B^* = \bigoplus_{n=0}^{\infty} B^n$ be graded rings. Then the *tensor product of graded rings* $A^* \otimes B^*$ is the graded ring $C^* = \bigoplus_{n=0}^{\infty} C^n$ where

$$C^n = \bigoplus_{i+j=n} A^i \otimes B^j$$

with the multiplication given by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|b_1|\cdot|a_2|}(a_1 \cdot a_2) \otimes (b_1 \cdot b_2).$$

Here $|b_1|$ is the degree of $b_1 \in B^*$, i.e. $b_1 \in B^{|b_1|}$. If A^* and B^* are graded commutative, so is $A^* \otimes B^*$.

Lemma. *The cross product*

$$\mu : H^k(X,A;R) \otimes H^l(Y,B;R) \to H^{k+l}(X \times Y, A \times Y \cup X \times B;R)$$

is a homomorphism of graded rings.

Proof. Using the definitions of the cup and cross products and their properties we have

$$\begin{aligned}\mu\big((a \times b) \cdot (c \times d)\big) &= (-1)^{|b|\cdot|c|}\mu\big((a \cup c) \otimes (b \cup d)\big) = (-1)^{|b|\cdot|c|}p_1^*(a \cup c) \cup p_2^*(b \cup d)\\ &= (-1)^{|b|\cdot|c|}p_1^*(a) \cup p_1^*(c) \cup p_2^*(b) \cup p_2^*(d)\\ &= p_1^*(a) \cup p_2^*(b) \cup p_1^*(c) \cup p_2^*(d) = \mu(a \otimes b) \cup \mu(c \otimes d).\end{aligned}$$

□

7.5. Künneth formulas tell us how to compute the graded R-modules $H_*(X\times Y;R)$ or $H^*(X\times Y;R)$ out of the graded modules $H_*(X;R)$ and $H_*(Y;R)$ or $H^*(X;R)$ and $H^*(Y;R)$, respectively. Under certain conditions it even determines the ring structure of $H^*(X\times Y;R)$.

Theorem (Künneth formula)**.** *Let (X,A) and (Y,B) be pairs of CW-complexes. Suppose that $H^k(Y,B;R)$ are free finitely generated R-modules for all k. Then*

$$\mu : H^*(X,A;R)\otimes H^*(Y,B;R)\to H^*(X\times Y, A\times Y\cup X\times B;R)$$

is an isomorphism of graded rings.

Example. $H^*(S^k\times S^l)\cong \mathbb{Z}[\alpha,\beta]/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by elements α^2, β^2, $\alpha\beta=(-1)^{kl}\beta\alpha$ and $\deg\alpha=k$, $\deg\beta=l$.

Proof. Consider the diagram

$$\begin{tikzcd}
H^*(X,A)\otimes_R H^*(Y) \arrow[rr] \arrow[dd, "\mu"] & & H^*(X)\otimes_R H^*(Y) \arrow[dl] \arrow[dd, "\mu"] \\
& H^*(A)\otimes_R H^*(Y) \arrow[ul, "\delta^*\otimes\mathrm{id}"'] \arrow[dd, "\mu"] & \\
H^*(X\times Y, A\times Y) H^*(X\times Y) \arrow[rr] & & H^*(X\times Y) \arrow[dl] \\
& H^*(A\times Y) \arrow[ul, "\delta^*"'] &
\end{tikzcd}$$

where the upper and the lower triangles come from the long exact sequences for pairs (X,A) and $(X\times Y, A\times Y)$, respectively. The right rhomb commutes as a consequence of the naturality of the cross product. We prove that the left rhomb also commutes.

Let φ and ψ be cocycles in $C^*(A)$ and $C^*(Y)$, respectively. Let Φ be a cocycle in $C^*(X)$ extending φ. Then $p_1^*\Phi\cup p_2^*\psi\in C^*(X\times Y)$ extends $p_1^*\varphi\cup p_2^*\psi\in C^*(A\times Y)$. Using the definition of the connecting homomorphism in cohomology (see Remark 5.6 B) we get

$$\begin{aligned}
\mu\big((\delta^*\otimes\mathrm{id})([\varphi]\otimes[\psi])\big) &= \mu[\delta\Phi\otimes\psi] = p_1^*[\delta\Phi]\cup p_2^*[\psi],\\
\delta^*\big(\mu([\varphi]\otimes[\psi])\big) &= \delta^*[p_1^*\varphi\cup p_2^*\psi] = [\delta(p_1^*\Phi\cup p_2^*\psi)] = p_1^*[\delta\Phi]\cup p_2^*[\psi].
\end{aligned}$$

First, we prove the statement of Theorem for a finetedimensional CW-complex X and $A=B=\emptyset$ using the induction by the dimension of X and Five Lemma. If $\dim X=0$, X is a finite discrete set and the statement of Theorem is true. Suppose that Theorem holds for spaces of dimension $n-1$ or less. Let $\dim X=n$. It suffices to show that

$$\mu : H^*(X^n, X^{n-1})\otimes H^*(Y)\to H^*(X^n\times Y, X^{n-1}\times Y)$$

is an isomorphism and than to use Five Lemma in the diagram above with $A=X^{n-1}$ to prove the statement for $X=X^n$. X^n/X^{n-1} is homeomorphic to $\bigsqcup_i D_i^n/\bigsqcup_i\partial D_i^n$.

To prove that

$$\mu : H^*\big(\bigvee_i S_i^n\big) \otimes H^*(Y) \to H^*\big(\bigvee_i S_i^n \times Y\big)$$

is an isomorphism, we use again the diagram above for $X = \bigsqcup_i D_i^n$ and $A = \bigsqcup_i \partial D_i^n$ and the induction with respect to n.

So we have proved the theorem for X a finite dimensional CW-complex and $A = B = \emptyset$. Using once more our diagram and Five Lemma, we can easily prove Theorem for any pairs (X, A), $(Y, \emptyset)$ with X of finite dimension. For X of infinite dimension, we have to prove $H^i(X) = H^i(X^n)$ for $i < n$ which is equivalent to $H^i(X/X^n) = 0$. We omit the details and refer the reader to [Hatcher], pages 220 – 221. □

7.6. Application of the cup product. In this paragraph we show how to use the cup product to prove that S^{2k} is not an H-space. A space X is called an *H-space* if there is a map $m : X \times X \to X$ called a multiplication and an element $e \in X$ called a unit such that $m(e, x) = m(x, e) = x$ for all $x \in X$.

Suppose that there is a multiplication $m : S^{2k} \times S^{2k} \to S^{2k}$ with a unit e. According to Example after Theorem 7.5

$$H^*(S^{2k} \times S^{2k}; \mathbb{Z}) = \mathbb{Z}[\alpha, \beta]/\mathcal{I}$$

where $\mathcal{I}$ is the ideal generated by relations $\alpha^2 = 0$, $\beta^2 = 0$ and $\alpha\beta = \beta\alpha$. The last relation is due to the fact that the dimension of the sphere is even. Moreover, $\alpha = \gamma \otimes 1$ and $\beta = 1 \otimes \gamma$ where $\gamma \in H^{2k}(S^{2k}; \mathbb{Z})$ is a generator. Let us compute $m^* : H^*(S^{2k}; \mathbb{Z}) \to H^*(S^{2k} \times S^{2k}; \mathbb{Z})$. We have

$$m^*(\gamma) = a\alpha + b\beta, \quad a, b \in \mathbb{Z}.$$

Since the composition

$$S^{2k} \xrightarrow{\mathrm{id} \times e} S^{2k} \times S^{2k} \xrightarrow{m} S^{2k}$$

is the identity, we get that $a = 1$. Similarly, $b = 1$. Now compute $m^*(\gamma^2)$:

$$0 = m^*(0) = m^*(\gamma^2) = \big(m^*(\gamma)\big)^2 = (\alpha + \beta)^2 = 2\alpha\beta \neq 0,$$

a contradiction. Does this proof go through for odd dimensional spheres?

7.7. Künneth formula in homological algebra. Consider two chain complexes (C_*, ∂_C), (D_*, ∂_D) of R-modules. Suppose there is an integer N such that $C_n = D_n = 0$ for all $n < N$. Then their tensor product is the chain complex $(C_* \otimes D_*, \partial)$ with

$$(C_* \otimes D_*)_n = \bigoplus_{i+j=n} C_i \otimes D_i$$

and the boundary operator on $C_i \otimes D_j$

$$\partial(c \otimes d) = \partial_C c \otimes d + (-1)^i c \otimes \partial_D d.$$

It is easy to make sure that $\partial\partial = 0$.

Next we can define the graded R-module $C_* * D_*$ as

$$(C_* * D_*)_n = \bigoplus_{i+j=n} \mathrm{Tor}_1^R(C_i, D_j).$$

A ring R is called *hereditary* if any submodule of a free R-module is free. Examples of hereditary rings are $\mathbb{Z}$ and all fields.

Theorem (Algebraic Künneth formula)**.** *Let R be a hereditary ring and let C_* and D_* be chain complexes of R-modules. If C_* is free, then the homology groups of $C_* \otimes D_*$ are determined by the splitting short exact sequence*

$$0 \to (H_*(C) \otimes H_*(D))_n \xrightarrow{l} H_n(C_* \otimes D_*) \to (H_*(C) * H_*(D))_{n-1} \to 0$$

where $l([c] \otimes [d]) = [c \otimes d]$. This sequence is natural but the splitting is not.

Notice that for the chain complex

$$D_n = \begin{cases} 0 & \text{for } n \neq 0, \\ G & \text{for } n = 0 \end{cases}$$

the Küneth formula gives the universal coefficient theorem for homology groups, see Theorem 6.11 B.

The proof of the Künneth formula is similar to the proof of the universal coefficient theorem and we omit it.

7.8. Eilenberg-Zilbert Theorem. To be able to apply the previous Künneth formula in topology we have to show that the singular chain complex $C_*(X \times Y)$ of a product $X \times Y$ is chain homotopy equivalent to the tensor product of the singular chain complexes $C_*(X) \otimes C_*(Y)$.

Theorem (Eilenberg-Zilbert)**.** *Up to chain homotopy there are unique natural chain homomorphisms*

$$\begin{aligned} \Phi :& C_*(X) \otimes C_*(Y) \to C_*(X \times Y), \\ \Psi :& C_*(X \times Y) \to C_*(X) \otimes C_*(Y) \end{aligned}$$

such that for 0-simplices σ and τ

$$\Phi(\sigma \otimes \tau) = (\sigma, \tau), \quad \Psi(\sigma, \tau) = \sigma \otimes \tau.$$

Moreover, such chain homomorphisms are chain homotopy equivances: there are natural chain homotopies such that

$$\Psi\Phi \sim \mathrm{id}_{C_* X \otimes C_*(Y)}, \quad \Phi\Psi \sim \mathrm{id}_{C_*(X \times Y)}.$$

For the proof of this theorem see [Dold], IV.12.1. The chain homomorphism Ψ is called the Eilenberg-Zilbert homomorphism and denoted EZ. It enables a different and more abstract approach to the definitions of the cross and cup products. The cross product is

$$\mu([\alpha] \otimes [\beta]) = [(\alpha \otimes \beta) \circ EZ]$$

for cocycles $\alpha \in C^*(X;R)$ and $\beta \in C^*(Y;R)$ and the cup product is

$$([\varphi] \otimes [\psi]) = [(\varphi \otimes \psi) \circ EZ \circ \Delta_*]$$

for cocycles $\varphi, \psi \in C^*(X;R)$ and the diagonal $\Delta : X \to X \times X$. In our definition in 7.1 we have used for $EZ \circ \Delta_*$ the homomorphism

$$\sigma \to \sigma/[v_0, v_1, \dots, v_k] \otimes \sigma/[v_k, \dots, v_n].$$

The properties of EZ can be used for a different proof of the graded commutativity of the cup product.

7.9. Künneth formulas in topology. The following statement is an immediate consequence of the previous paragraph.

Theorem A (Künneth formula for homology)**.** *Let R be a hereditary ring. The homology groups of the product of two spaces X and Y are determined by the following splitting short exact sequence*

$$0 \to (H_*(X;R) \otimes H_*(Y;R))_n \xrightarrow{l} H_n(X \times Y;R) \to (H_*(X;R) * H_*(Y;R))_{n-1} \to 0$$

where $l([c] \otimes [d]) = [c \otimes d]$. This sequence is natural but the splitting is not.

For cohomology groups one can prove

Theorem B (Künneth formula for cohomology groups)**.** *Let R be a hereditary ring. The cohomology groups of the product of two spaces X and Y are determined by the following splitting short exact sequence*

$$0 \to (H^*(X;R) \otimes H^*(Y;R))_n \xrightarrow{\mu} H^n(X \times Y;R) \to (H^*(X;R) * H^*(Y;R))_{n+1} \to 0.$$

This sequence is natural but the splitting is not.

For the proof and other forms of Künneth formulas see [Dold], Chapter VI, Theorem 12.16 or [Spanier], Chapter 5, Theorems 5.11. and 5.12.

CZ.1.07/2.2.00/28.0041
Centrum interaktivních a multimediálních studijních opor pro inovaci výuky a efektivní učení

evropský sociální fond v ČR
EVROPSKÁ UNIE
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