INTRODUCTION TO ALGEBRAIC TOPOLOGY

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9. Poincaré duality

Many interesting spaces used in geometry are closed oriented manifolds. Poincaré duality expresses a remarkable symmetry between their homology and cohomology.

9.1. Manifolds. A *manifold of dimension* n is a Hausdorff space M in which each point has an open neighbourhood U homeomorphic to $\mathbb{R}^n$. The dimension of M is characterized by the fact that for each $x \in M$, the local homology group $H_i(M, M - \{x\}; \mathbb{Z})$ is nonzero only for $i = n$ since by excision and homotopy equivalence

$$H_i(M, M - \{x\}; \mathbb{Z}) \cong H_i(U, U - \{x\}; \mathbb{Z}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}) \cong \tilde{H}_{i-1}(S^{n-1}; \mathbb{Z}).$$

A compact manifold is called *closed*.

Example. Examples of closed manifolds are spheres, real and complex projective spaces, orthogonal groups $O(n)$ and $SO(n)$, unitary groups $U(n)$ and $SU(n)$, real and complex Stiefel and Grassmann manifolds. The real Stiefel manifold $V_{n,k}$ is the space of k-tuples of orthonormal vectors in $\mathbb{R}^n$. The real Grassmann manifolds $G_{n,k}$ is the space of k-dimensional vector subspaces of $\mathbb{R}^n$.

9.2. Orientation of manifolds. Consider a manifold M of dimension n. A *local orientation* of M in a point $x \in M$ is a choice of a generator $\mu_x \in H_n(M, M-\{x\}; Z) \cong \mathbb{Z}$.

To shorten our notation we will use $H_i(M|A)$ for $H_i(M, M - A; \mathbb{Z})$ and $H^i(M|A)$ for $H^i(M; M - A; \mathbb{Z})$ if $A \subseteq M$.

An *orientation* of M is a function assigning to each point $x \in M$ a local orientation $\mu_x \in H_n(M|x)$ such that each point has an open neighbourhood B with the property that all local orientations μ_y for $y \in B$ are images of an element $\mu_B \in H_n(M|B)$ under the map $\rho_{y*} : H_n(M|B) \to H_n(M|x)$ where $\rho_y : (M, M - \{x\}) \to (M, M - B)$ is the natural inclusion.

If an orientation exists on M, the manifold is called *orientable*. A manifold with a chosen orientation is called *oriented*.

Proposition. *A connected manifold M is orientable if it is simply connected, i. e. every map $S^1 \to M$ is homotopic to a constant map.*

For the proof one has to know more about covering spaces and fundamental group. See [Hatcher], Proposition 3.25, pages 234 – 235.

In the same way we can define an R-orientation of a manifold for any commutative ring R. Every manifold is $\mathbb{Z}_2$-oriented.

9.3. Fundamental class. A *fundamental class* of a manifold M with coefficients in R is an element $\mu \in H_n(M;R)$ such that $\rho_{x*}(\mu)$ is a generator of $H_n(M|x;R) = R$ for each $x \in M$ where $\rho_x : (M, \emptyset) \to (M, M - \{x\})$ is the obvious inclusion. It is usual to denote the fundamental class of the manifold M by $[M]$. We will keep this notation.

If a fundamental class of M exists, it determines uniquely the orientation $\mu_x = \rho_{x*}([M])$ of M.

Theorem. *Let M be a closed manifold of dimension n. Then:*

(a) *If M is R-orientable, the natural map $H_n(M;R) \to H_n(M|x;R) = R$ is an isomorphism for all $x \in M$.*
(b) *If M is not R-orientable, the natural map $H_n(M;R) \to H_n(M|x;R) = R$ is injective with the image $\{r \in R;\ 2r = 0\}$ for all $x \in M$.*
(c) *$H_i(M;R) = 0$ for all $i > n$.*

(a) implies immediately that very oriented closed manifold has just one fundamental class. It is a suitable generator of $H_n(M;R)$.

The theorem will follow from a more technical statement:

Lemma. *Let M be n-manifold and let $A \subseteq M$ be compact. Then:*

(a) *$H_i(M|A;R) = 0$ for $i > n$ and $\alpha \in H_n(M|A;R)$ is zero iff its image $\rho_{x*}(\alpha) \in H_n(M|x;R)$ is zero for all $x \in M$.*
(b) *If $x \mapsto \mu_x$ is an R-orientation of M, then there is $\mu_A \in H_n(M|A;R)$ whose image in $H_n(M|x;R)$ is μ_x for all $x \in A$.*

To prove the theorem put $A = M$. We get immediately (c) of the theorem. Further, the lemma implies that an oriented manifold M has a fundamental class $[M] = \mu_M$ and any other element in $H_n(M;R)$ has to be its multiple in R. So we obtain (a) of the theorem. For the proof of (b) we refer to [Hatcher], pages 234 – 236.

Proof of Lemma. Since R does not play any substantial role in our considerations, we will omit it from our notation. We will omit also stars in notation of maps induced in homology. The proof will be divided into several steps.

(1) Suppose that the statements are true for compact subsets A, B and $A \cap B$ of M. We will prove them for $A \cup B$ using the Mayer-Vietoris exact sequence:

$$0 \to H_n(M|A \cup B) \xrightarrow{\Phi} H_n(M|A) \oplus H_n(M|B) \xrightarrow{\Psi} H_n(M|A \cap B)$$

where $\Phi(\alpha) = (\rho_A \alpha, \rho_B \alpha)$, $\Psi(\alpha, \beta) = \rho_{A\cap B}\alpha - \rho_{A\cap B}\beta$.

$H_i(M|A \cup B) = 0$ for $i > n$ is immediate from the exact sequence. Suppose $\alpha \in H_n(M|A \cup B)$ restricted to $H_n(M|x)$ is zero for all $x \in A \cup B$. Then $\rho_A\alpha$ and $\rho_B\alpha$ are zeroes. Since Φ is a monomorphism, α has to be also zero.

Take μ_A and μ_B such that their restrictions to $H_n(M|x)$ are orientations. Then the restrictions to points $x \in A \cap B$ are the same. Hence also the restrictions to $A \cap B$ coincide. It means $\Psi(\mu_A, \mu_B) = 0$ and the Mayer-Vietoris exact sequence yields the existence of α in $H_n(M|A\cup B)$ such that $\Phi(\alpha)=(\mu_A,\mu_B)$. Therefore α reduces to a generator of $H_n(M|x)$ for all $x\in A\cup B$, and consequently, $\alpha=\mu_{A\cup B}$.

(2) If $M=\mathbb{R}^n$ and A is a compact convex set in a disc D containing an origin 0, the lemma is true since the composition given by inclusions

$$H_i(\mathbb{R}^n|D)\longrightarrow H_i(\mathbb{R}^n|A)\longrightarrow H_i(\mathbb{R}^n|0)$$

is an isomorhism.

(3) If $M=\mathbb{R}^n$ and A is finite simplicial complex in $\mathbb{R}^n$, then $A=\bigcup_{i=1}^m A_i$ where A_i are convex compact sets. Using (1) and induction by m we can prove that the lemma holds in this case as well.

(4) Let $M=\mathbb{R}^n$ and A is an arbitrary compact subset. Let $\alpha\in H_i(\mathbb{R}^n|A)$ be represented by a relative cycle $z\in Z_i(\mathbb{R}^n,\mathbb{R}^n-A)$. Let $C\subset\mathbb{R}^n-A$ be the union of images of the singular simplices in ∂z. Since C is compact, $\mathrm{dist}(C,A)>0$, and consequenly, there is a finite simplicial complex $K\supset A$ such that $C\subset\mathbb{R}^n-K$. (Draw a pisture.) So the chain z defines also an element $\alpha_K\in H_i(\mathbb{R}^n|K)$ which reduces to $\alpha\in H_i(\mathbb{R}^n|A)$. If $i>n$, then by (3) $\alpha_K=0$ and consequently also $\alpha=0$.

Suppose that $i=n$ and that α reduces to zero in each point $x\in A$. K can be chosen in such a way that every its point lies in a simplex of K together with a point of A. Consequently, α_K reduces to zero not only for all $x\in A$ but for all $x\in K$. (Use the case (2) to prove it.) By (3) $\alpha_K=0$, and therefore also $\alpha=0$.

The proof of existence of $\mu_A\in H_n(\mathbb{R}^n|A)$ in the statement (b) is easy. Take $\mu_B\in H_n(\mathbb{R}^n|B)$ for a ball $B\supset A$ and its reduction is μ_A.

(5) Let M be a general manifold and A a compact subset in an open set U homeomorphic to $\mathbb{R}^n$. Now by excision

$$H_i(M|A)\cong H_i(U|A)\cong H_i(\mathbb{R}^n|A)$$

and we can use (4).

(6) Let M be a manifold and A an arbitrary compact set. Then A can be covered by open sets $V_1,V_2,\dots,V_m$ such that the closure of V_i lies in an open set U_i homeomorphic to $\mathbb{R}^n$. Then by (5) the lemma holds for $A_i=A\cap\bar{V}_i$. By (1) and induction it holds also for $\bigcup_{i=1}^m A_i=A$. $\square$

9.4. Cap product. Let X be a space. On the level of chains and cochains the *cap product*

$$\cap: C_n(X;R)\otimes C^k(X;R)\to C_{n-k}(X;R)$$

is given for $0\le k\le n$ by

$$\sigma\cap\varphi=\varphi(\sigma/[v_0,v_1,\dots,v_k])\sigma/[v_k,v_{k+1},\dots,v_n]$$

where σ is a singular n-simplex, $\varphi: C_k(X;R)\to R$ is a cochain and $\sigma/[v_0,v_1,\dots,v_k]$ is the composition of the inclusion of Δ^k into the indicated face of Δ^n with σ, and is given by zero in the remaining cases.

The proof of the following statement is similar as in the case of cup product and is left to the reader as an exercise.

Lemma A. *For $\sigma \in C_n(X;R)$ and $\varphi \in C^k(X;R)$*

$$\partial(\sigma \cap \varphi) = (-1)^k(\partial\sigma \cap \varphi - \sigma \cap \delta\varphi)$$

This enables us to define

$$\cap : H_n(X;R) \otimes H^k(X;R) \to H_{n-k}(X;R)$$

by

$$[\sigma] \cap [\varphi] = [\sigma \cap \varphi]$$

for all cycles σ and cocycles φ. In the same way one can define

$$\cap : H_n(X,A;R) \otimes H^k(X;R) \to H_{n-k}(X,A;R)$$
$$\cap : H_n(X,A;R) \otimes H^k(X,A;R) \to H_{n-k}(X;R)$$

for any pair (X,A) and

$$\cap : H_n(X,A \cup B;R) \otimes H^k(X,A;R) \to H_{n-k}(X,B;R)$$

for A, B open in X or subcomplexes of CW-complex X.

Exercise. Show the correctness of all the definitions above and prove the following lemma.

Lemma B (Naturality of cup product). *Let $f : (X,A) \to (Y,B)$. Then*

$$f_*(\alpha \cap f^*(\beta)) = f_*(\alpha) \cap \beta$$

for all $\alpha \in H_n(X,A;R)$ and $\beta \in H^k(Y;R)$.

9.5. Poincaré duality. Now we have all the tools needed to state the Poincaré duality for closed manifolds.

Theorem (Poincaré duality). *If M is a closed R-orientable manifold of dimension n with fundamental class $[M] \in H_n(M;R)$, then the map $D : H^k(M;R) \to H_{n-k}(M;R)$ defined by*

$$D(\varphi) = [M] \cap \varphi$$

is an isomorphism.

Exercise. Use Poincaré duality to show that the real projective spaces of even dimension are not orientable.

This theorem is a consequence of a more general version of Poincaré duality. To state it we introduce the notion of direct limit and cohomology with compact support.

9.6. Direct limits. A *direct set* is a partially ordered set I such that for each pair $\iota, \kappa \in I$ there is $\lambda \in I$ such that $\iota \leq \lambda$ and $\kappa \leq \lambda$.

Let G_ι be a system od Abelian groups (or R-modules) indexed by elements of a directed set I. Suppose that for each pair $\iota \leq \kappa$ of indices there is a homomorphism $f_{\iota\kappa} : G_\iota \to G_\kappa$ such that $f_{\iota\iota} = \mathrm{id}$ and $f_{\kappa\lambda} f_{\iota\kappa} = f_{\iota\lambda}$. Then such a system is called *directed.*

Having a directed system of Abelian groups (or R-modules) we will say that $a \in G_\iota$ and $b \in G_\kappa$ are equivalent ($a \simeq b$) if $f_{\iota\lambda}(a) = f_{\kappa\lambda}(b)$ for some $\lambda \in I$. The direct limit of the system $\{G_\iota\}_{\iota \in I}$ is the Abelian group (R-module) of classes of this equivalence

$$\varinjlim G_\iota = \bigoplus_{\iota \in I} G_\iota / \simeq .$$

Moreover, we have natural homomorphism $j_\iota : G_\iota \to \varinjlim G_\iota$.

The direct limit is characterized by the following universal property: Having a system of homomorphism $h_\iota : G_\iota \to A$ such that $h_\iota = h_\kappa f_{\iota\kappa}$ whenever $\iota \leq \kappa$, there is just one homomorphism

$$H : \varinjlim G_\iota \to A$$

such that $h_\iota = H j_\iota$.

It is not difficult to prove that direct limits preserve exact sequences.

In a system of sets the ordering is usually given by inclusions.

Lemma. *If a space X is the union of a directed set of subspaces X_ι with the property that each compact set in X is contained in some X_ι, the natural map*

$$\varinjlim H_n(X_\iota; R) \to H_n(X; R)$$

is an isomorphism.

The proof is not difficult, we refer to [Hatcher], Proposition 3.33, page 244.

9.7. Cohomology groups with compact support. Consider a space X with a directed system of compact subsets. For each pair (L, K), $K \subseteq L$, the inclusion $(X, X - L) \hookrightarrow (X, X - K)$ induces homomorphism $H^k(X|K; R) \to H^k(X|L; R)$. We define the cohomology groups with compact support as

$$H^k_c(X; R) = \varinjlim H^k(X|K; R).$$

If X is compact, then $H^k_c(X; R) = H^k(X; R)$.

For cohomology with compact support we get the following lemma which does not hold for ordinary cohomology groups.

Lemma. *If a space X is the union of a directed set of open subspaces X_ι with the property that each compact set in X is contained in some X_ι, the natural map*

$$\varinjlim H^k_c(X_\iota; R) \to H^k_c(X; R)$$

is an isomorphism.

Proof. The definition of natural homomorphism in the lemma is based on the following fact: Let U be an open subset in V. For any compact set $K \subset U$ the inclusion $(U, U - K) \hookrightarrow (V, V - K)$ induces by excision an isomorphism

$$H^k(V|K; R) \to H^k(U|K; R).$$

Its inverse can be composed with natural homomorphism $H^k(V|K; R) \to H^k_c(V; R)$. By the universal property of direct sum there is just one homomorphism

$$H^k_c(U; R) \to H^k_c(V; R).$$

So on inclusions of open sets H_c^k behaves as covariant functor and this makes the definition of the natural homomorphism in the lemma possible. The proof that it is an isomorphism (based on excision) is left to the reader. $\square$

9.8. Generalized Poincaré duality. Let M be an R-orientable manifold of dimension n. Let $K \subseteq M$ be compact. Let $\mu_K \in H_n(M|K;R)$ be such a class that its reduction to $H_n(M|x;R)$ gives a generator for each $x \in K$. The existence of such a class is ensured by Lemma in 9.3. Define

$$D_K : H^k(M|K) \to H_{n-k}(M;R) : \quad D_K(\varphi) = \mu_K \cap \varphi.$$

If $K \subset L$ are two compact subsets of M, we can easily prove using naturality of cap product that

$$D_L(\rho^*\varphi) = D_K(\varphi)$$

for $\varphi \in H^k(M|K;R)$ and $\rho : (M, M-L) \hookrightarrow (M, M-K)$. It enables us to define the *generalized duality map*

$$D_M : H_c^k(M;R) \to H_{n-k}(M;R) : \quad D_M(\varphi) = \mu_K \cap \varphi$$

since each element $\varphi \in H_c^k(M;R)$ is contained in $H^k(M|K;R)$ for some compact set $K \subseteq M$.

Theorem (Duality for all orientable manifolds). *If M is an R-orientable manifold of dimension n, then the duality map*

$$D_M : H_c^k(M;R) \to H_{n-k}(M;R)$$

is an isomorphism.

The proof is based on the following

Lemma. *If a manifolds M be a union of two open subsets U and V, the following diagram of Mayer-Vietoris sequences*

$$\begin{array}{ccccccc}
H_c^k(U\cap V) & \longrightarrow & H_c^k(U)\oplus H_c^k(V) & \longrightarrow & H_c^k(M) & \longrightarrow & H_c^{k+1}(U\cap V) \\
\downarrow{\scriptstyle D_{U\cap V}} & & \downarrow{\scriptstyle D_U\oplus D_V} & & \downarrow{\scriptstyle D_M} & & \downarrow{\scriptstyle D_{U\cap V}} \\
H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U)\oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \longrightarrow & H_{n-k-1}(U\cap V)
\end{array}$$

commutes up to signs.

The proof of this lemma is analogous as the proof of commutativity of the diagram in the proof of Theorem 8.4 on Thom isomorphism. So we omit it referring the reader to [Hatcher], Lemma 3.36, pages 246 – 247 or to [Bredon], Chapter VI, Lemma 8.2, pages 350 – 351.

Proof of Poincaré Duality Theorem. We will use the following two statements

(A) If $M = U \cup V$ where U and V are open subsets such that D_U, D_V and $D_{U\cap V}$ are isomorphisms, then D_M is also an isomorphism.

(B) If $M = \bigcup_{i=1}^{\infty} U_i$ where U_i are open subsets such that $U_1 \subset U_2 \subset U_3 \subset \dots$ and all D_{U_i} are isomorphisms, then D_M is also an isomorphism.

The former is an immediate consequence of the previous lemma and Five Lemma. To obtain the latter apply the direct limit to the short exact sequences

$$0 \to H^k_c(U_i) \xrightarrow{D_{U_i}} H_{n-k}(U_i) \to 0$$

and use the lemmas in 9.6 and 9.7. The proof of Duality Theorem will be carried out in four steps.
(1) For $M = \mathbb{R}^n$ we have

$$H^k_c(\mathbb{R}^n) \cong H^k(\Delta^n, \partial\Delta^n), \quad H_n(\mathbb{R}^n|\Delta^n) \cong H_n(\Delta^n, \partial\Delta^n).$$

Take the generator $\mu \in H_n(\Delta^n, \partial\Delta^n)$ represented by the singular simplex given by identity. The only nontriavial case is $k = n$. In this case for a generator

$$\varphi \in H^n((\Delta^n \partial\Delta^n)) = \mathrm{Hom}(H_n((\Delta^n \partial\Delta^n), R)$$

we get $\mu \cap \varphi = \varphi(\mu) = \pm 1$. So the duality map is an isomorphism.
(2) Let $M \subset \mathbb{R}^n$ be open. Then M is a countable union of open convex sets V_i which are homeomorphic to $\mathbb{R}^n$. Using the previous step and induction in statement (A) we show that the duality map is an isomorphism for every finite union of V_i. The application of statement (B) yields that the duality map D_M is an isomorphism as well.
(3) Let M be a manifold which is a countable union of open sets U_i which are homeomorphic to $\mathbb{R}^n$. Now we can proceed in the same way as in (2) using its result instead of the result in (1).
(4) For general M we have to use Zorn lemma. See [Hatcher], page 248. □

Corollary. *The Euler characteristic of a closed manifold of odd dimension is zero.*

Proof. For M orientable we get from Poincaré duality and the universal coefficient theorem that

$$\begin{aligned} \operatorname{rank} H_{n-k}(M;\mathbb{Z}) &= \operatorname{rank} H^k(M;\mathbb{Z}) = \operatorname{rank} \mathrm{Hom}\, H_k(M;Z) \\ &= \operatorname{rank} H_k(M;\mathbb{Z}) \end{aligned}$$

Hence $\chi(M) = \sum_{i=0}^n (-1)^i \operatorname{rank} H_i(M;\mathbb{Z}) = 0$ for n odd.

If M is not orientable, we get from the Poincaré duality with $\mathbb{Z}_2$ coefficients that

$$\sum_{i=0}^n (-1)^i \dim H_i(M;\mathbb{Z}_2) = 0.$$

Here the dimension is considered over $\mathbb{Z}_2$. Applying the universal coefficient theorem one can show that the expression on the left hand side equals to $\chi(M)$. See [Hatcher], page 249. □

Remark. Consider an oriented closed smooth manifold M. The orientation of the manifold induces an orientation of the tangent bundle τ_M and we get the following relation between the Euler class of τ_M, the fundamental class of M and the Euler characteristic of M:

$$\chi(M) = e(\tau_M) \cap [M].$$

Particulary, for spheres of even dimension we get that the Euler class of their tangent bundle is twice a generator of $H^n(S^n;\mathbb{Z})$. For the proof see [MS], Corollary 11.12.

9.9. Duality and cup product. One can easily show that for $\alpha \in C_n(X;R)$, $\varphi \in C^k(X;R)$ and $\psi \in C^{n-k}(X;R)$ we have

$$\psi(\alpha \cap \varphi) = (\varphi \cup \psi)(\alpha).$$

For a closed R-orientable manifold M we define bilinear form

$$(*) \qquad H^k(M;R) \times H^{n-k}(M;R) \to R: \ (\varphi,\psi) \mapsto (\varphi \cup \psi)[M].$$

A bilinear form $A \times B \to R$ is called *regular* if induced linear maps $A \to \operatorname{Hom}(B,R)$ and $B \to \operatorname{Hom}(A,R)$ are isomorphisms.

Theorem. *Let M be a closed R-orientable manifold. If R is a field, then the bilinear form $(*)$ is regular.*
If $R = \mathbb{Z}$, then the bilinar form

$$H^k(M;\mathbb{Z})/\operatorname{Torsion} H^k(M;\mathbb{Z}) \times H^{n-k}(M;\mathbb{Z})/\operatorname{Torsion} H^{n-k}(M;\mathbb{Z}) \to \mathbb{Z}$$

induced by $(*)$ is regular.

Proof. Consider the homomorphism

$$H^{n-k}(M;R) \xrightarrow{h} \operatorname{Hom}(H_{n-k}(M;R);R) \xrightarrow{D^*} \operatorname{Hom}(H^k(M;R),R).$$

Here $h(\psi)(\beta) = \psi(\beta)$ for $\beta \in H_{n-k}(M;R)$ and $\psi \in H^{n-k}(M;R)$ and D^* is the dual map to duality. The homomorphism h is an isomorphism by the universal coefficient theorem and D^* is an isomorphism since so is D. Now it suffices to prove that the composition D^*h is the homomorphism induced from the bilinear form $(*)$. For $\psi \in H^{n-k}(M;R)$ and $\varphi \in H^k(M;R)$ we get

$$(D^*h(\psi))(\varphi) = (h(\psi))\, D(\varphi) = (h(\psi))([M] \cap \varphi) = \psi([M] \cap \varphi) = (\varphi \cup \psi)[M].$$

$\square$

This theorem gives us a further tool for computing the cup product structure in cohomology of closed manifolds.

Corollary. *Let M be a closed orientable manifold of dimension n. Then for every $\varphi \in H^k(M;\mathbb{Z})$ of infinite order which is not of the form $\varphi = m\varphi_1$ for $m > 1$, there is $\psi \in H^{n-k}(M;\mathbb{Z})$ such that $\varphi \cup \psi$ is a generator of $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$.*

Example. We will prove by induction that $H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[\omega]/\langle \omega^{n+1} \rangle$ where $\omega \in H^2(\mathbb{CP}^n;\mathbb{Z})$ is a generator. For $n = 1$ the statement is clear. Suppose that it holds for $n-1$. From the long exact sequence for the pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ we get that

$$H^i(\mathbb{CP}^n;\mathbb{Z}) \cong H^i(\mathbb{CP}^{n-1};\mathbb{Z})$$

for $i \leq 2n-1$. Now, using the consequence above for $\varphi = \omega$ we obtain that ω^n is a generator of $H^{2n}(\mathbb{CP}^n;\mathbb{Z})$.

9.10. Manifolds with boundary. A *manifold with boundary* of dimension n is a Hausdorff space M in which each point has an open neighbourhood homeomorphic either to $\mathbb{R}^n$ or to the half-space

$$\mathbb{R}^n_+ = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n;\ x_n \geq 0\}.$$

The boundary ∂M of the manifold M is formed by points which have all neighbourhoods of the second type. The boundary of a manifold of dimension n is a manifold of dimension $n-1$. In a similar way as for a manifold we can define orientation of a manifold with boundary and its fundamental class $[M] \in H_n(M; \partial M; R)$.

Theorem. *Suppose that M is a compact R-orientable n-dimensional manifold whose boundary ∂M is decomposed as a union of two compact $(n-1)$-dimensional manifolds A and B with common boundary $\partial A = \partial B = A \cap B$. Then the cap product with the fundamental class $[M] \in H_n(M, \partial M; R)$ gives the isomorphism*

$$D_M : H^k(M, A; R) \to H_{n-k}(M, B; R).$$

For the proof and many other applications of Poincaré duality we refer to [Hatcher], Theorem 3.43 and pages 250 – 254, and [Bredon], Chapter VI, Sections 9 and 10, pages 355 – 366.

9.11. Alexander duality. In this paragraph we introduce another version of duality.

Theorem (Alexander duality). *If K is a proper compact subset of S^n which is a deformation retract of an open neighbourhood, then*

$$\tilde{H}_i(S^n - K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z}).$$

Proof. For $i \neq 0$ and U a neighbourhood of K we have

$$\begin{aligned}
H_i(S^n - K) &\cong H_c^{n-i}(S^n - K) && \text{by Poincaré duality}\\
&\cong \varinjlim{}_U H^{n-i}(S^n - K, U - K) && \text{by definition}\\
&\cong \varinjlim{}_U H^{n-i}(S^n, U) && \text{by excision}\\
&\cong \varinjlim{}_U \tilde{H}^{n-i-1}(U) && \text{connecting homomorphism}\\
&\cong \tilde{H}^{n-i-1}(K) && K \text{ is a def. retract of some } U
\end{aligned}$$

First three isomorphisms are natural and exist also for $i = 0$. So using these facts we have

$$\begin{aligned}
\tilde{H}_0(S^n - K) &\cong \operatorname{Ker}\left(H_0(S^n - K) \to H_0(\mathrm{pt})\right)\\
&\cong \operatorname{Ker}\left(H_0(S^n - K) \to H_0(S^n)\right)\\
&\cong \operatorname{Ker}\left(\varinjlim H^n(S^n, U) \to H^n(S^n)\right)\\
&\cong \varinjlim \operatorname{Ker}\left(H^n(S^n, U) \to H^n(S^n)\right)\\
&\cong \varinjlim H^{n-1}(U) = H^{n-1}(K).
\end{aligned}$$

$\square$

Corollary. *A closed nonorientable manifold of dimension n cannot be embedded as a subspace into* $\mathbb{R}^{n+1}$.

Proof. Suppose that M can be embedded into $\mathbb{R}^{n+1}$. Then it can be embedded also in S^{n+1}. By Alexander duality

$$H_{n-1}(M;\mathbb{Z}) \cong H^1(S^{n+1} - M;\mathbb{Z}).$$

According to the universal coefficient theorem

$$H^1(S^{n+1} - M;\mathbb{Z}) \cong \mathrm{Hom}(H_1(S^{n+1} - M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_0(S^{n+1} - M;\mathbb{Z}))$$

is a free Abelian group. On the other hand

$$\mathbb{Z}_2 = H_n(M;\mathbb{Z}_2) \cong H_n(M;\mathbb{Z}) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}(H_{n-1}(M,\mathbb{Z}),\mathbb{Z}_2).$$

According to (b) of Theorem 9.3 the tensor product has to be zero, and since $H_{n-1}(M;\mathbb{Z})$ is free, the second summand has to be also zero, which is a contradiction. $\square$

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