INTRODUCTION TO ALGEBRAIC TOPOLOGY

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10. Homotopy groups

In this section we will define homotopy groups and derive their basic properties. While the definition of homotopy groups is relatively simple, their computation is complicated in general.

10.1. Homotopy groups. Let I^n be the n-dimensional unit cube and ∂I^n its boundary. For $n = 0$ we take I^0 to be one point and ∂I^0 to be empty. Consider a space X with a basepoint x_0. Maps $(I^n, \partial I^n) \to (X, x_0)$ are the same as the maps of the quotient $(S^n = I^n/\partial I^n, s_0 = \partial I^n/\partial I^n) \to (X, x_0)$. We define the n-th *homotopy group* of the space X with the basepoint x_0 as

$$\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)] = [(I^n, \partial I^n), (X, x_0)].$$

$\pi_0(X, x_0)$ is the set of path connected components of X with the component containing x_0 as a distinguished element. For $n \geq 1$ we can introduce a sum operation on $\pi_n(X, x_0)$

$$(f+g)(t_1, t_2, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & t_1 \in [0, \frac{1}{2}], \\ g(2t_1 - 1, t_2, \dots, t_n) & t_1 \in [\frac{1}{2}, 1]. \end{cases}$$

This operation is well defined on homotopy classes. It is easy to show that $\pi_n(X, x_0)$ is a group with identity element represented by the constant map to x_0 and with the inverse represented by

$$-f(t_1, t_2, \dots, t_n) = f(1 - t_1, t_2, \dots, t_n).$$

For $n \geq 2$ the groups $\pi_n(X, x_0)$ are commutative. The proof is indicated by the following pictures.

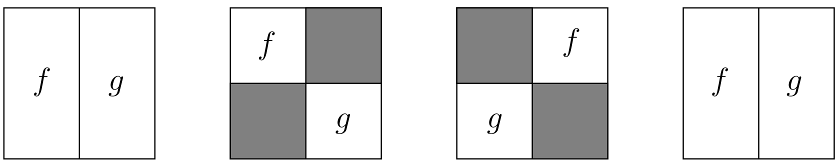

Figure 10.1. $f + g \sim g + f$

In the interpretation of $\pi_n(X, x_0)$ as $[(S^n, s_0), (X, x_0)]$, the sum $f + g$ is the composition

$$S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X$$

where c collapses the equator S^{n-1} of S^n to a point $s_0 \in S^{n-1} \subset S^n$.

Any map $F : (X, x_0) \to (Y, y_0)$ induces the homomorphism $F_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ by composition

$$F_*([f]) = [Ff].$$

Hence π_n is a functor from Top_* to the category of Abelian groups Ab for $n \geq 2$, to the category of groups G for $n = 1$ and to the category of sets with distiguished element Set_* for $n = 0$.

10.2. Relative homotopy groups. Consider I^{n-1} as a face of I^n with the last coordinate $t_n = 0$. Denote J^{n-1} the closure of $\partial I^n - I^{n-1}$. Let (X, A) be a pair with basepoint $x_0 \in A$. For $n \geq 1$ we define the n-th *relative homotopy group of the pair* (X, A) as

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}, s_0), (X, A, x_0)] = [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)].$$

A sum operation on $\pi_n(X, A, x_0)$ is defined by the same formula as for $\pi_n(X, x_0)$ only for $n \geq 2$. (Explain why this definition does not work for $n = 1$.) Similarly as in the case of absolute homotopy groups one can show that $\pi_n(X, A, x_0)$ is a group for $n \geq 2$ which is commutative if $n \geq 3$.

Sometimes it is useful to know how the representatives of zero (neutral element) in $\pi_n(X, A, x_0)$ look like. We say that two maps $f, g : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ are *homotopic rel* S^{n-1} if there is a homotopy h between f and g such that $h(x, t) = f(x) = g(x)$ for all $x \in S^{n-1}$ and all $t \in I$.

Proposition. *A map $f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ iff it is homotopic rel S^{n-1} to a map with image in A.*

Proof. Suppose that $f \sim g$ rel S^{n-1} and $g(D^n) \subseteq A$. Then $g = g \circ \mathrm{id}_{D^n}$ is homotopic to the constant map $g \circ \mathrm{const}$ into $x_0 \in A$. Hence $[f] = [g] = 0$.

Let f be homotopic to the constant map via homotopy $h : D^n \times I \to X$. Have a look at the picture and consider the subset

$$C = \{(x, t) \in D^n \times I;\ 2\|x\| \leq 2 - t\}$$

of $D^n \times I$ simultaneously with a vertical retraction $r : D^n \times I \to C$ and a horisontal homeomorphism $q : C \to D^n \times I$.

The maps can be defined in the following way:

$$r(x, t) = \begin{cases} (x, t) & \text{for } 2\|x\| \leq 2 - t, \\ (x, 2(1 - \|x\|) & \text{for } 2\|x\| \geq 2 - t \end{cases}$$

and

$$q(x, t) = \left(\frac{2}{2 - t} x, t \right).$$

Now $H = h \circ q \circ r : D^n \times I \to X$ is a homotopy between $H(x, 0) = h(x, 0) = f(x)$ and $H(x, 1) = g(x)$ where

$$g(D^n) = H(D^n \times I) = h(D^n \times \{1\} \cup S^{n-1} \times I) \subseteq A$$

and H is a homotopy rel S^{n-1}. □

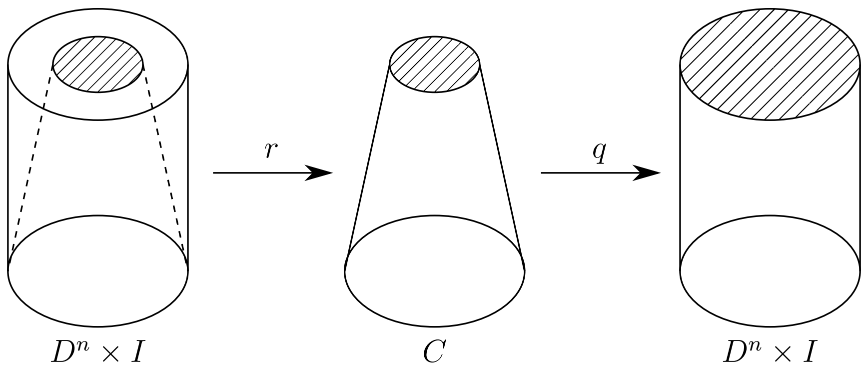

FIGURE 10.2. Retraction r and homeomorphism q

A map $F : (X, A, x_0) \to (Y, B, y_0)$ induces again the homomorphism $F_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$. Since $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$ the functor π_n on Top_* can be extended to a functor from Top_*^2 to Abelian groups Ab for $n \geq 3$, to the category of groups G for $n = 2$ and to the category Set_* of sets with distinguished element for $n = 1$.

From definitions it is clear that homotopic maps induce the same homomorphisms between homotopy groups. Hence homotopy equivalent spaces have the same homotopy groups. Particularly, contractible spaces have trivial homotopy groups.

10.3. Long exact sequence of a pair. Relative homotopy groups fit into the following long exact sequence of a pair.

Theorem. *Let (X, A) be a pair of spaces with a distinguished point $x_0 \in A$. Then the sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\delta} \pi_{n-1}(A, x_0) \to \dots$$

where $i : A \hookrightarrow X$, $j : (X, x_0) \hookrightarrow (X, A)$ are inclusions and δ comes from restriction, is exact.

More generally, any triple $B \subseteq A \subseteq X$ induces the long exact sequence

$$\cdots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\delta} \pi_{n-1}(A, B, x_0) \to \dots$$

Proof. We will prove only the version for the pair (X, A). δ is defined on $[f] \in \pi_n(X, A, x_0)$ by

$$\delta[f] = [f/I^{n-1}].$$

Exactness in $\pi_n(X, x_0)$. According to the previous proposition $j_* i_* = 0$, hence $\operatorname{Im} i_* \subseteq \operatorname{Ker} j_*$. Let $[f] \in \operatorname{Ker} j_*$ for $f : (I^n, \partial I^n) \to (X, x_0)$. Using again the previous proposition $f \sim g \text{ rel } \partial I^n$ where $g : I^n \to A$. Hence $[f] = i_*[g]$.

Exactness in $\pi_n(X, A, x_0)$. $\delta j_* = 0$, hence $\operatorname{Im} j_* \subseteq \operatorname{Ker} \delta$. Let $[f] \in \operatorname{Ker} \delta$, i. e. $f(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ and $f/I^{n-1} \sim \text{const}$. Then according to HEP there is $f_1 : (I^n, \partial I^n, J^n) \to (X, x_0, x_0)$ homotopic to f. Therefore $[f_1] \in \pi_n(X, x_0)$ and $[f] = j_*[f_1]$.

Exactness in $\pi_n(A, x_0)$. Let $[F] \in \pi_{n+1}(X, A, x_0)$. Then $i \circ F/I^n : I^n \to X$ is a map homotopic to the constant map to x_0 through the homotopy F. (Draw a picture.)

Let $f : (I^n, \partial I^n) \to (A, x_0)$ and $f \sim 0$ through the homotopy $F : I^n \times I \to X$ such that $F(x, 0) = f(x) \in A$, $F/J^n = x_0$. Hence $[F] \in \pi_{n+1}(X, A, x_0)$ and $\delta[F] = [f]$. $\square$

Remark. The boundary operator for a triple (X, A, B) is the composition

$$\pi_n(X, A) \xrightarrow{\delta} \pi_n(A) \xrightarrow{j_*} \pi_{n-1}(A, B).$$

10.4. Changing basepoints. Let X be a space and $\gamma : I \to X$ a path connecting points x_0 and x_1. This path associates to $f : (I^n, \partial I^n) \to (X, x_1)$ a map $\gamma \cdot f : (I^n, \partial I^n) \to (X, x_0)$ by shrinking the domain of f to a smaller concentric cube in I^n and inserting the path γ on each radial segment in the shell between ∂I^n and the smaller cube.

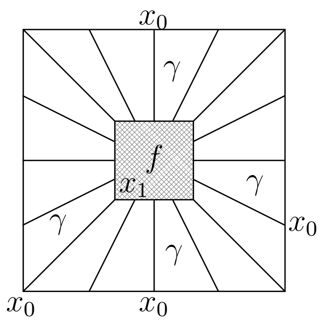

FIGURE 10.3. The action of γ on f

It is not difficult to prove that this assigment has the following properties:

(1) $\gamma \cdot (f + g) \sim \gamma \cdot f + \gamma \cdot g$ for $f, g : (I^n, \partial I^n) \to (X, x_1)$,

(2) $(\gamma + \kappa) \cdot f \sim \gamma \cdot (\kappa \cdot f)$ for $f : (I^n, \partial I^n) \to (X, x_2)$, $\gamma(0) = x_0$, $\gamma(1) = x_1 = \kappa(0)$, $\kappa(1) = x_2$.

(3) If $\gamma_1, \gamma_2 : I \to X$ are homotopic rel $\partial I = \{0, 1\}$, then $\gamma_1 \cdot f \sim \gamma_2 \cdot f$.

Hence, every path γ defines an isomorphism

$$\gamma : \pi_n(X, \gamma(1)) \to \pi_n(X, \gamma(0)).$$

Particulary, we have a left *action of the group* $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

10.5. Fibrations. Fibration is a dual notion to cofibration. (See 1.7.) It plays an important role in homotopy theory.

A map $p : E \to B$ has the *homotopy lifting property*, shortly HLP, with respect to a pair (X, A) if the following commutative diagram can be completed by a map $X \times I \to E$

$$\begin{tikzcd}
X \times \{0\} \cup A \times I \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\
X \times I \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

A map $p : E \to B$ is called a *fibration* (sometimes also Serre fibration or weak fibration), if it has the homotopy lifting property with respect to all disks $(D^k, \emptyset)$.

Theorem. *If $p : E \to B$ is a fibration, then it has homotopy lifting property with respect to all pairs of CW-complexes (X, A).*

Proof. The proof can be carried out by induction from $(k-1)$-skeleton to k-skeleton similarly as in the proof of Theorem 2.7 if we show that $p : E \to B$ has the homotopy lifting property with respect to the pair $(D^k, \partial D^k = S^{k-1})$. The HLP for this pair follows from the fact that the pair $(D^k \times I, D^k \times \{0\} \cup S^{k-1} \times I)$ is homeomorphic to the pair $(D^k \times I, D^k \times \{0\})$, see the picture below, and the fact that p has homotopy lifting property with respect to the pair $(D^k, \emptyset)$.

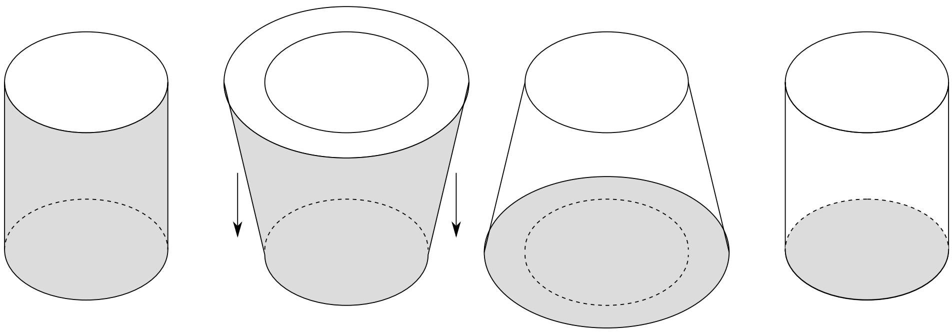

FIGURE 10.4. Homeomorphism $(D^n \times I, D^n \times \{0\} \cup S^n \times I) \to (D^n \times I, D^n \times \{0\})$

$\square$

Proposition. *Every fibre bundle (E, B, p) is a fibration.*

Proof. For the definition of a fibre bundle see 8.1. Let U_α be an open covering of B with trivializations $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$. We would like to define a lift of a homotopy $G : I^k \times I \to B$. (We have replaced D^k by I^k.) The compactness of $I^k \times I$ implies the existence of a division

$$0 = t_0 < t_1 < \cdots < t_m = 1, \quad I_j = [t_{j-1}, t_j],$$

such that $G(I_{j_1} \times \cdots \times I_{j_{k+1}})$ lies in some U_α. Now we make a lift $H : I^k \times I \to E$ of G, first on $(I_1)^{k+1}$ and then we add successively the other small cubes. We need retractions r of cubes $C \times I_{j_{k+1}} = \prod_{i=1}^{k+1} I_{j_i}$ to a suitable part of the boundary $C \times \{0\} \cup A \times I_{j_{k+1}}$ where H is already defined. A is a CW-subcomplex of the cube C and we are in the following situation

$$\begin{tikzcd} C \times \{0\} \cup A \times I \arrow[r, "g"] \arrow[d, "i"'] & U_\alpha \times F \arrow[d, "p_1"] \\ C \times I \arrow[r, "G"'] \arrow[ur, dashed, "H"] & U_\alpha \end{tikzcd}$$

Now, we can define

$$H(x,t) = (G(x,t), p_2 \circ g \circ r)(x,t)$$

where $p_2 : U_\alpha \times F \to F$ is a projection. $\square$

Example. Here you are several examples of fibre bundles.
(1) The projection $p : S^n \to \mathbb{RP}^n$ determines a fibre bundle with the fibre S^0.
(2) The projection $p : S^{2n+1} \to \mathbb{RC}^n$ determines a fibre bundle with the fibre S^1.
(3) The special case is so called Hopf fibration

$$S^1 \to S^3 \to \mathbb{CP}^1 = S^2.$$

(4) Similarly, as complex projective space we can define quaternionic projective space $\mathbb{HP}^n$. The definition determines the fibre bundle

$$S^3 \to S^{4n+3} \to \mathbb{HP}^n.$$

(5) The special case of the previous fibre bundle is the second Hopf fibration

$$S^3 \to S^7 \to \mathbb{HP}^1 = S^4.$$

(6) Similarly, the Cayley numbers enable to define another Hopf fibration

$$S^7 \to S^{15} \to S^8.$$

(7) Let H be a Lie subgroup of G. Then we get a fibre bundle given by the projection $p : G \to G/H$ with the fibre H.
(8) Let $n \geq k > l \geq 1$. Then the projection

$$p : V_{n,k} \to V_{n,l}, \quad p(v_1, v_2, \ldots, v_k) = (v_1, v_2, \ldots, v_l)$$

determines a fibre bundle with the fibre $V_{n-l,k-l}$.
(9) Natural projection $p : V_{n,k} \to G_{n,k}$ is a fibre bundle with the fibre $O(k)$.

10.6. Long exact sequence of a fibration. Consider a fibration $p : E \to B$. Take a basepoint $b_0 \in B$, put $F = p^{-1}(b_0)$ and choose $x_0 \in F$.

Lemma. *For all $n \geq 1$*

$$p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$$

is an isomorphism.

Proof. First, we show that p_* is an epimorphism. Consider $f : (I^n, \partial I^n) \to (B, b_0)$. Let $k : J^{n-1} \to E$ be the constant map into x_0. Since p is a fibration the commutative diagram

$$\begin{tikzcd} J^{n-1} = I^{n-1} \times \{1\} \cup \partial I^{n-1} \times I \arrow[r, "k"] \arrow[d] & E \arrow[d, "p"] \\ I^{n-1} \times I \arrow[r, "f"'] \arrow[ur, dashed, "g"] & B \end{tikzcd}$$

can be completed by $g : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$. Hence $p_*[g] = [f]$.

Now we prove that p_* is a monomorphism. Consider $f : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ such that $p_*[f] = 0$. Then there is a homotopy $G : (I^n \times I, \partial I^n \times I) \to (B, b_0)$ between

pf and the constant map into b_0. Denote the constant map into x_0 by k. Since p is a fibration, we complete the following commutative diagram:

$$\begin{tikzcd} J^{n-1} \times I \cup I^n \times \{0\} \cup I^n \times \{1\} \arrow[r, "k\cup f\cup k"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "G"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

by $H : (I^n \times I, \partial I^n \times I, J^{n-1} \times I) \to (E, B, x_0)$ which is a homotopy between f and the constant map k. □

The notion of exact sequence can be enlarged to groups and also to the category Set_* of sets with distinquished elements. Here we have to define $\operatorname{Ker} f = f^{-1}(b_0)$ for $f : (A, a_0) \to (B, b_0)$.

Theorem. *If $p : E \to B$ be a fibration with a fibre $F = p_{-1}(b_0)$, $x_0 \in F$ and B is path connected, then the sequence*

$$\cdots \to \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\delta} \pi_{n-1}(F, x_0) \to \dots$$
$$\cdots \to \pi_0(F) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B).$$

is exact.

Proof. Substitute the isomorphism $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ into the exact sequence for the pair (E, F). In this way we get the required exact sequence ending with

$$\cdots \to \pi_0(F, x_0) \to \pi_0(E, x_0).$$

We can prolong it by one term to the right. The exactness in $\pi_0(E, x_0)$ follows from the fact that every path in B ending in b_0 can be lifted to a path in E ending in F. □

The direct definition of $\delta : \pi_n(B, b_0) \to \pi_{n-1}(F, x_0)$ is given by

$$\delta[f] = [g/I^{n-1}]$$

where g is the lift in the diagram

$$\begin{tikzcd} J^{n-1} \arrow[r, "x_0"] \arrow[d] & E \arrow[d, "p"] \\ I^n \arrow[r, "f"'] \arrow[ur, dashed, "g"] & B \end{tikzcd}$$

Some applications of this long exact sequence to computations of homotopy groups will be given in Section 14.

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