INTRODUCTION TO ALGEBRAIC TOPOLOGY

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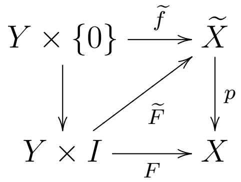
11. Fundamental group

The *fundamental group* of a space is the first homotopy group. In this section we describe two basic methods how to compute it.

11.1. Covering space. A *covering space* of a space X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ such that $(\widetilde{X}, X, p)$ is a fibre bundle with a discrete fibre.

In the previous section we have proved that every fibre bundle has homotopy lifting property with respect to CW-complexes. In the case of covering spaces the lifts of homotopies are unique:

Proposition. *Let $p : \widetilde{X} \to X$ be a covering space and let Y be a space. Given a homotopy $F : Y \times I \to X$ and a map $\widetilde{f} : Y \times \{0\} \to \widetilde{X}$ such that $F(-, 0) = p\widetilde{f}$, there is a unique homotopy $\widetilde{F} : Y \times I \to \widetilde{X}$ making the following diagram commutative:*



Proof. Since the proof follows the same lines as the proof of the analogous proposition in 10.5, we outline only the main steps.

1. Using compactness of I we show that for each $y \in Y$ there is a neighbourhood U such that $\widetilde{F}$ can be defined on $U \times I$.
2. $\widetilde{F}$ is uniquely determined on $\{y\} \times I$ for each $y \in Y$.
3. The lifts of F defined on $U_1 \times I$ and $U_2 \times I$ concide on $(U_1 \cap U_2) \times I$.

□

From the uniquiness of lifts of loops and their homotopies starting at a fixed point we get immediately the following

Corollary. *The group homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ induced by a covering space $(\widetilde{X}, X, p)$ is injective. The image subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$ consists of loops in X based at x_0 whose lifts in $\widetilde{X}$ starting at $\widetilde{x}_0$ are loops.*

11.2. Group actions. A *left action* of a discrete group G on a space Y is a map

$$G \times Y \to Y, \quad (g, y) \mapsto g \cdot y$$

such that $1 \cdot y = y$ and $(g_1 g_2) \cdot y = g_1 \cdot (g_2 \cdot y)$. We will call this action *properly discontinuous* if each point $y \in Y$ has an open neighbourhood U such that $g_1 U \cap g_2 U \neq \emptyset$ implies $g_1 = g_2$.

An action of a group G on a space Y induces the equivalence $x \sim y$ if $y = g \cdot x$ for some $g \in G$. The *orbit space* Y/G is the factor space $Y/\sim$.

A space Y is called *simply connected* if it is path connected and $\pi_1(Y, y_0)$ is trivial for some (and hence all) base point y_0.

The following theorem provides a useful method for computation of fundamental groups.

Theorem. *Let Y be a path connected space with a properly discontinuous action of a group G. Then*

(1) *The natural projection $p : Y \to Y/G$ is a covering space.*
(2) *$G \cong \pi_1(Y/G, p(y_0))/p_*\pi_1(Y, y_0)$. Particularly, if Y is simply connected, then $\pi_1(Y/G) \cong G$.*

Proof. Let $y \in Y$ and let U be a neighbourhood of y from the definition of properly discontinuous action. Then $p^{-1}(p(U))$ is a disjoint union of gU, $g \in G$. Hence $(Y, Y/G, p)$ is a fibre bundle with the fibre G.

Applying the long exact sequence of homotopy groups of this fibration we obtain

$$0 = \pi_1(G, 1) \to \pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(Y/G; p(y_0)) \xrightarrow{\delta} \pi_0(G) = G \to \pi_0(Y) = 0.$$

In general π_0 of a fibre is only the set with distinguished point. However, here it has the group structure given by G. Using the definition of δ from 10.3 one can check that δ is a group homomorphism. Consequently, the exact sequence implies that

$$G \cong \pi_1(Y/G, p(y_0))/p_*\pi_1(Y, y_0).$$

□

Example A. $\mathbb{Z}$ acts on real numbers $\mathbb{R}$ by addition. The orbit space is $\mathbb{R}/\mathbb{Z} = S^1$. According to the previous theorem

$$\pi_1(S^1, s) = \mathbb{Z}.$$

The fundamental group of the sphere S^n with $n \geq 2$ is trivial. The reason is that any loop $\gamma : S^1 \to S^n$ is homotopic to a loop which is not a map onto S^n and S^n without a point is contractible.

Next, the group $\mathbb{Z}_2 = \{1, -1\}$ has an action on S^n, $n \geq 2$ given by $(-1) \cdot x = -x$. Hence

$$\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2.$$

Example B. The abelian group $\mathbb{Z} \oplus \mathbb{Z}$ acts on $\mathbb{R}^2$

$$(m, n) \cdot (x, y) = (x + m, y + n).$$

The factor $\mathbb{R}^2/(\mathbb{Z} \oplus \mathbb{Z})$ is two dimensional torus $S^1 \times S^1$. Its fundamental group is $\mathbb{Z} \oplus \mathbb{Z}$.

Example C. The group G given by two generators α, β and the relation $\beta^{-1}\alpha\beta = \alpha^{-1}$ acts on $\mathbb{R}^2$ by

$$\alpha \cdot (x, y) = (x + 1, y), \quad \beta \cdot (x, y) = (1 - x, y + 1).$$

The factor $\mathbb{R}^2/G$ is the Klein bottle. Hence its fundamental group is G.

11.3. Free product of groups. As a set the *free product* $*_\alpha G_\alpha$ *of groups* G_α, $\alpha \in I$ is the set of finite sequences $g_1 g_2 \dots g_m$ such that $1 \neq g_i \in G_{\alpha_i}$, $\alpha_i \neq \alpha_{i+1}$, called words. The elements g_i are called letters. The group operation is given by

$$(g_1 g_2 \dots g_m) \cdot (h_1 h_2 \dots h_n) = (g_1 g_2 \dots g_m h_1 h_2 \dots h_n)$$

where we take $g_m h_1$ as a single letter $g_m \cdot h_1$ if both elements belong to the same group G_α. It is easy to show that $*_\alpha G_\alpha$ is a group with the empty word as the identity element. Moreover, for each $\beta \in I$ there is the natural inclusion $i_\beta : G_\beta \hookrightarrow *_\alpha G_\alpha$.

Up to isomorhism the free product of groups is characterized by the following universal property: Having a system of group homomorphism $h_\alpha : G_\alpha \to G$ there is just one group homomorphism $h : *_\alpha G_\alpha \to G$ such that $h_\alpha = h i_\alpha$.

Exercise. Describe $\mathbb{Z}_2 * \mathbb{Z}_2$.

11.4. Van Kampen Theorem. Suppose that a space X is a union of path connected open subsets U_α each of which contains a base point $x_0 \in X$. The inclusions $U_\alpha \hookrightarrow X$ induce homomorphisms $j_\alpha : \pi_1(U_\alpha) \to \pi_1(X)$ which determine a unique homomorphism $\varphi : *_\alpha \pi_1(U_\alpha) \to \pi_1(X)$.

Next, the inclusions $U_\alpha \cap U_\beta \hookrightarrow U_\alpha$ induce the homomorphisms $i_{\alpha\beta} : \pi_1(U_\alpha \cap U_\beta) \to \pi_1(U_\alpha)$. We have $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$. Consequently, the kernel of φ contains elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega^{-1})$ for any $\omega \in \pi_1(U_\alpha \cap U_\beta)$.

Van Kampen Theorem provides the full description of the homomorphism φ which enables us to compute $\pi_1(X)$ using groups $\pi_1(U_\alpha)$ and $\pi_1(U_\alpha \cap U_\beta)$.

Theorem (Van Kampen Theorem)**.** *If X is a union of path connected open sets U_α each containing a base point $x_0 \in X$ and if each intersection $U_\alpha \cap U_\beta$ is path connected, then the homomorhism $\varphi : *_\alpha \pi_1(U_\alpha) \to \pi_1(X)$ is surjective. If in addition each intersection $U_\alpha \cap U_\beta \cap U_\gamma$ is path connected, then the kernel of φ is the normal subgroup N in $*_\alpha \pi_1(U_\alpha)$ generated by elements $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega^{-1})$ for any $\omega \in \pi_1(U_\alpha \cap U_\beta)$. So φ induces an isomorphism*

$$\pi_1(X) \cong *_\alpha \pi_1(U_\alpha)/N.$$

Example. If X_α are path connected spaces, then

$$\pi_1(\bigvee X_\alpha) = *_\alpha \pi_1(X_\alpha).$$

Outline of the proof of Van Kampen Theorem. For simplicity we suppose that X is a union of only two open subsets U_1 and U_2.

Surjectivity of φ. Let $f : I \to X$ be a loop starting at $x_0 \in U_1 \cup U_2$. This loop is up to homotopy a composition of several paths, for simplicity suppose there are three such that $f_1 : I \to U_1$, $f_2 : I \to U_2$ and $f_3 : I \to U_1$ with end points succesively

$x_0, x_1, x_2, x_0 \in U_1 \cap U_2$. Since $U_1 \cap U_2$ is path connected there are paths $g_1 : I \to U_1 \cap U_2$ and $g_2 : I \to U_1 \cap U_2$ from x_0 to x_1 and x_2, respectively. Then the loop f is up to homotopy the composition of loops $f_1 - g_1 : I \to U_1$, $g_1 + f_2 - g_2 : I \to U_2$ and $g_2 + f_3 : I \to U_1$. Consequently, $[f] \in \pi_1(X)$ lies in the image of φ.

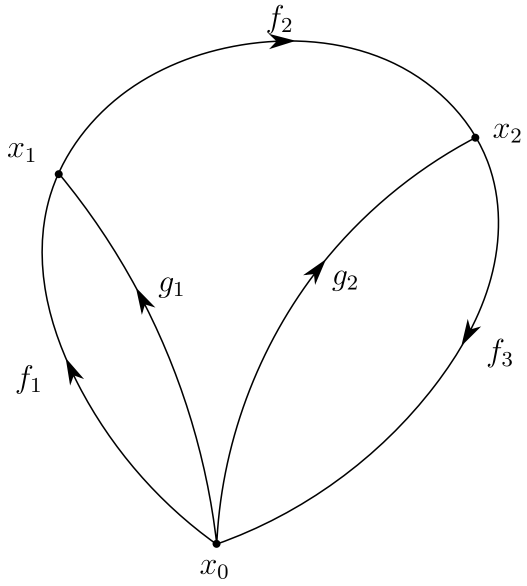


FIGURE 11.1. $[f] = [f_1 + f_2 + f_3] = [f_1 - g_1] + [g_1 + f_2 - g_2] + [g_2 + f_3]$

Kernel of φ. Suppose that the image under φ of a word with m letters $[f_1][g_1][f_2]\dots$, where $[f_i] \in \pi_1(U_1)$, $[g_i] \in \pi_1(U_2)$, is zero in $\pi_1(X)$. Then there is a homotopy $F : I \times I \to X$ such that

$$F(s,0) = f_1 + g_1 + f_2 + \dots, \quad F(s,1) = x_0, \quad F(0,t) = F(1,t) = x_0$$

where we suppose that f_i is defined on $[\frac{2i-2}{m}, \frac{2i-1}{m}]$ and g_i is defined on $[\frac{2i-1}{m}, \frac{2i}{m}]$. Since $I \times I$ is compact, there is an integer n, a multiple of m, such that

$$F\left(\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]\right)$$

is a subset in U_1 or U_2. Using homotopy extension property, we can construct a homotopy from F to $\widetilde{F}$ rel J^1 such that again

$$\widetilde{F}\left(\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]\right)$$

is a subset in U_1 or U_2, and moreover,

$$\widetilde{F}\left(\frac{i}{n}, \frac{j}{n}\right) = x_0.$$

Further, $\widetilde{F}(s,0) = f'_1 + g'_1 + f'_2 + \dots$ where $f'_i \sim f_i$, $g'_i \sim g_i$ in U_1 and U_2, respectively, rel the boundary of the domain of definition. We want to show that the word $[f'_1]_1[g'_1]_2[f'_2]_1 \dots$ belongs to N. Here $[\ ]_i$ stands for an element in $\pi_1(U_i)$.

We can decompose

$$I \times I = \bigcup_i M_i$$

where M_i is a maximal subset with the properties:

(1) M_i is a union of several squares $[\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}]$.
(2) $\operatorname{int} M_i$ is path connected.
(3) $\widetilde{F}(M_i)$ is a subset in U_1 or U_2.

For simplicity suppose that we have four sets M_i as indicated in the picture.

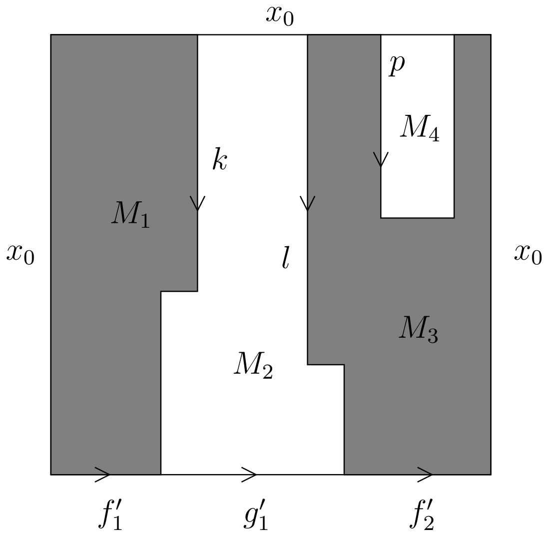


FIGURE 11.2. $[f_1']_1[g_1']_2[f_2']_1 \in \operatorname{Ker}\varphi$

In this situation there are three loops k, l and p starting at x_0 and lying in $U_1 \cap U_2$. They are defined by $\widetilde{F}$ on common boundary of M_1 and M_2, M_2 and M_3, M_3 and M_4, respectively. Now, we get

$$\begin{aligned}[f'_1]_1[g'_1]_2[f'_2]_1 &= [k]_1[-k+l]_2[-l+p]_1 = [k]_1[-k]_2[l]_2[-l]_1[p]_1 \\ &= [k]_1[-k]_2[l]_2[-l]_1 \in N.\end{aligned}$$

□

Corollary. *Let X be a union of two open subsets U and V where V is simply connected and $U \cap V$ is path connected. Then*

$$\pi_1(X) = \pi_1(U)/N$$

where N is the normal subgroup in $\pi_1(U)$ generated by the image of $\pi_1(U \cap V)$.

Exercise. Use the previous statement to compute the fundamental group of the Klein bottle and other 2-dimensional closed surfaces.

11.5. Fundamental group and homology. Here we compare the fundamental group of a space with the first homology group. We obtain a special case of Hurewitz theorem, see 13.6.

Theorem. *By regarding loops as 1-cycles, we obtain a homomorphism* $h : \pi_1(X, x_0) \to H_1(X)$*. If* X *is path connected, then* h *is surjective and its kernel is the commutator subgroup of* $\pi_1(X)$*. So* h *induces isomorphism from the abelization of* $\pi_1(X, x_0)$ *to* $H_1(X)$*.*

For the proof we refer to [Hatcher], Theorem 2A.1, pages 166–167.

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