INTRODUCTION TO ALGEBRAIC TOPOLOGY

MARTIN ČADEK

14. Short overview of some further methods in homotopy theory

We start this sections with two examples of computations of homotopy groups. These computations demonstrate the fact that the possibilities of the methods we have learnt so far are very restricted. Hence we outline some further (still very classical) methods which enable us to prove and compute more.

14.1. Homotopy groups of Stiefel manifolds. Let $n \geq 3$ and $n > k \geq 1$. The Stiefel manifold $V_{n,k}$ is $(n-k-1)$-connected and

$$\pi_{n-k}(V_{n,k}) = \begin{cases} \mathbb{Z} & \text{for } k = 1, \\ \mathbb{Z} & \text{for } k \neq 1 \text{ and } n-k \text{ even}, \\ \mathbb{Z}_2 & \text{for } k \neq 1 \text{ and } n-k \text{ odd}. \end{cases}$$

Proof. The statement about connectivity follows from the long exact sequence for the fibration

$$V_{n-1,k-1} \to V_{n,k} \to V_{n,1} = S^{n-1}$$

by induction.

As for the second statement, it is sufficient to prove that

$$\pi_{n-2}(V_{n,2}) = \begin{cases} \mathbb{Z} & \text{for } n \text{ even}, \\ \mathbb{Z}_2 & \text{for } n \text{ odd} \end{cases}$$

and to use the induction in the long exact sequence for the fibration above.

We have the fibration

$$S^{n-2} = V_{n-1,1} \to V_{n,2} \xrightarrow{p} V_{n,1} = S^{n-1}$$

which corresponds to the tangent vector bundle of the sphere S^{n-1}. If n is even, there is a nonzero vector field on S^{n-1}. This field is a map $s : S^{n-1} \to V_{n,2}$ such that $ps = \mathrm{id}_{S^{n-1}}$. Such a map is called a section and its existence ensures that the map $p_* : \pi_{n-1}(V_{n,2}) \to \pi_{n-1}(S^{n-1})$ is an epimorphism. Hence we get the following part of the long exact sequence

$$\pi_{n-1}(V_{n,2}) \xrightarrow{\text{epi}} \pi_{n-1}(S^{n-1}) \xrightarrow{0} \pi_{n-2}(S^{n-2}) \xrightarrow{\cong} \pi_{n-2}(V_{n,2}) \to 0.$$

Consequently, $\pi_{n-2}(V_{n,2}) = \mathbb{Z}$.

The case n odd is more complicated. We need the fact that the Euler class of tangent bundle of S^{n-1} is twice a generator $\iota \in H^{n-1}(S^{n-1})$. We obtain the following part of the Gysin exact sequence for cohomology groups with integer coefficients

$$0 \to H^{n-2}(V_{n,2}) \xrightarrow{0} H^0(S^{n-1}) \xrightarrow{\cup 2\iota} H^{n-1}(S^{n-1}) \to H^{n-1}(V_{n,2}) \to 0.$$

From this sequence and the universal coefficient theorem we get that

$$\begin{aligned} 0 = H^{n-2}(V_{n,2};\mathbb{Z}) &\cong \mathrm{Hom}(H_{n-2}(V_{n,2}),\mathbb{Z}) \\ \mathbb{Z}_2 \cong H^{n-1}(V_{n,2}) &\cong \mathrm{Hom}(H_{n-1}(V_{n,2}),\mathbb{Z}) \oplus \mathrm{Ext}(H_{n-2}(V_{n,2}),\mathbb{Z}) \end{aligned}$$

which implies that $H_{n-2}(V_{n,2};\mathbb{Z}) \cong \mathbb{Z}_2$. The Hurewicz theorem now yields $\pi_{n-1}(V_{n,2}) \cong \mathbb{Z}_2$. □

14.2. Hopf fibration. Consider the Hopf fibration

$$S^1 \to S^3 \xrightarrow{\eta} S^2$$

defined in 10.5. From the long exact sequence for this fibration we get

$$\pi_i(S^2) \cong \pi_i(S^3) \quad \text{for } i \geq 2.$$

Particularly,

$$\pi_3(S^2) \cong \mathbb{Z}$$

with $[\eta]$ as a generator (since [id] is a generator of $\pi_3(S^3)$). By the Freudenthal theorem $\mathbb{Z} \cong \pi_3(S^2) \xrightarrow{\text{epi}} \pi_4(S^3) \xrightarrow{\cong} \pi_1^s$. The methods we have learnt so far give us only that $\pi_4(S^3) \cong \pi_1^s$ is a factor of $\mathbb{Z}$ with $\Sigma\eta$ as a generator.

Exercise. Try to compute as much as possible from the long exact sequences for the other two Hopf fibrations in 10.5.

14.3. Composition methods were developed in works of I. James and the Japanese school of H. Toda in the 1950-ies and are described in the monograph [Toda]. They enable us to find maps which determine the generators of homotopy groups $\pi_{n+k}(S^n)$ for k not very big (approximately $k \leq 20$). For these purposes various types of compositions and products are used.

Having two maps $f : S^i \to S^n$ and $g : S^n \to S^m$ their composition $gf : S^i \to S^m$ determines an element $[gf] \in \pi_i(S^m)$ which depends only on $[f]$ and $[g]$. If the target of f is different from the source of g, we can use suitable multiple suspensions to be able to make compositions. For instance, if $f : S^6 \to S^4$ and $g : S^7 \to S^3$ we can make composition $g \circ (\Sigma^3 f) : S^9 \to S^3$. (Here Σ stands for reduced suspension.) In this way we get a bilinear map $\pi_a^s \times \pi_b^s \to \pi_{a+b}^s$.

More complicated tool is the *Toda bracket*. Consider three maps

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

preserving distinquished points such that $gf \sim 0$ and $hg \sim 0$. Then gf can be extended to a map $F : \widetilde{C}W \to Y$ and hg can be extended to a map $G : \widetilde{C}X \to Z$. ($\widetilde{C}$ stands for reduced cone.) Define $\langle f, g, h\rangle : \Sigma W = \widetilde{C}_+W \cup \widetilde{C}_-W \to Z$ as $G \circ \widetilde{C}f$ on $\widetilde{C}_+W$ and $h \circ F$ on $\widetilde{C}_-W$.

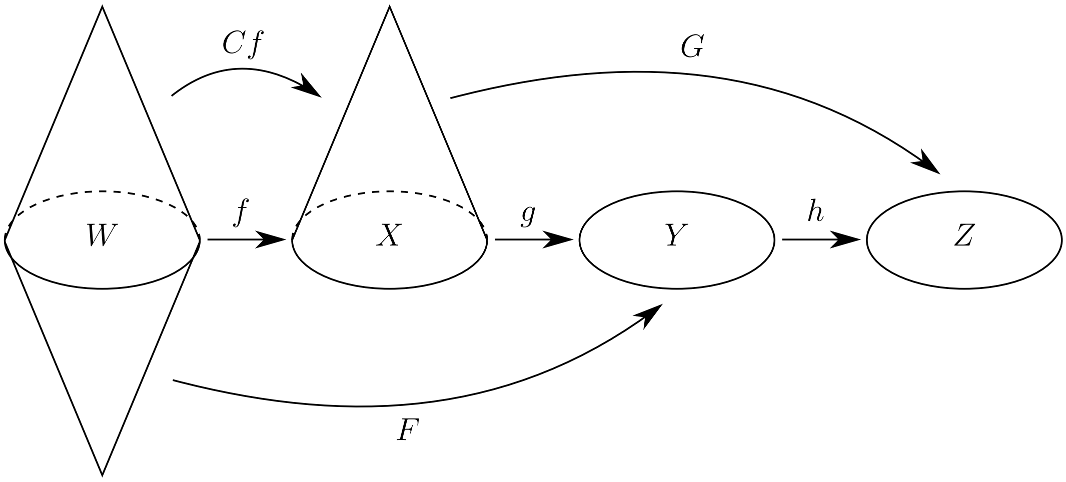

FIGURE 14.1. Definition of Toda bracket $< f, g, h >$.

This definition depends on homotopies $gf \sim 0$ and $hg \sim 0$. So it defines a map from $\pi_i^s \times \pi_j^s \times \pi_k^s$ to cosets of $\pi_{i+j+k+1}^s$. See [Toda] and also Exercise 39 in [Hatcher], Chapter 4.2.

The *Whitehead product* $[\ ,\ ] : \pi_i(X) \times \pi_j(X) \to \pi_{i+j-1}(X)$ is defined as follows: $f : I^i \to X$ and $g : I^j \to X$ define the map $f \times g : I^{i+j} = I^i \times I^j \to X$ and we put $[f, g] = (f \times g)/\partial I^{i+j}$.

Having a map $f : S^{2n-1} \to S^n$, $n \geq 2$, we can construct a CW-complex $C_f = S^n \cup_f e^{2n}$ with just one cell in the dimensions 0, n and $2n$. Denote the generators of $H^n(C_f; \mathbb{Z})$ and $H^{2n}(C_f; \mathbb{Z})$ by α and β, respectively. Then the *Hopf invariant* of f is the number $H(f)$ such that

$$\alpha^2 = H(f)\beta.$$

The Hopf invariant determines a homomorphism $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$.

For the Hopf map $\eta : S^3 \to S^2$ we have $C_\eta \cong \mathbb{CP}^2$, consequently

$$H(\eta) = 1.$$

For $\mathrm{id} : S^2 \to S^2$ we can make the Whitehead product $[\mathrm{id}, \mathrm{id}] : S^3 \to S^2$ and compute (see [Hatcher], page 474) that

$$H([\mathrm{id}, \mathrm{id}]) = \pm 2.$$

Since $\pi_3(S^2) \cong \mathbb{Z}$, we get $[\mathrm{id}, \mathrm{id}] = \pm 2\eta$. One can show (see [Hatcher], page 474 and Corollary 4J.4) that the kernel of the suspension homomorphism $\pi_3(S^2) \to \pi_4(S^3)$ is generated just by $[\mathrm{id}, \mathrm{id}]$. By the Freudental theorem this suspension homomorphism is an epimorphism which implies that

$$\pi_4(S^3) \cong \mathbb{Z}_2.$$

Consequently, $\pi_1^s \cong \mathbb{Z}_2$.

Remark. J. F. Adams proved in [Adams1] that the only maps with the odd Hopf invariant are the maps coming from the Hopf fibrations $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$.

Another important tool for composition methods is the EHP exact sequence for the homotopy groups of S^n, S^{n+1} and S^{2n}:

$$\pi_{3n-2}(S^n) \xrightarrow{E} \pi_{3n-1}(S^{n+1}) \xrightarrow{H} \pi_{3n-2}(S^{2n}) \xrightarrow{P} \pi_{3n-3}(S^n) \to \dots$$
$$\dots \to \pi_i(S^n) \xrightarrow{E} \pi_{i+1}(S^{n+1}) \xrightarrow{H} \pi_i(S^{2n}) \xrightarrow{P} \pi_{i-1}(S^n) \to \dots$$

Here E stands for suspension, H refers to a generalized Hopf invariant and P is defined with connection to the Whitehead product. See [Whitehead], Chapter XII or [Hatcher], page 474.

For $n = 2$ the EHP exact sequence yields

$$\pi_4(S^2) \xrightarrow{E} \pi_5(S^3) \xrightarrow{H} \pi_4(S^4) \xrightarrow{P} \pi_3(S^2) \xrightarrow{E} \pi_4(S^3) \to 0.$$

Since $\pi_4(S^3) \cong \mathbb{Z}_2$, $\pi_3(S^2) \cong \mathbb{Z}$ and $\pi_4(S^4) \cong \mathbb{Z}$, we obtain that P is a multiplication by 2 and $H = 0$. From the long exact sequence for the Hopf fibration (see 14.2) we get that $\pi_4(S^2) \cong \pi_4(S^3) \cong \mathbb{Z}_2$ with the generator $\eta(\Sigma\eta)$. So $\pi_5(S^3)$ is either $\mathbb{Z}_2$ or 0. By a different methods one can show that

$$\pi_5(S^3) \cong \mathbb{Z}_2$$

with the generator $(\Sigma\eta)(\Sigma^2\eta)$.

14.4. Cohomological methods have been playing an important role in homotopy theory since they were introduced in the 1950-ies.

By the methods used in proofs in Section 12 we can construct so called *Eilenberg-McLane spaces* $K(G, n)$ for any $n \geq 0$ and any group G, Abelian if $n \geq 2$. These spaces are up to homotopy equivalence uniquely determined by their homotopy groups

$$\pi_i(K(G,n)) = \begin{cases} 0 & \text{for } i \neq n, \\ G & \text{for } i = n. \end{cases}$$

Moreover, these spaces provide the following homotopy description of reduced singular cohomology groups

$$[(X,*),(K(G,n),*)] \xrightarrow{\cong} \widetilde{H}^n(X;G).$$

To each $[f] \in [(X,*),(K(G,n),*)]$ we assign

$$f^*(\iota) \in \widetilde{H}^n(X;G)$$

where ι is the generator of

$$\widetilde{H}^n(K(G,n);G) \cong \mathrm{Hom}(\widetilde{H}_n(K(G,n);\mathbb{Z}),G) \cong \mathrm{Hom}(G,G)$$

corresponding to id_G.

A system of homomorphisms $\theta_X : \widetilde{H}^n(X;G_1) \to \widetilde{H}^m(X;G_2)$ which is natural, i. e. $f^*\theta_Y = \theta_X f^*$ for all $f : X \to Y$, is called a *cohomology operation*. A system of cohomology operations $\theta_j : \widetilde{H}^{n+j} \to \widetilde{H}^{m+j}$ is called *stable* if it commutes with suspensions $\Sigma\theta_j = \theta_{j+1}\Sigma$.

The most important stable cohomology operations for singular cohomology are the Steenrod squares and the Steenrod powers:

$$\begin{aligned} Sq^i &: \widetilde{H}^n(X;\mathbb{Z}_2) \to \widetilde{H}^{n+i}(X;\mathbb{Z}_2) \\ P_p^i &: \widetilde{H}^n(X;\mathbb{Z}_p) \to \widetilde{H}^{n+2i(p-1)}(X;\mathbb{Z}_p) \quad \text{for } p \neq 2 \text{ a prime.} \end{aligned}$$

For their definition and properties see [SE] or [Hatcher], Section 4.L. These operations can be also interpreted as homotopy classes of maps between Eilenberg-McLane spaces, for instance

$$Sq^i : K(\mathbb{Z}_2, n) \to K(\mathbb{Z}_2, n+i).$$

Example A. We show how the Steenrod squares can be used to prove that some maps are not homotopic to a trivial one. Consider the Hopf map $\eta : S^3 \to S^2$. We know that $C_\eta = \mathbb{CP}^2$ and $H^2(\mathbb{CP}^2;\mathbb{Z}_2)$ and $H^4(\mathbb{CP}^2;\mathbb{Z}_2)$ have generators α and α^2. Since one of the properties of the Steenrod squares is

$$Sq^n x = x^2 \quad \text{for } x \in H^n(X;\mathbb{Z}_2),$$

we get that $Sq^2\alpha = \alpha^2 \neq 0$. Using this fact we show that $[\Sigma\eta] \in \pi_4(S^3)$ is nontrivial.

For reduced cones and reduced suspensions one can prove that

$$\widetilde{C}_{\Sigma\eta} = \Sigma\widetilde{C}_\eta \simeq \Sigma\mathbb{CP}^2.$$

Then $\Sigma\alpha : \Sigma\mathbb{CP}^2 \to K(\mathbb{Z}_2, 3)$ and $\Sigma\alpha^2 : \Sigma\mathbb{CP}^2 \to k(\mathbb{Z}_2; 5)$ represent generators in $H^3(\Sigma\mathbb{CP}^2;\mathbb{Z}_2)$ and $H^5(\Sigma\mathbb{CP}^2;\mathbb{Z}_2)$, respectively. Now

$$Sq^2(\Sigma\alpha) = \Sigma(Sq^2\alpha) = \Sigma\alpha^2 \neq 0.$$

If $\Sigma\eta$ were homotopic to a constant map, we would have $\widetilde{C}_{\Sigma\eta} = S^3 \vee S^5$, and consequently, $Sq^2(\Sigma\alpha) = 0$ since Sq^2 is trivial on S^3.

Example B. We outline how to compute $\pi_{n+1}(S^n)$ using cohomological methods. A generator $\alpha \in H^n(S^n)$ induces up to homotopy a map $S^n \to K(\mathbb{Z}, n)$. Further, $H^n(K(\mathbb{Z}, n);\mathbb{Z}) \cong \mathbb{Z}$ with a generator ι and $H^{n+2}(K(\mathbb{Z}, n);\mathbb{Z}_2) \cong \mathbb{Z}_2$ with the generator $Sq^2\rho\iota$ where $\rho : H^n(X;\mathbb{Z}) \to H^n(X;\mathbb{Z}_2)$ is induced by reduction mod 2. $Sq^2\rho\iota$ induces up to homotopy a map

$$K(\mathbb{Z}, n) \xrightarrow{Sq^2\rho\iota} K(\mathbb{Z}_2, n+2).$$

Consider the fibration

$$\Omega K(\mathbb{Z}_2, n+2) \to PK(\mathbb{Z}_2, n+2) \to K(\mathbb{Z}_2, n+2)$$

where PX is the space of all maps $p : I \to X$, $p(1) = x_0$ and ΩX is the space of all maps $\omega : I \to X$, $\omega(0) = \omega(1) = x_0$. (These maps are called loops in X.) One can show that $\Omega K(\mathbb{Z}_2, n+2)$ has a homotopy type of $K(\mathbb{Z}_2, n+1)$. The pullback of the fibration above by the map $Sq^2\rho\iota : K(\mathbb{Z}, n) \to K(\mathbb{Z}_2, n+2)$ is the fibration

$$K(\mathbb{Z}_2, n+1) \to E \xrightarrow{p} K(\mathbb{Z}, n).$$

Since $Sq^2\rho\alpha = 0$ in $H^{n+2}(S^n;\mathbb{Z})$, one can show that the map $\alpha : S^n \to K(\mathbb{Z}, n)$ can be lifted to a map $f : S^n \to E$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ S^n \arrow[ur, "f"] \arrow[r, "\alpha"'] & K(\mathbb{Z}, n) \end{tikzcd}$$

One can compute f^* in cohomology (using so called long Serre exact sequence) and then also f_* in homology. A modified version of the homology Whitehead theorem implies that f is an $(n+2)$-equivalence. Hence $f_* : \pi_{n+1}(S^n) \to \pi_{n+1}(E)$ is an isomorphism. Using the long exact sequence for the fibration $(E, K(\mathbb{Z}, n), p)$ we get

$$\mathbb{Z}_2 \cong \pi_{n+1}(K(\mathbb{Z}_2, n+1)) \xrightarrow{\cong} \pi_{n+1}(E) \cong \pi_{n+1}(S^n).$$

For more details see [MT].

The Steenrod operations form a beginning for the second course in algebraic topology which should contain spectral sequences, other homology and cohomology theories, spectra. We refer the reader to [Adams2], [Kochman], [MT], [Switzer], [Whitehead] or to the last sections of [Hatcher].

CZ.1.07/2.2.00/28.0041
Centrum interaktivních a multimediálních studijních opor pro inovaci výuky a efektivní učení

evropský sociální fond v ČR
EVROPSKÁ UNIE
MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY
OP Vzdělávání pro konkurenceschopnost
UNIVERSITAS MASARYKIANA BRUNENSIS

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ