HOMEWORK 2

Exercise 1. Finish the proof of the 5–lemma, i.e. prove that in the following commutative diagram of Abelian groups

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

where the rows are exact sequences and $a, b, , d, e$ are isomorphisms, the morphism c is an epimorphism.

Exercise 2. Given the short exact sequence of Abelian groups

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

the following conditions are equivalent:

(1) There exists $p\colon C \to B$ such that $j \circ p = id_C$.
(2) There exists $q\colon B \to A$ such that $q \circ i = id_A$.
(3) There are p, q as above such that $i \circ q + p \circ j = id_B$

We have shown that (1) $\Rightarrow$ (2) and (3). Prove that (2) $\Rightarrow$ (1) and (3)

Exercise 3. For the short exact sequence od chain complexes

$$0 \to A_* \to B_* \to C_* \to 0,$$

there is a long exact sequence of homology groups

$$\cdots \to H_{n+1}(C_*) \to H_n(A_*) \to H_n(B_*) \to H_n(C_*) \to H_{n-1}(A_*) \to \ldots$$

Prove the exactness in $H_n(C_*)$.