Exercise 1. *There is a lemma, that says: Given the following diagram, where rows are long exact sequences and m is an iso*

$$\begin{array}{ccccccccccc}
K_n & \xrightarrow{i} & L_n & \xrightarrow{j} & M_n & \xrightarrow{h} & K_{n-1} & \longrightarrow & L_{n-1} & \longrightarrow & M_{n-1} \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle m}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{K}_n & \xrightarrow{\overline{i}} & \overline{L}_n & \xrightarrow{\overline{j}} & \overline{M}_n & \longrightarrow & \overline{K}_{n-1} & \longrightarrow & \overline{L}_{n-1} & \longrightarrow & \overline{M}_{n-1}
\end{array}$$

we get a long exact sequence

$$K_n \xrightarrow{(i,f)} L_n \oplus \overline{K}_n \xrightarrow{g-\overline{i}} \overline{L}_n \xrightarrow{i \circ iso \circ \overline{j}} K_{n-1} \longrightarrow \cdots$$

We can denote $\partial = h \circ m^{-1} \circ \bar{j}$.
Show exactness in $L_n \oplus \bar{K}_n$ and also in $\bar{L}_n$.

Solution. We have $(g - \bar{i}) \circ (i, f) = \bar{i}f - gi = 0$ obviously. For $x \in L_n, y \in \bar{K}_n$ we have $(g - \bar{i})(x, y) = 0$, so $g(x) = \bar{i}(y)$. Now, let x be such that $j(x) = 0$, then there is $z \in K_n$ such that $i(z) = x$. Then, suppose $g(x) = a \in \bar{L}_n$, then by m being iso we know $\bar{j}(a) = 0$, so exists $y \in \bar{K}_n$ such that $\bar{i}(y) = a$. Since $f(z)$ and y have the same image, their difference has a preimage, i.e. exists $b \in \bar{M}_{n+1}$ such that $b \mapsto y - f(z)$. By iso then there exists $c \mapsto z$, or denote $h(c) = z$. Now, all of this is much easier with a picture (that I don't draw). Compute now:
$f(z + c) = f(z) + y - f(z) = y$ and $i(z + h(c)) = i(z) = x$, and we are done.

Exactness in $\bar{L}_n$ is easier. It holds $\partial \circ (g - \bar{i}) = 0$, so take $x \in \ker \partial$ (also, $x \in \bar{L}_n$). Now, $x \mapsto a$, by iso there is b in the upper row that maps to zero. Then there exists y such that $y \mapsto b$. Now we can work with $x - g(y)$. There exists also z such that, obviously, $z \mapsto x - g(y) \mapsto a - a = 0$. Get $x = g(y) + \bar{i}(z) = g(y) - \bar{i}(-z)$, that is we needed to express x as this difference, hence we are done. □

Exercise 2. *There is a long exact sequence of the triple (X, A, B), i.e. $(B \subseteq A \subseteq X)$:*

$$\cdots \to H_n(A, B) \xrightarrow{i} H_n(X, B) \xrightarrow{j_X} H_n(X, A) \xrightarrow{D_*} H_{n-1}(A, B) \to \cdots,$$

with $H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{j_A} H_{n-1}(A, B)$. We get this sequence from a special short exact sequence of chain complexes. Show that it is exact and that the triangle commutes, that is $D_* = j_A \circ \partial_*$.

Solution. The chain complex of a pair is a quotient. We have

$$0 \longrightarrow \frac{C_*(A)}{C_*(B)} \xrightarrow{i} \frac{C_*(X)}{C_*(B)} \xrightarrow{j} \frac{C_*(X)}{C_*(A)} \to 0.$$

Take $c \in C_*(A)$, then $ji[c] = j[ic] = [jic] = [c]$, but seen as a different class. So, $j \circ i = 0$ and inclusion im $i \subseteq \ker j$ holds. The other inclusion $\ker j \subseteq$ im i is obvious.

Analogous: $0 \to C_*(A) \to C_*(X) \to \dfrac{C_*(X)}{C_*(A)} \to 0$.

Now, show D_* equals the composition:

Note, if you are familiar with the definition, it's clear.

Take a chain complex in $C_*(X)$ with boundary 0, chain in $C_*(A)$, take preimage, boundary in $C_*(B)$, we have $[c] \in C_n(X, B)$ and its image $[c] \in C_n(X, A)$, same representative, but different equivalence. Take $\partial c \in C_{n-1}(X, B)$, it has preimage $[\partial c]_{A,B} \in C_{n-1}(A, B)$. (Drawing diagram and chase helps.) The equality of D_*, that it is composition, holds, basically thanks to j_A being inclusion. □

Exercise 3. *Apply previous exercise to the triple $(D^k, S^{k-1}, *)$, where $*$ is a point.*

Solution.

$$\cdots H_n(S^{k-1}, *) \to H_n(D^k, *) \to H_n(D^k, S^{k-1}) \xrightarrow{\varphi} H_{n-1}(S^{k-1}, *) \to H_{n-1}(D^k, *) \to \cdots,$$

and, note $H_n(*) = \mathbb{Z}$ for $n = 0$ and $H_n(*) = 0$ otherwise.

We also work with reduced homology groups: $\bar{H}_n(X) = H_n(X, x_0)$, $H_*(X) = \bar{H}_*(X) \oplus H_*(*)$. Since D^k is contractible, $H_n(D^k, *) = \bar{H}_n(D^k) = 0$ and $H_{n-1}(D^k, *) = 0$, so we have

$$\cdots H_n(S^{k-1}, *) \to 0 \to H_n(D^k, S^{k-1}) \xrightarrow{\varphi} H_{n-1}(S^{k-1}, *) \to 0 \to \cdots,$$

where φ is *iso*.

Reduced homology for pairs is the same as unreduced:

$$H_n(D^k, S^{k-1}) = \bar{H}_n(D^k, S^{k-1}) \cong \bar{H}_{n-1}(S^{k-1}).$$

With this we know that $H_n(D^k, S^{k-1}) = \bar{H}_n(D^k, S^{k-1}) = \mathbb{Z}$ for $n = k$ and it is 0 for $n \neq k$.

Note also, that $(D^k, S^{k-1}) \cong (\Delta^k, \partial\Delta^k)$, this might be useful later on. □

Exercise 4. *Show that the chain in $C_k(\Delta^k, \partial\Delta^k)$ given by $\mathrm{id}\colon \Delta^k \to \Delta^k$ is the representative of the generator of*

$$H_k(\Delta^k, \partial\Delta^k) \cong \mathbb{Z}.$$

(Use induction and the long exact sequence for triple.)

Solution. First denote $\wedge^{k-1}$ boundary without interior of one face. Then work with the triple $\Delta^k, \partial\Delta^k, \wedge^{k-1}$, so the sequence needed is as follows:

$$0 \to H_k(\Delta^k, \partial\Delta^k) = \mathbb{Z} \to H_{k-1}(\partial\Delta^k, \wedge^{k-1}) \to 0,$$

where we have the zeroes because $\Delta^k, \wedge^{k-1}$ are contractible to points. Use excision theorem, $H_*(X - C, A - C) \cong H_*(X, A)$, where C is the boundary with bottom cut out (imagine upper part of the letter Δ, i.e. triangle). We know that $H_{k-1}(\Delta^{k-1}, \partial\Delta^{k-1}) = \mathbb{Z}$ and this is isomorphic (by excision theorem) to $H_{k-1}(\partial\Delta^k, \wedge^{k-1})$. So, everything is $\mathbb{Z}$. We want to show that images of id_{k-1} and id_k are the same, this is actually the inductive step.

Suppose that the generator in $H_{k-1}(\Delta^{k-1}, \partial\Delta^{k-1})$ is given by $\mathrm{id}_{k-1}\colon \Delta^{k-1} \to \Delta^{k-1}$. Then $\mathrm{id}_k\colon \Delta^k \to \Delta^k$ is cycle again, represents element $[\mathrm{id}_k] \in H_k(\Delta^k, \partial\Delta^k)$.
The Beginning of the Induction *(coming to theaters this summer)*:

$$H_1([-1,1],\{-1,1\}) \xrightarrow{\partial_*} H_0(\{-1,1\},\{1\}),$$

take $\mathrm{id}\colon [-1,1] \to [-1,1]$ as a chain complex, $1-(-1) = [-1]$, $[-1]$ generator. □

Exercise 5. *Using the Mayor-Vietoris exact sequence compute the homology groups of the torus. (note: Vietoris died in 2002, aged 110, remarkable)*

Solution. It goes as union, intersection, pair and we will want to determine the union. First draw two disks with holes (these glued together give a torus). Call one interior of the disk A and the other B. We work with $X = A \cup B$, it is not a problem, that we need to work with A, B open, as from the point of view of homology it doesn't matter.

The sequence is

$$\cdots \to H_n(A\cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A\cap B) \to \cdots,$$

review: $X = A \cup B$ is torus, $A \cap B = S^1 \sqcup S^1$ disjoint union, $H_n(A) = H_n(B) = \mathbb{Z}$ for $n = 0,1$ and 0 otherwise, $H_n(A\cap B) = H_n(A) \oplus H_n(B) = \mathbb{Z}\oplus\mathbb{Z}$ for $n = 0,1$ and 0 otherwise. We can therefore continue with this sequence:

$$H_2(A)\oplus H_2(B) \to H_2(X) \to H_1(A\cap B) \xrightarrow{f} H_1(A)\oplus H_1(B) \to H_1(X) \to$$

$$\to H_0(A\cap B) \xrightarrow{g} H_0(A)\oplus H_0(B) \to H_0(X) \to 0,$$

and we can rewrite it as

$$0 \to H_2(X) \to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{f} \mathbb{Z}\oplus\mathbb{Z} \to H_1(X) \to$$

$$\to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{g} \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to 0,$$

where we use the fact, that torus is connected, so $H_0(X) = \mathbb{Z}$. Now we want to compute $H_2(X)$ and $H_1(X)$.

We know $H_2(X)$ is $\ker f$, $\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$, $(a,b) \mapsto (a+b, a+b)$, then $(a,-a)$ is in the kernel, so $H_2(X) = \mathbb{Z}[a,-a] = \mathbb{Z}$.

For the $H_1(X)$ group use the fact, that $\ker g$ is $\mathbb{Z}$ (it has same idea, basically). Now consider the sequence

$$0 \to \mathbb{Z} \hookrightarrow H_1(X) \to \ker g = \mathbb{Z} \to 0,$$

which splits, so $H_1(X) = \mathbb{Z}\oplus\mathbb{Z}$. We are done.

Sphere with two handels might be a homework.

□