Exercise 1 (Five lemma). *Let*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow\cong & & \downarrow\cong & & \downarrow f & & \downarrow\cong & & \downarrow\cong \\ \overline{A} & \longrightarrow & \overline{B} & \longrightarrow & \overline{C} & \longrightarrow & \overline{D} & \longrightarrow & \overline{E} \end{array}$$

be a commutative diagram of modules with exact rows. Show that the middle homomorphism f is an isomorphism.

Solution. We will show the surjectivity of f, injectivity is dual. Let $\overline{c} \in \overline{C}$ be arbitrary and suppose it maps to $\overline{d} \in \overline{D}$. This $\overline{d}$ corresponds to some $d \in D$, and since $\overline{d}$ maps to 0 in $\overline{E}$ by exactness, d also has to map to $0 \in E$ by commutativity. By exactness, there exists $c \in C$ which maps to d. Since $\overline{c}$ and $f(c)$ both map to $\overline{d}$ by commutativity, there is (by exactness) some $\overline{b} \in \overline{B}$ which maps to $f(c) - \overline{c}$. This $\overline{b}$ corresponds to some $b \in B$, which maps to some $c' \in C$. Then by commutativity, $f(c - c') = f(c) - (f(c) - \overline{c}) = \overline{c}$, as desired. (Note that instead of the four vertical maps being isomorphisms, we only needed the surjectivity of $B \to \overline{B}$, $D \to \overline{D}$ and the injectivity of $E \to \overline{E}$). □

Exercise 2. *Show that for a finite CW-complex X and $H^*(Y)$ being finitely generated free group in all dimensions, the cross product*

$$H^*(X) \otimes H^*(Y) \xrightarrow{\mu} H^*(X \times Y)$$

is an isomorphism. (In fact, the same is true for X being an infinite CW-complex.)

Solution. First let $X = pt$ be a point. Then $H^*(pt) = \mathbb{Z}$ with $1 \in H^0(pt)$ and $pt \times Y$ is homeomorphic to Y, hence

$$H^*(pt) \otimes H^*(Y) = \mathbb{Z} \otimes H^*(Y) \cong H^*(Y) \cong H^*(pt \times Y).$$

Now let $X = p_1 \sqcup p_2 \sqcup \cdots \sqcup p_k$ be a finite disjoint union of points (i.e, a discrete set). Then $H^*(X) = \bigotimes_{i=1}^k \mathbb{Z}$, hence

$$\begin{aligned} H^*(X) \otimes H^*(Y) = \left(\bigoplus_{i=1}^k \mathbb{Z} \right) \otimes H^*(Y) \cong \bigoplus_{i=1}^k (\mathbb{Z} \otimes H^*(Y)) \cong \bigoplus_{i=1}^k H^*(Y) \\ \cong H^*(\underbrace{Y \sqcup Y \sqcup \cdots \sqcup Y}_{n \text{ times}}) \cong H^*(X \times Y) \end{aligned}$$

(We should also show that the isomorphism is indeed given by μ, but if $e_1, \ldots, e_k \in H^0(X)$ are such that

$$e_i(p_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

it's not hard to see that

$$\mu(e_i \otimes a) = (0, 0, \ldots, \underbrace{a}_{i\text{-th place}}, 0, \ldots, 0)$$

using projections and the definition of the cup product.)

Now we can proceed inductively:

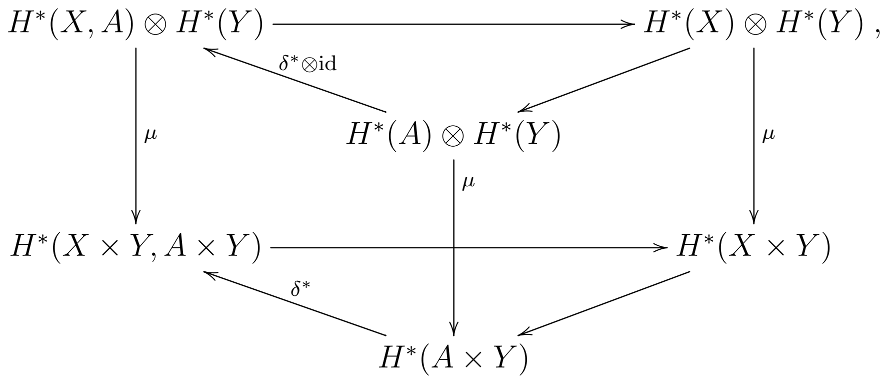
i) We now know that the theorem is true for X of dimension 0.

ii) Since D^n is homotopy equivalent to a point, the theorem is also true for $X = \bigsqcup_{\alpha=1}^k D^n_\alpha$.

iii) Suppose that the theorem holds for finite CW-complexes of dimension $n-1$. Then it is also true for pairs $(\bigsqcup D^n_\alpha, \bigsqcup S^{n-1}_\alpha)$.

iv) Since $H^*(X,A) \cong H^*(X/A)$ for a subcomplex $A \subseteq X$, the theorem also holds for $\bigvee S^n_\alpha \cong \bigsqcup D^n_\alpha / \bigsqcup S^{n-1}_\alpha$.

v) Now let $X = X^{(n)}$ be an n-dimensional CW-complex and consider the diagram



where the lower triangle represents a long exact cohomology sequence by definition, and the same is true for the upper triangle (since free modules are flat). We know that the theorem holds for $X^{(n-1)}$, and also for $X^{(n)}/X^{(n-1)} \cong \bigvee S^{n-1}_\alpha$. Therefore we can use the Five lemma after unfolding the diagram in the appropriate dimensions, from which it follows that the theorem also holds for $X^{(n)}$. This completes the induction.

□

Exercise 3. *Compute the cohomology rings of $\mathbb{C}P^2 \times S^6$ and $\mathbb{C}P^2 \vee S^6$.*

Solution. We have $H^*(\mathbb{C}P^2) = \mathbb{Z}[w]/\langle w^3 \rangle$ for $w \in H^2$ and $H^*(S^6) = \mathbb{Z}[a]/\langle a^2 \rangle$ for $a \in H^6$, hence

$$H^*(\mathbb{C}P^2 \times S^6) = \mathbb{Z}[w]/\langle w^3 \rangle \otimes \mathbb{Z}[a]/\langle a^2 \rangle \cong \mathbb{Z}[w,a]/\langle w^3, a^2 \rangle.$$

Next, it is true in general that $\overline{H}^*(X \wedge Y) \cong \overline{H}^*(X) \oplus \overline{H}^*(Y)$ is an isomoprhism of graded rings (this can be proven proven straight from the definitions, but it takes some time). Since $\mathbb{C}P^2 \vee S^6$ is connected, we have $H^*(\mathbb{C}P^2 \vee S^6) \cong \overline{H}^*(\mathbb{C}P^2) \oplus \overline{H}^*(S^6) \oplus \mathbb{Z}$. Now $w \cup a \in H^8 = 0$ (more generally, we could use that fact that $(w,0) \cup (0,a) = (0,0)$). Therefore

$$H^*(\mathbb{C}P^2 \vee S^6) \cong \mathbb{Z}[w,a]/\langle w^3, a^2, wa \rangle.$$

□

Exercise 4. *Show that the $\mathbb{C}P^2 \vee S^6$ is not homotopy equivalent to $\mathbb{C}P^3$.*

Solution. It suffices to show that the cohomology rings of these spaces are not isomorphic (note that the additive group structure is not enough to distinguish them). We already know that

$$H^*(\mathbb{C}P^2 \vee S^6) \cong \mathbb{Z}[w, a]/\langle w^3, a^2, wa\rangle$$

and we have $H^*(\mathbb{C}P^3) = \mathbb{Z}[b]/\langle b^4\rangle$ for $b \in H^2$. Any isomorphism would have to map w and b to $\pm$ each other (these are the respective generators in dimension 2), but $w^3 = 0$ while $b^3 \neq 0$, so this is not possible. □