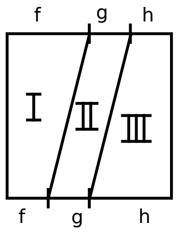
Define the n-th homotopy group of the space X with the base point x_0 as the group of homotopy classes of the cts[1] maps $(I^n, \partial I^n) \to (X, x_0)$ with the operation given by prescription:

$$(f+g)(t_1,\dots,t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & 0 \le t_1 \le \frac{1}{2}, \\ g(2t_1 - 1, t_2, \dots, t_n) & \frac{1}{2} \le t_1 \le 1. \end{cases}$$

Denote it $\pi_n(X, x_0)$.

Exercise 1. *Show the operation on $\pi_n(X, x_0)$ is associative.*

Solution. We want to show $(f+g)+k \sim f+(g+k)$. We will find prescription for the homotopy by the following diagram:



In the following notation[2], understand $f(t_1)$ as $f(t_1, t_2, \dots, t_n)$ for all $t_2, \dots, t_n \in I$.

$$h(s,t_1) = \begin{cases} f(\frac{4}{1+s}t_1) & (s,t_1) \in \mathrm{I} & (t_1 \in [0, \frac{1}{4} + s\frac{1}{4}], s \in [0,1]) \\ g(4t_1 - (1+s)) & (s,t_1) \in \mathrm{II} & (t_1 \in [\frac{1}{4} + s\frac{1}{4}, \frac{1}{2} + s\frac{1}{4}], s \in [0,1]) \\ k(\frac{4}{2-s}t_1 - \frac{2+s}{2-s}) & (s,t_1) \in \mathrm{III} & (t_1 \in [\frac{1}{2} + s\frac{1}{4}, 1], s \in [0,1]) \end{cases}$$

□

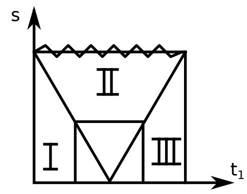
Exercise 2. *Show that the element given by prescription*

$$(-f)(t_1,\dots,t_n) = f(1-t_1, t_2, \dots, t_n)$$

is really the inverse element of f.

Solution. We want to show $f + (-f) \sim const$. The constant will be (function given by) the point $f(2t_1) = f(0)$. Again let us draw a diagram (the square (its boundary) in the middle sign the same value; the wavy line sign one value too).

$$h(s,t_1) = \begin{cases} f(2t_1) & (s,t_1) \in \mathrm{I} \\ f(\frac{1-s}{2}) = (-f)(\frac{1+s}{2}) & (s,t_1) \in \mathrm{II} \\ (-f)(2t_1 - 1) = f(2t_1) & (s,t_1) \in \mathrm{III}, \end{cases}$$



where $\mathrm{II} = \{(s,t_1) \mid s \in [0,1], t_1 \in [\frac{1-s}{2}, \frac{1+s}{2}]\}$. □

Remark. One can see, that proving by pictures is much more pleasant.

[1] continuous

[2] and some other following notations

There is a long exact sequence:

$$\cdots \to \pi_{n+1}(X,A,x_0) \xrightarrow{\partial} \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \to \dots$$

Exercise 3. *Show the exactness of this sequence in $\pi_n(X,A,x_0)$ and $\pi_n(A,x_0)$.*

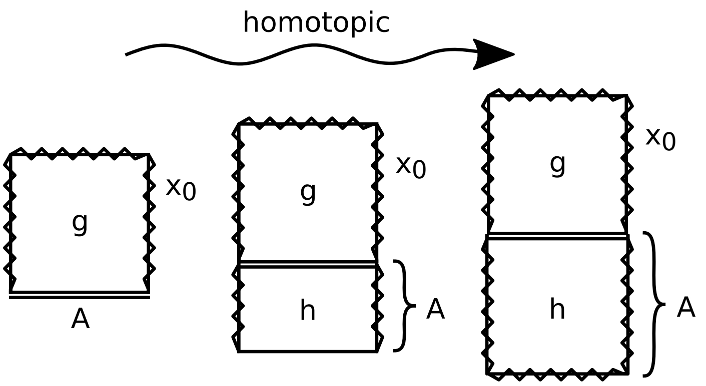
Solution. At first we will show the exactness in $\pi_n(X,A,x_0)$.

Let us show the inclusion "$\operatorname{im} j_* \subseteq \ker \partial$". Take an arbitrary $f \in \pi_n(X,x_0)$, thus $f\colon (I^n,\partial I^n) \to (X,x_0)$. From definition $j_*(f) = j \circ f\colon (I^n,\partial I^n,J^{n-1}) \to (X,A,x_0)$, where $J^{n-1} = I^n - I^{n-1}$, and $\partial([f]) = [f|_{I^{n-1}}] = const$, since f is constant on whole $\partial I^n \supseteq I^{n-1}$.

"$\operatorname{im} j_* \supseteq \ker \partial$": Take an arbitrary $g \in \ker\partial \subseteq \pi_n(X,A,x_0)$, thus $g\colon (I^n,\partial I^n,J^{n-1}) \to (X,A,x_0)$. Since $g \in \ker\partial$, there is the homotopy $h\colon (I^{n-1},\partial I^{n-1}) \times I \longrightarrow (A,x_0)$ such that $h(x,0) = g|_{I^{n-1}}(x)$ and $h(x,1) = const$. Because $h(x,t) \in A$ and $h(x',t) = x_0$ for all $x \in I^{n-1}$, $x \in \partial I^{n-1}$ and $t \in [0,1]$, we can take $f \in \pi_n(X,x_0)$ defined by

$$f(x,t) = \begin{cases} g(x,2t) & \text{for } t \in [0,\frac{1}{2}] \\ h(x,2t-1) & \text{for } t \in [\frac{1}{2},1]. \end{cases}$$

It is not hard to prove that $j_*(f)$ is homotopic to g, see picture below.



Now, let us show the exactness in $\pi_n(A,x_0)$.

"$\operatorname{im}\partial \supseteq \ker i_*$" Let $f \in \ker i_* \subseteq \pi_n(A,x_0)$ be an arbitrary. Because $i_* f \sim const$, we have homotopy $h\colon (I^n,\partial I^n) \times I \to (X,x_0)$ such that $h(x,0) = f(x)$ and $h(x,1) = x_0$. It holds $h \in \pi_{n+1}(X,A,x_0)$, since $h(x,0) \in A$, $h(x,1) = x_0$ and $h(x',t) = x_0$ for all $x \in I^n$ and $x' \in \partial I^n$.

"$\operatorname{im}\partial \subseteq \ker i_*$" Let $h \in \pi_{n+1}(X,A,x_0)$ be an arbitrary. Denote $h|_{I^n} = f$. Then h gives the homotopy $i_* f \sim const$ in (X,x_0), since $h(x,0) = f(x)$, $h(x,1) = x_0$ and $h(x',t) = x_0$ for all $x' \in \partial I^n$ and $x \in I^n$. □

A map $p\colon E \to B$ is called a fibration if it has the homotopy lifting property for all $(D^n,\emptyset)$:

$$\begin{tikzcd} D^n \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

If p is a fibration then it has the homotopy lifting property also for all pairs (X, A) of CW-complexes:

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Recall that $p\colon E \to B$ is a fiber bundle with fibre F if there are open subsets U_α such that $B = \bigcup_\alpha U_\alpha$ and the following diagram commutes for all U_α:

$$\begin{tikzcd} U_\alpha \times F \arrow[r, "homeo."] \arrow[dr, "pr_1"'] & p^{-1}(U_\alpha) \arrow[d, "p"] \\ & U_\alpha \end{tikzcd}$$

Exercise 4. *Show that every fibre bundle is a fibration.*

Solution. At first consider a trivial fibre bundle $E = B \times F$. Take an arbitrary commutative diagram of the form:

$$\begin{tikzcd} D^n \times \{0\} \arrow[r, "f"] \arrow[d] & B \times F \arrow[d, "pr"] \\ D^n \times I \arrow[r, "h"] & B \end{tikzcd}$$

Then $h(-, 0) = f$ and we can define $H\colon D^n \times I \to B \times F$ by $H(x, t) = \big(h(x, t), g(x)\big)$. One can see that the diagram commutes with H too.

Now, let $p\colon E \to B$ be an arbitrary fibre bundle with fiber F and $B = \bigcup_\alpha U_\alpha$. We can take I^n instead of D^n and consider a diagram:

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "h"] & B \end{tikzcd}$$

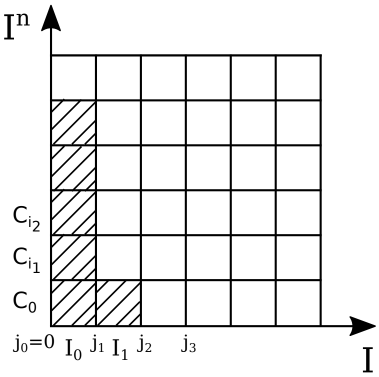
Because I^n is compact, we can divide $I^n \times I$ to finitely many subcubes $C_i \times I_k$ where $I_i = [j_k, j_{k+1}]$ such that $h(C_i \times I_k) \subseteq U_\alpha$ for some α. Since $U_\alpha \times F \to B$ makes a trivial bundle, we can use the same approach as above for each subcube. Since we know $H|_{C_i \times \{0\}} = f|_{C_i \times \{0\}}$, we can find the lift H for all cubes in the first "column" (see the picture below) in the same way as for the trivial case:

$$\begin{tikzcd} C_i \times \{0\} \arrow[r, "f"] \arrow[d] & U_\alpha \times F \arrow[d, "pr"] \\ C_i \times I_0 \arrow[r, "h"] \arrow[ur, dashed, "H"] & U_\alpha \end{tikzcd}$$

Since we know $H|_{C_i \times \{j_1\}}$ now, we can continue with the second "column":

$$\begin{tikzcd}
C_i \times \{j_1\} \arrow[r, "f"] \arrow[d] & U_\alpha \times F \arrow[d, "pr"] \\
C_i \times I_1 \arrow[r, "h"] \arrow[ur, dashed, "H"] & U_\alpha
\end{tikzcd}$$

Thus, we can proceed through all columns in this way until we will get H on the whole $I^n \times I$. The illustration of this situation[3]:



□

Exercise 5. *Show the structure of the fibre bundle $S^n \xrightarrow{p} \mathbb{R}\mathrm{P}^n$.*

Solution. The fibre is $S^0 = \{-1, 1\}$, since $x, -x \longmapsto [x]$. Now, we want to find a neighbourhood U of $[x]$ such that $p^{-1}(U) = U \times S^0$. Set $U = \{[x+v] \mid v \in [x]^\perp\}$ then we have homeomorphism $\varphi\colon U \times S^0 \longrightarrow p^{-1}(U) \subseteq S^n$ given by $\varphi([x+v], 1) = \frac{x+v}{\|x+v\|}$ and $\varphi([x+v], -1) = \frac{-x-v}{\|x+v\|}$. We can cover the whole $\mathbb{R}\mathrm{P}^n$ by the open subsets $U_i = \{(x_0 : x_1 : \cdots : x_n) \mid x_i \neq 0\}$. □

Exercise 6. *Show the structure of the fibre bundle $S^{2n+1} \xrightarrow{p} \mathbb{C}\mathrm{P}^n$ with the fibre S^1.*

Solution. Let us look on the special case $S^1 \hookrightarrow S^3 \to \mathbb{C}\mathrm{P}^1 \cong S^2$ called "Hopf fibration". Realise that we can consider $S^3 \subseteq \mathbb{C}^2$, so we can (locally) define the projection $S^3 \to \mathbb{C}\mathrm{P}^1$ by $(z_1, z_2) \mapsto \frac{z_1}{z_2}$.

In the general case, realise that we can consider $S^{2n+1} \subseteq \mathbb{C}^{n+1}$. Take $U_0 = \{[z_0 : z_1 : \cdots : z_n] \mid z_0 \neq 0\} \subseteq \mathbb{C}\mathrm{P}^n$. We can consider $U_0 = \{[1 : z_1 : \cdots : z_n]\}$. Then the map $U_0 \times S^1 \to S^{2n+1}$ is given by

$$[(1 : z_1 : \cdots : z_n), e^{it}] \longmapsto \frac{(e^{it}, e^{it} z_1, \ldots, e^{it} z_n)}{\|(e^{it}, e^{it} z_1, \ldots, e^{it} z_n)\|}$$

We can do the same for other U_i from the covering of $\mathbb{R}\mathrm{P}^n$. □

[3] it is drawn as planar, but it should be n-dimensional