Exercise 53. *Long exact sequence of the fibration (Hopf) $S^1 \to S^3 \to \mathbb{C}P^1 = S^2$.*

Solution. This is an important example of a fibration, it deserves our attention. First we write long exact sequence

$$\pi_3(S^1) \xrightarrow{i_*} \pi_3(S^3) \xrightarrow{j_*} \pi_3(S^2) \xrightarrow{\partial} \pi_2(S^1) \to \pi_2(S^3) \to \pi_2(S^2) \to$$
$$\to \pi_1(S^1) \to \pi_1(S^3) \to \pi_1(S^2) \to \cdots$$

and also for $\mathbb{Z} \to \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$, that is $\pi_n(\mathbb{Z}, 0) \to \pi_n(\mathbb{R}) \to \pi_n(S^1) \to \pi_{n-1}(\mathbb{Z})$ for $n > 0$. Since $\mathbb{Z}$ is discrete, $(S^n, s_0) \to (\mathbb{Z}, 0), s_0$ goes to a base point 0, therefore the map is constant . Hence we get $\pi_n(\mathbb{Z}, 0) = 0$ for $n \geq 1$. Also, $\pi_n(\mathbb{R})$ is zero as well, because $\mathbb{R}$ is homotopy equivalent to point. We get that this whole sequence are zeroes for $n \geq 2$. What we are left with is

$$0 \to \pi_1(S^1) \to \pi_0(\mathbb{Z}) \to \pi_0(\mathbb{R}) = 0,$$

so $\pi_1(S^1) \cong \mathbb{Z}$. Continue now with the updated long exact sequence:

$$0 \to \pi_3(S^3) \to \pi_3(S^2) \to 0 \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) = \mathbb{Z} \to \pi_1(S^3) \cdots,$$

and $\pi_1(S^3) = 0$, because $\pi_k(S^n) = 0$ for $k < n$ considering the map $S^k \to S^n$ that can be deformed into cellular map and so it is not surjective. So we get

$$\pi_3(S^3) \cong \pi_3(S^2), \quad \pi_2(S^2) \cong \pi_1(S^1) = \mathbb{Z}.$$

It implies that $\pi_2(S^2) \cong \mathbb{Z}$.

Let us remark that if $\pi_3(S^3) = \mathbb{Z}$, then $\pi_3(S^2) = \mathbb{Z}$. Later we will prove that

$$\pi_n(S^n) \cong H_n(S^n) \cong \mathbb{Z}.$$

□

Remark. *(Story time) Eduard Čech was the first who defined higher homotopy groups (1932, Höherdimensionale Homotopiegruppen) but the community od these groups didn't support the study as they were considered not interesting. Were they mistaken? The rest of this remark is left as an exercise for the reader.*

Remark. *(For geometers) For G Lie group and H its subgroup we have a fibre bundle $H \hookrightarrow G \to G/H$.*

As an example consider otronormal group $O(n)$, we have inclusions $O(1) \subseteq O(2) \subseteq \cdots O(n)$. Then

$$O(n-k) \to O(n) \to O(n)/O(n-k) = V_{n,k}$$

is a fibre bundle, we call $V_{n,k}$ Stiefel manifold, k-tuples of orthonormal vectors in $\mathbb{R}^n$.

Also, we can take $O(k) \to O(n)/O(n-k) \to O(n)/(O(k) \times O(n-k))$, Grassmannian manifold (k-dimensional subspaces in $\mathbb{R}^n$):

$$O(k) \to V_{n,k} \to G_{n,k},$$

is also a fibre bundle.

Exercise 54. *Long exact sequence of the fibration $F \to E \to B$ ends with*

$$\pi_1(B) \to \pi_0(F, x_0) \to \pi_0(E, x_0) \to \pi_0(B).$$

Show exactness in $\pi_0(E)$.

Solution. We showed $\pi_n(B) \cong \pi_n(E, F)$ for $n \geq 0$ which gave us the exactness of the fibration sequence from the exactness of the sequence of the pair (E, F) till $\pi_0(E, x_0)$.

Denote $S^0 = \{-1, 1\}$ and consider the composition

$$(\{-1, 1\}, -1) \to (F, x_0) = p^{-1}(b_0) \to (E, x_0) \to (B, p(x_0) = b_0),$$

where -1 goes to x_0 and b_0 and 1 goes to F but that is $p^{-1}(b_0)$ so we get the constant map.

For the other part of the exactness we will use homotopy lifting property.

Consider $f : (\{-1, 1\}, -1) \to (E, x_0)$ such that pf is homotopic to the constant map into b_0. It means that $pf(1)$ is connected with b_0 by a curve. Have a diagram

$$\begin{tikzcd}
(1) \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\
(1) \times I \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

and remark that $f(1)$ is connected by a curve with $x \in E$ such that $p(x) = b_0$. So $x \in F$ and f is homotopic to the map $g : (\{-1, 1\}, -1) \to (F, x_0)$ which maps -1 into x_0 and 1 into x.

□

Exercise 55. *Covering: $G \to X \to X/G$, with action of G on X properly discontinuous, where X is path connected.*

Take

$$\pi_1(G, 1) \to \pi_1(X) \to \pi_1(X/G) \to \pi_0(G) \to \pi_0(X),$$

that is

$$0 \to \pi_1(X) \to \pi_1(X/G) \xrightarrow{\partial} \pi_0(G) \to 0.$$

Show that ∂ is a group homomorphism.

Solution. First note that $\pi_0(G)$ is G taken as a set. We recall that for $F \to E \to B$ we had

$$\begin{tikzcd}
\pi_n(E) \arrow[r, "j_*"] & \pi_n(E, F, x_0) \arrow[r, "\bar{\partial}"] \arrow[d, "p_*", "\cong"'] & \pi_{n-1}(F) \\
& \pi_n(B, b_0) \arrow[ur, dashed, "\partial"'] &
\end{tikzcd}$$

We showed the $\cong$ finding a map going from $\pi_n(B, b_0) \to \pi_n(E, F, x_0)$. Also,

$$f \colon (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0),$$

$\bar{\partial}[F] = [f/\partial I^n]$. The prescription for $\partial\colon \pi_n(B, b_0) \to \pi_{n-1}(F, x_0)$: take $f\colon (I^n, \partial I^n) \to (B, b_0)$ and make a lift $F\colon (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$, $\partial[f] = F/I^{n-1}\colon (I^{n-1}, \partial I^{n-1} \to (F, x_0)$. The diagram is:

$$\begin{tikzcd}
I^{n-1} \times \{1\} \cup \partial I^{n-1} \times I \arrow[r, "const"] \arrow[d, hook] & E \quad x_0 \arrow[d, "p"] \\
I^{n-1} \times I \arrow[r] \arrow[ur, dashed, "F"] & B \quad b_0
\end{tikzcd}$$

and $pF(I^{n-1} \times \{0\}) = b_0$, so $F(I^{n-1} \times \{0\}) \subseteq p^{-1}(b_0) = F$.

Now, $\pi_1(X/G) \xrightarrow{\partial} \pi_0(G)$, $f\colon (I, \partial I) \to (X/G, [x_0])$. Take two closed curves ω, τ at $[x_0]$ and lift them to curves $\bar{\omega}, \bar{\tau}$ in X starting in x_0. We denote the end point of $\bar{\omega}$ by $g_1 x_0$ and the end point of $\bar{\tau}$ by $g_2 x_0$. Then the operation $\cdot$ (that is not a composition $\circ$) is:

$$\begin{aligned}
\overline{\omega \cdot \tau} &= \bar{\omega} \cdot g_1 \bar{\tau} \\
\overline{\omega \cdot \tau}(0) &= x_0 \\
\overline{\omega \cdot \tau}(1/2) &= g_1 x_0 \\
\overline{\omega \cdot \tau}(1) &= g_2(g_1 x_0)
\end{aligned}$$

And it's a homomorphism.

□

Exercise 56. *Van Kampen theorem - Applications.*

Klein bottle K. Model as a square with identified sides as seen in Fig 1,2,3.

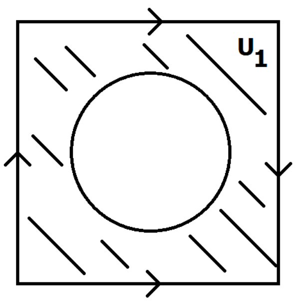

Fig 1

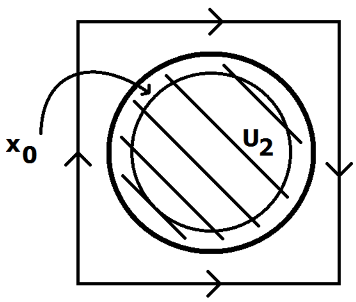

Fig 2

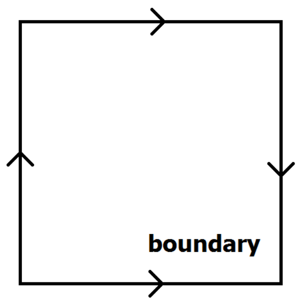

Fig 3

We denote open sets U_1, U_2 (disc) and point x_0 as in figure 1 and 2. (some of the notation in the solution is established in the theorem)

Solution. Set U_1 is homotopy equivalent to the boundary (Fig 3), and this boundary is in fact a wedge of two circles, so $U_1 \simeq S^1 \vee S^1$.

We can compute: $\pi_1(U_2, x_0) = \{1\}$ by contractibility, $\pi_1(U_1, x_0) = \pi_1(S^1 \vee S^1, x_0) =$ free group on two generators α, β as was already shown in lecture.

Then $\pi_1(U_1, x_0) * \pi_1(U_2, x_0) =$ free group on two generators α, β, also $\pi_1(U_1 \cap U_2, x_0) = \mathbb{Z}$.

Now, the intersection $U_1 \cap U_2 \simeq (S^1, x_0)$ and we take generator ω, $i_{2,1*}(\omega^{-1}) = 1$ in $\pi_1(U_2, x_0)$, $i_{1,2*}(\omega) = \alpha\beta\alpha^{-1}\beta$.

So, kernel of φ (the map from the theorem) is generated by element $\alpha\beta\alpha^{-1}\beta$
$\pi_1(K)$ is the group with two generators α, β and one relation $\alpha\beta\alpha^{-1}\beta = 1$ $\square$