Exercise 57. *Recall definitions of n-connectness and n-equivalence. Prove the following lemma: Inclusion $A \hookrightarrow X$ is n-equivalence if and only if (X, A) is n-connected.*

Solution. "$\Leftarrow$" Take long exact sequence:

$$\to \pi_n(A, x_0) \xrightarrow{f_1} \pi_n(X, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0) \xrightarrow{f_2} \pi_{n-1}(X) \to \pi_{n-1}(X, A) \to$$

and use the assumption that $\pi_i(X, A, x_0) = 0$ for $i \leq n$. Then we get that f_1 is epimorphism and f_2 is isomorphism.

"$\Rightarrow$" Reasoning is the same as in the other direction, the only thing we need to realize is $\pi_0(A, x_0) \xrightarrow{\cong} \pi_0(X, x_0)$. □

Exercise 58. *Show $\pi_k(S^\infty) = 0$ for all k, where S^∞ is colim S^n.*

Solution. We have $S^1 \subset S^2 \subset S^3 \subset \cdots \subset S^n \subset \cdots S^\infty$. Take element in $\pi_k(S^\infty)$, that is $f\colon S^k \to S^\infty$. We know that $f(S^k)$ is compact in S^∞. Consider other CW-complex structure than $e^0 \cup e^k$ for S^k that is $S^k = \bigcup_{i=0}^k e_1^i \cup e_2^i$ (two hemispeheres). Then the following holds: $S^\infty = \bigcup_{i=0}^\infty e_1^i \cup e_2^i$. So $f(S^k) \subseteq (S^\infty)^{(N)} = S^N$ for some N where $(S^\infty)^{(N)}$ is N-skeleton of S^∞. Now,

$$f\colon S^k \to S^N \to S^{N+1} \hookrightarrow S^\infty,$$

so the composition $S^k \to S^{N+1}$ is a map that is not onto and therefore f is homotopic to constant map. (map into a disc is homotopic to constant map, disc is contractible) Then we have $[f] = 0$. Thus we have proved $\pi_k(S^\infty) = 0$. □

Exercise 59. *Compute homotopy groups of $\mathbb{R}P^\infty$.*

Solution. Suprisingly use previous exercise: We can view $\mathbb{R}P^\infty$ as lines going through origin in S^∞, or...just take $S^\infty/_{\mathbb{Z}/2}$, where the action is $x \mapsto -x$. So we work with the following fibration (we don't write the distinguished points as they are not needed) $\mathbb{Z}/2 \to S^\infty \to \mathbb{R}P^\infty$ and the long exact sequence

$$\pi_n(\mathbb{Z}/2) \to \pi_n(S^\infty) \to \pi_n(\mathbb{R}P^\infty) \xrightarrow{\partial} \pi_{n-1}(\mathbb{Z}/2) \to \pi_{n-1}(S^\infty),$$

where for all $n \geq 2$ we have all zeroes, for $n = 1$ consider $0 \to \pi_1(\mathbb{R}P^\infty) \xrightarrow{\partial} \pi_0(\mathbb{Z}/2) \to \pi_0(S^\infty)$. Since $\pi_0(S^\infty) = 0$ and $\pi_0(\mathbb{Z}/2) = \mathbb{Z}/2$, we get that the homomorphism ∂ (it is homomorphism, really, we did it in previous tutorial, but it's still a homomorphism independetly on whether we did it or not) is an isomorphism of groups. By connectness we also know the π_0 group. So the final results are:

$$\pi_n(\mathbb{R}P^\infty) = 0 \text{ for } n \geq 2,\ \pi_1(\mathbb{R}P^\infty) = \mathbb{Z}/2,\ \pi_0(\mathbb{R}P^\infty) = 0.$$

□

Exercise 60. *Show that the spaces $S^2 \times \mathbb{R}P^\infty$ and $\mathbb{R}P^2$ have the same homotopy groups but they are not homotopy equivalent.*

Solution. Here, also use pre^{2}vious exercise. It is known that $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$. With this we can compute

$$\pi_0(S^2 \times \mathbb{R}P^\infty) = 0,\ \pi_1(S^2 \times \mathbb{R}P^\infty) = \pi_1(S^2) \times \pi_1(\mathbb{R}P^\infty) = \{0\} \times \mathbb{Z}/2 \cong \mathbb{Z}/2,$$

$$\text{for } n \geq 2\, \pi_n(S^2 \times \mathbb{R}P^\infty) = \pi_n(S^2) \times \{0\} \cong \pi_n(S^2.$$

Now consider $\mathbb{R}P^2$ as $S^2/_{\mathbb{Z}/2}$, work with the fibration (scheme as follows)

$$\begin{array}{cccccc}
 & \mathbb{Z}/2 \longrightarrow & S^2 \longrightarrow & \mathbb{R}P^2 & & \\
 & \pi_n(\mathbb{Z}/2) \longrightarrow & \pi_n(S^2) \longrightarrow & \pi_n(\mathbb{R}P^2) \longrightarrow & \pi_{n-1}(\mathbb{Z}/2) & \\
n \geq 2 & 0 & \pi_n(S^2) & \cong \pi_n(\mathbb{R}P^2) & 0 & \\
n = 1 & & 0 & \pi_1(\mathbb{R}P^2) & \cong \mathbb{Z}/2 & 0
\end{array}$$

and $\pi_0(\mathbb{R}P^2) = 0$. Thus we showed that these two spaces have the same homotopy groups. Marvelous. How to show, that they are not homotopy equivalent? Use cohomology group! That's right. It is well known (or we should already know) that

$$H^*(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2[\alpha]/\langle \alpha^3 \rangle, \alpha \in H^1 \text{ and}$$

$$H^*(S^2 \times \mathbb{R}P^\infty; \mathbb{Z}/2) = H^*(S^2; \mathbb{Z}/2) \otimes H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[\beta]/\langle \beta^2 \rangle \otimes \mathbb{Z}/2[\gamma], \beta \in H^2.$$

The former space obviously has no non-zero elements of order 4, while the latter has a non-zero element of order 4. This is impossible for homotopy equivalent spaces. We are done.

□

Exercise 61. *Extension lemma: Let (X, A) be a pair of CW-complexes, Y a space with $\pi_{n-1}(Y, y_0) = 0$ whenever there is a cell of dimension n in $X - A$. Then every map $f\colon A \to Y$ can be extended to a map $F\colon X \to Y$.*

Solution. Set $X_{-1} = A, X_0 = X^{(0)} \cup A, X_k = X^{(k)} \cup A$, and $f = f_{-1}\colon X_{-1} \to Y, f_0\colon X_0 \to Y$ which extends f_{-1}, $f_0(x_0)$ to any point in Y.

We have $f_{k-1}\colon X_{k-1} \to Y$ and want to define $f_k\colon X_k \to Y$. Also, $\pi_{k-1}(Y, \bullet) = 0$.

Consider following diagram:

$$\begin{tikzcd}
(D^k, S^{k-1}) \arrow[r, "\varphi"] \arrow[dr, "f_{k-1} \circ \varphi"'] & (X_k, X_{k-1}) \arrow[d, "f_{k-1}"] \\
 & Y
\end{tikzcd}$$

The map $f_{k-1} \circ \varphi$ is in $\pi_{k-1}(Y, \bullet)$ so it is homotopic to constant map, so we define f_k on D^k as a constant map. Now,

$$\begin{tikzcd} D^k \times \{1\} \cup S^{k-1} \times I \arrow[r] \arrow[d, hook] & Y \\ D^k \times I \arrow[ur, dashed, "H"'] & \end{tikzcd}$$

and go back to the first diagram:

$$\begin{tikzcd} (D^k, S^{k-1}) \arrow[r, "\varphi"] \arrow[dr, "g"'] & (X_k, X_{k-1}) \arrow[d, "f_k"] \\ & Y \end{tikzcd}$$

We extend f_{k-1} to $f_k \colon X_k \to Y$ and proceed to infinity *(and beyond)*, as we always do. $\square$

Exercise 62. *Compare $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$, when distinguished points (are / are not) path connected. Use proof with pillows.*

Solution. First the case where $X = S^1 \sqcup S^2$ and $x_0 \in S^1$ and $x_1 \in S^2$. Then $\pi_1(X, x_0) = \pi_1(S^1) = \mathbb{Z}$ but $\pi_1(X, x_1) = \pi_1(S^2) = 0$. If distinguished points are not path connected, homotopy groups can be different, so consider now ω a curve connecting x_0 and x_1, $\omega \colon I \to X$, $\omega(0) = x_0$ and $\omega(1) = x_1$. We have $\pi_n(X, x_0) \to \pi_n(X, x_1)$ $f \colon I^n \to X$, $\partial I^n \mapsto x_0$, $g \colon I^n \to X$, $\partial I^n \mapsto x_1$, as seen in Figure 1, $x_0 - x_1$ segments. Proofs with pillows!

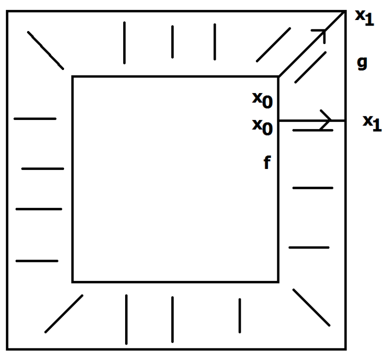

Figure 1

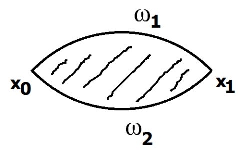

Figure 2

Denote action $f \mapsto \omega \cdot f$, then $f_1 \sim f_2 \Rightarrow \omega \cdot f_1 \sim \omega \cdot f_2$. We are not satisfying algebraist at the moment, only geometers. Let us try to do something about that.

Figure 2 shows other pillow and that $\omega_1 \sim \omega_2 \Rightarrow \omega_1 \cdot f \sim \omega_2 \cdot f$, we can imagine the segment as in Figure 2, two curves.

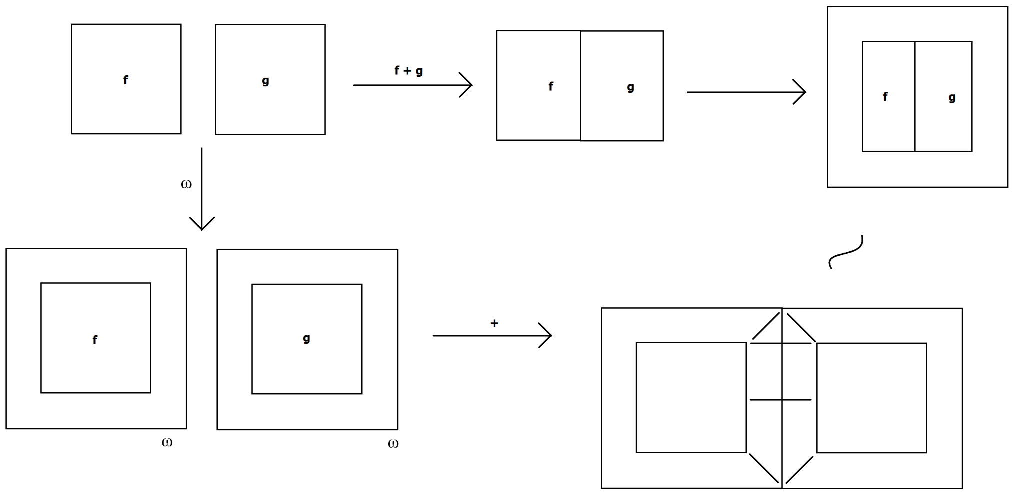

Figure 3

Map given by ω is a bijection $\omega\colon x_0 \to x_1$, $\omega_2(\omega_1 f) \sim (\omega_1\omega_2)f$, $\omega^{-1}(\omega f) \sim (\omega\omega^{-1})f \simeq f$, so the map $\pi_n(X, x_0) \to \pi_n(X, x_1)$ is bijection. Figure 3 tries to explain homomorphism.

We get that x_0, x_1 are in the same path component and if $\omega\colon x_0 \to x_1$ is a curve, then $\pi_n(X, x_0) \to \pi_n(X, x_1)$ is an isomorphism. In particular, if X is simply connected, then every curve gives the same isomorphism. $\square$