Exercise 69. *Let (X, A) be a pair. Show that the following diagram commutes:*

$$\begin{array}{ccc}
\pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_n(A, x_0) \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\
H_n(X, A) & \xrightarrow[\partial_*]{} & H_{n-1}(A)
\end{array}$$

where ∂ is the boundary homomorphism, h is the Hurewicz homomorphism and ∂_* is the connecting homomorphism.

Solution. Take $[f] \in \pi_n(X, A, x_0)$, that is $f\colon (D^n, D^{n-1}, s_0) \to (X, A, x_0)$. Then $\partial[f] = [f/S^{n-1}]$ and $h\partial[f] = h[f/S^{n-1}] = (f/S^{n-1})_*(b)$, where b is a generator in $H_{n-1}(S^{n-1})$. (We recall the definition of the Hurewicz homomorphism: if $g\colon S^{n-1} \to A$, $g_*\colon H_{n-1}(S^{n-1}) \to H_{n-1}(A)$ and $h[g] = g_*(b) \in H_{n-1}(A)$.)

Let $a \in H_n(D^n, S^{n-1})$ be a generator such that $\partial_* a = b$. We proceed using commutativity of the following diagram:

$$\begin{array}{ccc}
H_n(D^n, S^{n-1}) & \xrightarrow[\cong]{\partial_*} & H_{n-1}(S^{n-1}) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f/S^{n-1})_*} \\
H_n(X, A) & \xrightarrow[\partial_*]{} & H_{n-1}(A)
\end{array}$$

Now $\partial_* h[f] = \partial_*(f_* a) = (f/S^{n-1})_*(\partial_* a) = h\partial[f]$ which concludes the proof. □

Exercise 70. *Homology version of Whitehead theorem: If X and Y are simply connected CW-complexes and $f : X \to Y$ induces an isomorphism on all homology groups, then f is a homotopy equivalence. Show that $f_*\colon \pi_n(X) \to \pi_n(Y)$ is an iso for all n and then use the homotopy version of Whitehead theorem.*

Solution. First, for consider $f\colon X \hookrightarrow Y$ an inclusion. For pair (Y, X) we have the following long exact sequences connected by Hurewicz homomorphisms:

$$\begin{array}{ccccccc}
\pi_{n+1}(Y, X) & \longrightarrow & \pi_n(Y) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_n(Y, X) \\
{\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow \\
H_{n+1}(Y, X) & \xrightarrow[0]{} & H_n(Y) & \xrightarrow[\cong]{} & H_n(X) & \xrightarrow[0]{} & H_n(Y, X)
\end{array}$$

Since $H_n(X) \to H_n(Y)$ are isomorphisms, all $H_n(Y, X) = 0$. Since Y and X are simply connected, we have $\pi_1(Y, X) = 0$. Now, $\pi_2(Y, X) \xrightarrow{h} H_2(Y, X)$ is an iso by Hurewicz theorem. Since $H_2(Y, X) = 0$, $\pi_2(X, Y) = 0$ as well. With the same argument we can proceed inductively to get $\pi_n(Y, X) = 0$ for all n. Then $f_* : \pi_n(X) \to \pi_n(Y)$ are isomorphisms.

Now, for general map $f\colon X \to Y$ we use the argument with mapping cylinder M_f. Do you remember this one from the first lecture? This construction you thought would not be interesting? Well, hold your hats, its usage is advantageous here!

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{f} & \downarrow{\scriptstyle i_X} & \searrow^{f} & \\ Y & \xrightarrow[i_Y]{} & M_f & \xleftarrow[r]{} & Y \end{array}$$

We know that $f_*\colon H_n(X) \to H_n(Y)$ is an iso and (as i_Y is homotopy equivalence) $i_{Y*}\colon H_n(Y) \to H_n(M_f)$ is an iso. Thus, we conclude that $i_{X*}\colon H_n(X) \to H_n(M_f)$ is an iso. That is, i_* is homotopy equivalence and therefore f is also homotopy equivalence. (Let us remark that for this exercise you should revisit Whitehead theorem, at.pdf study text is also recommended) □

Exercise 71. *(application of Whitehead theorem) Show that $(n-1)$-connected compact manifold of dim n is homotopy equivalent to S^n $(n \geq 2)$.*

Solution. Take manifold M as two parts: disk and its complement. Map the disk to $S^n = D^n/S^{n-1}$, that is the identity on the interior of the disk and the complement goes to one point. We have $f\colon M \to S^n$ and for $i < n$ by Hurewicz theorem (M is $(n-1)$-connected) we get an iso $f_* : H_i(M) \cong H_i(S^n) = 0$. The same iso we obtain for $i > n$ (both H_i are zero).

We conclude our application with this spectacular diagram where all arrows are iso, reasoning goes from the bottom arrows by excision, then with definiton of fundamental class we get the top arrow (our main focus) an iso. (we denote interior of B as B^o)

$$\begin{array}{ccc} H_n(M) & \longrightarrow & H_n(S^n) \\ \downarrow & & \downarrow \\ H_n(M, M-(B^n)^o) & \longrightarrow & H_n(S^n, S^n-(B^n)^o) \\ \downarrow & & \downarrow \\ H_n(disk, \partial disk) & \xrightarrow[\text{id}]{} & H_n(disk, \partial disk) \end{array}$$

□

Remark. *(Story time) We have got that a compact manifold 3-dim (compact) which is 2-connected is homotopy equivalent to S^3. There is another result: Every 3-dim manifold simply connected compact manifold is homeomorphic to S^3. This latter result (it might seem we are close to proving it) is actually famous Poincaré conjecture, one of Millenium Prize Problems and it was already solved by Grigori Perelman in 2002. Interesting story and interesting mathematician for sure. It is well known (as we like to say) that Perelman declined Fields medal (among other prizes).*

Exercise 72. *Find a map f with Hopf invariant $H(f) = 2$.*

Solution. We study a space X with a basepoint e. Denote construction $J_2(X) = X \times X/\sim$, where $(x, e) \sim (e, x)$. Apply this idea to S^n. We get a projection $p\colon S^n \times S^n \to J_2(S^n)$. On the left we have one 0-cell, two n-cells and one 2n-cell, while on the right we have one of each. We get that $J_2(S^n)$ has to be a space of the form C_f, so $H^n(J_2) = \mathbb{Z}$ given by a and $H^{2n}(J_2) = \mathbb{Z}$ given by b and $H^n(S^n \times S^n) = \mathbb{Z} \oplus \mathbb{Z}$ (generators a_1, a_2) and $H^{2n}(S^n \times S^n) = \mathbb{Z}$ (with b_0). Now, $p^*\colon H^i(J_2) \to H^i(S^n \times S^n)$ and $p^*(a) = a_1 + a_2, p^*(b) = b_0$.

$$\begin{aligned} a^2 &= H(f)b \\ p^*(a^2) &= H(f)p^*(b) \\ (a_1 + a_2)^2 &= H(f)b_0 \\ (a_1^2 + a_1a_2 + a_2a_1 + a_2^2) &= H(f)a_1a_2 \\ 2a_1a_2 &= H(f)a_1a_2 \\ H(f) &= 2 \end{aligned}$$

because $b_0 = a_1a_2$ and by evenness of the dimension $a_1a_2 = a_2a_1$. □