Věta o rozkladu A na součin dolní a horní trojúhel. matice.

Dk.: - indukcí vzhledem k m - řád matice.

$m=1$: $A=(a_{11})$, $a_{11}\neq 0$ volíme $L=(1)$, $R=(a_{11})$

nechť tvrzení platí pro $m-1$, A – řádu m

$$A=\begin{pmatrix} A_{m-1} & b_m \\ c_m^T & a_{mm}\end{pmatrix}$$

A_{m-1} … prvních $m-1$ řádků a sloupců

$$b_m=\begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{m-1,m}\end{pmatrix},\quad c_m=\begin{pmatrix} a_{m,1} \\ a_{m,2} \\ \vdots \\ a_{m,m-1}\end{pmatrix},\quad \det(A_{m-1})\neq 0$$

$$A_{m-1}=\underbrace{L_{m-1}\cdot R_{m-1}}_{\text{regulární}}$$

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$$L=\begin{pmatrix} L_{m-1} & 0 \\ l_m^T & l_{mm}\end{pmatrix},\quad R=\begin{pmatrix} R_{m-1} & r_m \\ 0 & r_{mm}\end{pmatrix}$$

$$L\cdot R=\begin{pmatrix} L_{m-1}\cdot R_{m-1} & L_{m-1}\cdot r_m \\ l_m^T\cdot R_{m-1} & l_m^T\cdot r_m+l_{mm}\cdot r_{mm}\end{pmatrix}=\begin{pmatrix} A_{m-1} & b_m \\ c_m^T & a_{mm}\end{pmatrix}$$

$L_{m-1}\cdot r_m=b_m \Rightarrow r_m=L_{m-1}^{-1}\cdot b_m$

$l_m^T\cdot R_{m-1}=c_m^T \Rightarrow l_m^T=c_m^T\cdot R_m^{-1}$

$l_m^T\cdot r_m+l_{mm}r_{mm}=a_{mm}$

$l_{mm}\cdot r_{mm}=a_{mm}-l_m^T\cdot r_m$

Volíme např. $l_{mm}=1 \Rightarrow r_{mm}=a_{mm}-l_m^T\cdot r_m$

kvě 3-10:19

Choleského rozklad

$A=T^T\cdot T$, $A=(a_{ij})$, $T=\begin{pmatrix} t_{11} & t_{12} & t_{13} & \cdots & t_{1m} \\ 0 & t_{22} & t_{23} & \cdots & t_{2m} \\ 0 & 0 & t_{33} & \cdots & t_{3m} \\ & 0 & & \ddots & \\ & & & & t_{mm}\end{pmatrix}$

$$T^T\cdot T=\begin{pmatrix} t_{11} & 0 & 0 & & 0 \\ t_{12} & t_{22} & 0 & & \\ t_{13} & t_{23} & t_{33} & & \\ \vdots & \vdots & \vdots & \ddots & \\ t_{1m} & t_{2m} & t_{3m} & & t_{mm}\end{pmatrix}\begin{pmatrix} t_{11} & t_{12} & t_{13} & \cdots & t_{1m} \\ 0 & t_{22} & t_{23} & \cdots & t_{2m} \\ 0 & 0 & t_{33} & \cdots & t_{3m} \\ & & & \ddots & \\ & & & & t_{mm}\end{pmatrix}$$

$a_{11}=t_{11}^2 \Rightarrow t_{11}=\sqrt{a_{11}}$, $a_{1j}=t_{11}t_{1j} \Rightarrow t_{1j}=\frac{a_{1j}}{t_{11}}$, $j=2,\dots,m$

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$a_{22}=t_{12}^2+t_{22}^2 \Rightarrow t_{22}=\sqrt{a_{22}-t_{12}^2}$

$a_{2j}=t_{12}\cdot t_{1j}+t_{22}\cdot t_{2j} \Rightarrow t_{2j}=\frac{1}{t_{22}}(a_{2j}-t_{12}\cdot t_{1j})$, $j=3,\dots,m$

$\vdots$

$a_{ii}=t_{1i}^2+t_{2i}^2+\dots+t_{ii}^2 \Rightarrow t_{ii}=\sqrt{a_{ii}-\sum_{k=1}^{i-1}t_{ki}^2}$

$j>i$: $a_{ij}=t_{1i}\cdot t_{1j}+t_{2i}t_{2j}+\dots+t_{ii}t_{ij} \Rightarrow t_{ij}=\frac{1}{t_{ii}}\left(a_{ij}-\sum_{k=1}^{i-1}t_{ki}t_{kj}\right)$

$j=i+1,\dots,m$

kvě 3-10:42

Použití Chol. metody na systém lin. rovnic.

$Ax=b$, A - sym. $A=T^T T$

$T^T\underbrace{Tx}_{y}=b \;\rightarrow\; T^T y=b$ - systém s dolní Δ maticí, řešíme od $y_1,\dots,y_n$

Pak řešíme $Tx=y$ - systém s horní Δ maticí, řešíme od $x_n,\dots,x_1$

kvě 3-10:50

Př: $A=\begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ -1 & 4 & 8\end{pmatrix}$, $b=\begin{pmatrix} 4 \\ 1 \\ -8\end{pmatrix}$, $T=\begin{pmatrix} 1 & 2 & -1 \\ 0 & i\sqrt{2} & -i3\sqrt{2} \\ 0 & 0 & 5\end{pmatrix}$

$t_{11}=\sqrt{1}=1$

$t_{12}=2$

$t_{13}=-1$

$t_{22}=\sqrt{2-2^2}=\sqrt{-2}=i\sqrt{2}$

$t_{23}=\frac{1}{i\sqrt{2}}(4-2\cdot(-1))=\frac{6}{i\sqrt{2}}=$

$=-i3\sqrt{2}$

$t_{33}=\sqrt{8-(-1)^2-(-i3\sqrt{2})^2}=\sqrt{25}=5$

$T^T\cdot y=b$, $\begin{pmatrix} 1 & 0 & 0 \\ 2 & i\sqrt{2} & 0 \\ -1 & -i3\sqrt{2} & 5\end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3\end{pmatrix}=\begin{pmatrix} 4 \\ 1 \\ -8\end{pmatrix}$

$y_1=4$

$2y_1+i\sqrt{2}\,y_2=1$

$i\sqrt{2}\,y_2=-7$

$y_2=\frac{-7}{i\sqrt{2}}=\frac{7}{2}\sqrt{2}\,i$

$-4-i3\cancel{\sqrt{2}}\cdot\frac{7}{\cancel{2}}\cancel{\sqrt{2}}\,i+5y_3=-8$

$5y_3=-25$

$y_3=-5$

$T\cdot x=y$

$5x_3=-5 \Rightarrow x_3=-1$

$\cancel{i}\cancel{\sqrt{2}}\,x_2-\cancel{i}3\cancel{\sqrt{2}}(-1)=\frac{7}{2}\cancel{\sqrt{2}}\,\cancel{i} \Rightarrow x_2=\frac{7}{2}-3=\frac{1}{2}$

$x_1+2\cdot\frac{1}{2}-1(-1)=4$ $\quad x_1=2$

kvě 3-10:52

$$A=\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & \cdots \\ a_{21} & a_{22} & a_{23} & 0 & 0 & \cdots \\ 0 & a_{32} & a_{33} & a_{34} & 0 & \cdots \\ 0 & 0 & a_{43} & a_{44} & a_{45} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix} = \underbrace{\begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \\ & & \ddots & \ddots \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 1 & u_{12} & 0 & 0 & \cdots \\ 0 & 1 & u_{23} & 0 & \cdots \\ 0 & 0 & 1 & u_{34} & \cdots \\ & & & \ddots & \ddots \\ & 0 & & & \ddots \end{pmatrix}}_{U}$$

$a_{11}=l_{11}$

$a_{21}=l_{21}\cdot 1$

$a_{32}=l_{32}\cdot 1$

$\vdots$

$a_{i,i-1}=l_{i,i-1}$

$a_{12}=l_{11}\cdot u_{12} \Rightarrow u_{12}=\frac{a_{12}}{l_{11}}$

$a_{22}=l_{21}\, u_{12}+l_{22}\cdot 1 \Rightarrow l_{22}=a_{22}-l_{21}u_{12}$

$a_{23}=l_{21}\cdot 0+l_{22}\cdot u_{23} \Rightarrow u_{23}=\frac{a_{23}}{l_{22}}$

$a_{ii}=l_{i,i-1}\cdot u_{i-1,i}+l_{ii}\cdot 1 \Rightarrow l_{ii}=a_{ii}-l_{i,i-1}\cdot u_{i-1,i}$

$a_{i,i+1}=l_{i,i-1}\cdot 0+l_{ii}\cdot u_{i,i+1} \Rightarrow u_{i,i+1}=\frac{a_{i,i+1}}{l_{ii}}$

kvě 3-11:15

Croutova metoda

Př.: $A=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$, $L=\begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 3/2 & 0 & 0 \\ 0 & -1 & 4/3 & 0 \\ 0 & 0 & -1 & 5/4 \end{pmatrix}$, $U=\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Pozn: $\frac{d^2y(x)}{dx^2}=\frac{y(x+h)+y(x-y)-2y(x)}{h^2}+O(h^2)$

$u_{1,2}=\frac{a_{12}}{l_{11}}=\frac{-1}{2}=-\frac{1}{2}$

$l_{22}=a_{22}-l_{2,1}\cdot u_{1,2}=2-(-1)\cdot(-\frac{1}{2})=\frac{3}{2}$

$u_{2,3}=\frac{a_{23}}{l_{22}}=-\frac{1}{3/2}=-\frac{2}{3}$

$l_{33}=2-(-1)(-\frac{2}{3})=\frac{4}{3}$

$u_{34}=-\frac{1}{\frac{4}{3}}=-\frac{3}{4}$

$l_{44}=2-(-1)(-\frac{3}{4})=\frac{5}{4}$

kvě 3-11:30