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Analysis on the dynamics of a Cournot investment game with bounded rationality

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ABSTRACT

In this work, a dynamic system of investment game played by two firms with bounded rationality is proposed. It is assumed that each firm in any period makes a strategy for investment and uses local knowledge to make investment strategy according to the marginal profit observed in the previous period. Theoretic work is done on the existence of equilibrium solutions, the instability of the boundary equilibriums and the stability conditions of the interior equilibrium. Numerical simulations are used to provide experimented evidence for the complicated behaviors of the system evolution. It is observed that the equilibrium of the system can loose stability via flip bifurcation or Neimark–Sacher bifurcation and time-delayed feedback control can be used to stabilize the chaotic behaviors of the system.

1. Introduction

Cournot (1838) introduced earliest the mathematical model which describes production competitions in an oligopolistic market. In a classical Cournot model each participant uses a naïve expectation to guess that opponents' output remains at the same level as in the previous period and adopts an output strategy which solves the corresponding profit maximization problem. Since then, a great number of literatures have been devoted to enrich and expand the Cournot oligopoly game theory. Much work has paid attention to the stability and the complex phenomena in a dynamical Cournot game with this kind of naïve expectation [e.g., (Teocharis, 1960; Puu, 1991; Puu, 1996; Puu, 1998; Agiza, 1998; Kopel, 1996; Ahmed & Agiza, 1998; Agliari et al., 2000; Rosser, 2002)]. As a more sophisticated kind of learning rule with respect to naïve expectations, adaptive expectations or adaptive adjustments have been proposed in other dynamical models [e.g., (Okuguchi, 1970; Bischi & Kopel, 2001; Agiza et al., 1999; Rassenti et al., 2000; Szidarovszky & Okuguchi, 1988)]. In recent years, many researchers have paid attention to a kind of bounded rationality, with which a player (without complete information of the demand function) uses local knowledge to update output by the marginal profit. Bischi and Naimzada (1999) gave a general formula of the dynamical Cournot model with this form of bounded rationality, assuming that producers behave as local profit maximizers in a local adjustment process, "where each firm increases its output if it perceives a positive marginal profit and decreases its production if the perceived marginal profit is negative (Bischi & Naimzada, 1999)". Much work has been done on the dynamical Cournot games performed by players with this kind of marginal profit method. The models with homogeneous players (all players are boundedly rational players and use the marginal profit method to adjust strategies) are considered in Agiza et al. (2001), Agiza et al. (2002), Ahmed et al. (2000), and Bischi and Naimzada (1999). Some other work has focused on modeling the system with heterogeneous expectations. Agiza and Elsadany (2003) and Zhang et al. (2007) studied the dynamics of a Cournot duopoly game with one bounded rationality player and one naïve player. Agiza and Elsadany (2004) and Dubiel-Teleszynski (2011) considered a duopoly model in which one player has bounded rationality and the other has adaptive expectation. In the model by Fan et al. (2012), there is one player using the marginal profit method and one player adjusting production in terms of the market price in the previous period. Ding et al. (2009) studied the dynamics of a two-team Cournot game with heterogeneous players.

In these models for dynamical Cournot game, output is a key variable and each player is able to take any needed output updating for the purpose of local profit maximization; thus it is based on an implicit assumption that all players could provide sufficient quantity of products on the market. However, this implicit assumption may be impractical in an economy market where investment plays the most important role. For instance, in an emerging industry with immature development (e.g., a new energy market), it is unlikely for a firm to hold productivity large enough due to its lack of investment accumulation. As a strategic

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behavior in these economic activities, investment accumulation plays a significant role in achieving a good production level. Moreover, we know that even in a mature industry, the production capacity of a firm is greatly dependent on its large-scale investment stock. Only when the investment comes up to a certain level can a firm provide as much goods as the market demands. Therefore, during the developing period of an infant industry, the competition among producers lies mainly in their investment strategy. To obtain a competitive market share and get superiority over opponents, producers must consider their investment strategies in successive periods.

The main purpose of our work is to formulate a novel model, which puts investment decision as a substitute for output adjustment into the dynamical Cournot game. In our model, all producers are also assumed to have bounded rationality and make their investment decisions in line with the marginal profit in the previous period. That is to say, each firm will increase its investment if it perceives a positive marginal profit and decrease its investment if the perceived marginal profit is negative. It is analyzed as to how this novel dynamical Cournot game, in a local adjustment process, evolves to equilibrium or exhibits complicated dynamic behaviors.

This article is organized as the following. In Section 2, we model the dynamical investment game played by players with bounded rationality. In Section 3, we discuss the existence and local stability of the equilibrium points for the system. In Section 4, we show the dynamic features of this system with numerical simulations, including bifurcation diagram, phase portrait, stable region and sensitive dependence on initial conditions. In Section 5, time-delayed feedback control is used to stabilize the chaotic behaviors of the system.

2. The model

Our work focuses on firms' investment competition rather than their output competition. We pay attention to a duopoly investment game, where producers' investment choices are substituted for their output decisions discussed in classic Cournot games.

We consider a competition between two firms (players), labeled by $i = 1,2$, producing homogeneous goods. The strategy of each firm is to choose an investment in every period. Both players make their decisions in discrete periods $t = 0,1,2 \cdots$. We write $K_i(t-1)$ for firm i's capital stock in period $t-1$, and $x_i(t)$ for its investment in period t. We pay our attention mainly to relatively long economic periods and assume that the previous accumulated capital $K_i(t-1)$, after depreciating in a period, keeps a residual value $\theta K_i(t-1)$ ($0 < \theta < 1$, identical for both firms). Then the capital stock $K_i(t)$ for firm i is formulated as

$$K_i(t) = \theta K_i(t-1) + x_i(t), \quad i = 1,2. \tag{1}$$

We suppose that for firm i, its totally accumulated capital $K_i(t)$ determines its potential production capacity in period t and it produces at full capacity to make its output $q_i(t)$. Namely, $q_i(t)$ is assumed to be a function of $K_i(t)$. For simplicity, we consider a linear form of output function: $q_i(t) = B_iK_i(t)$, where B_i is a positive constant. From Eq. (1) we have

$$q_i(t) = B_i(\theta K_i(t-1) + x_i(t)), \quad i = 1,2. \tag{2}$$

For the price in the market, we consider a linear inverse demand function (Agiza & Elsadany, 2003; Agiza & Elsadany, 2004; Agiza et al., 2001; Bischi & Naimzada, 1999; Ding et al., 2009; Dubiel-Teleszynski, 2011; Rassenti et al., 2000; Szidarovszky & Okuguchi, 1988; Zhang et al., 2007):

$$p(t) = a - bQ(t), \tag{3}$$

where $a > 0, b > 0$, and $Q(t) = q_1(t) + q_2(t)$ is the total supply by both firms. We also suppose that the production cost function of each firm takes a linear form (Agiza & Elsadany, 2003; Agiza & Elsadany, 2004; Agiza et al., 2002; Agliari et al., 2000; Ahmed et al., 2000; Bischi & Naimzada, 1999; Rassenti et al., 2000):

$$C_i(q_i(t)) = c_iq_i(t), \quad i = 1,2, \tag{4}$$

where c_1 and c_2 are both positive.

With all the assumptions above, we get firm i's profit in period t as follows:

$$\begin{aligned}\pi_i(x_1(t),x_2(t)) &= q_i(t)p(t) - C_i(q_i(t)) - x_i(t)\\ &= (aB_i - B_ic_i)(\theta K_i(t-1) + x_i(t)) - bB_i^2(\theta K_i(t-1) + x_i(t))^2\\ &\quad - bB_iB_j(\theta K_i(t-1) + x_i(t))\left(\theta K_j(t-1) + x_j(t)\right) - x_i(t)\\ i \neq j, &\quad i,j = 1,2.\end{aligned} \tag{5}$$

By differentiating $\pi_i(x_1(t),x_2(t))$ $(i = 1,2)$, we obtain firm i's marginal profit with respect to its investment in period $t(i = 1,2$, respectively):

$$\begin{aligned}\phi_1(t) &= \frac{\partial \pi_1(x_1(t),x_2(t))}{\partial x_1(t)}\\ &= aB_1 - 1 - B_1c_1 - 2bB_1^2(\theta K_1(t-1) + x_1(t))\\ &\quad - bB_1B_2(\theta K_2(t-1) + x_2(t)),\end{aligned} \tag{6a}$$

$$\begin{aligned}\phi_2(t) &= \frac{\partial \pi_2(x_1(t),x_2(t))}{\partial x_2(t)}\\ &= aB_2 - 1 - B_2c_2 - bB_1B_2(\theta K_1(t-1) + x_1(t))\\ &\quad - 2bB_2^2(\theta K_2(t-1) + x_2(t)).\end{aligned} \tag{6b}$$

We suppose both players have bounded rationality and use the marginal profit method to update their investment strategy in the next period, as supposed in the existing work on the classical Cournot games for output competition [e.g., (Agiza et al., 2001; Agiza et al., 2002; Agiza & Elsadany, 2003; Agiza & Elsadany, 2004; Ahmed et al., 2000; Bischi & Naimzada, 1999; Ding et al., 2009; Dubiel-Teleszynski, 2011; Fan et al., 2012; Zhang et al., 2007)]. It means that each firm will increase its investment flow in period $t+1$ if the marginal profit in the current period t is positive; otherwise the firm will decrease its investment. Then the investment adjustment mechanism for player i can be modeled as:

$$x_i(t+1) = x_i(t) + \alpha_i(x_i(t))\phi_i(t), \quad i = 1,2, \tag{7}$$

where $\alpha_i(x_i(t))$ is a positive function which gives the extent of firm i's investment variation based on its marginal profit. For simplicity, we also put the function $\alpha_i(x_i(t))$ in a linear form (Agiza & Elsadany, 2003; Agiza & Elsadany, 2004; Agiza et al., 2001; Agiza et al., 2002; Bischi & Naimzada, 1999; Ding et al., 2009; Dubiel-Teleszynski, 2011; Fan et al., 2012; Zhang et al., 2007): $\alpha_i(x_i(t)) = \alpha_i x_i(t)$, where α_i is a positive constant representing the relative adjustment speed. Then the dynamics (7) takes its form as:

$$x_i(t+1) = x_i(t) + \alpha_i x_i(t)\phi_i(t), \quad i = 1,2. \tag{8}$$

From Eqs. (1), (6a)–(6b) and (8), we obtain a four-dimensional discrete dynamic system:

$$\begin{cases} x_1(t+1) = x_1(t) + \alpha_1x_1(t)(aB_1 - 1 - B_1c_1 - 2bB_1^2(\theta K_1(t-1)\\ \qquad + x_1(t)) - bB_1B_2(\theta K_2(t-1) + x_2(t))),\\ x_2(t+1) = x_2(t) + \alpha_2x_2(t)(aB_2 - 1 - B_2c_2 - bB_1B_2(\theta K_1(t-1)\\ \qquad + x_1(t)) - 2bB_2^2(\theta K_2(t-1) + x_2(t))),\\ K_1(t) = \theta K_1(t-1) + x_1(t),\\ K_2(t) = \theta K_2(t-1) + x_2(t).\end{cases} \tag{9}$$

If we denote $K_i(t-1)$ by $I_i(t)$ (and hence $K_i(t)$ by $I_i(t+1)$), $i=1{,}2$, then we can rewrite system (9) as the following standard dynamics

$$\begin{cases} x_1(t+1) = x_1(t) + \alpha_1 x_1(t)(aB_1 - 1 - B_1c_1 - 2bB_1^2(\theta I_1(t) + x_1(t)) \\ \qquad - bB_1B_2(\theta I_2(t) + x_2(t))), \\ x_2(t+1) = x_2(t) + \alpha_2 x_2(t)(aB_2 - 1 - B_2c_2 - 2bB_2^2(\theta I_2(t) + x_2(t)) \\ \qquad - bB_1B_2(\theta I_1(t) + x_1(t))), \\ I_1(t+1) = \theta I_1(t) + x_1(t), \\ I_2(t+1) = \theta I_2(t) + x_2(t). \end{cases} \quad (10)$$

System (10) describes a duopoly game played by two boundedly rational players making decision in a process of dynamical investment. In the following sections, we are to investigate the dynamical properties of this model.

3. Equilibrium points and stability

Let $x_i(t+1) = x_i(t)$ and $I_i(t+1) = I_i(t)$ $(i=1{,}2)$ in system (10), then we get

$$\begin{cases} x_1(t)\Big(aB_1 - 1 - B_1c_1 - 2bB_1^2(\theta I_1(t) + x_1(t)) - bB_1B_2(\theta I_2(t) + x_2(t))\Big) = 0, \\ x_2(t)\Big(aB_2 - 1 - B_2c_2 - 2bB_2^2(\theta I_2(t) + x_2(t)) - bB_1B_2(\theta I_1(t) + x_1(t))\Big) = 0, \\ (\theta - 1)I_1(t) + x_1(t) = 0, \\ (\theta - 1)I_2(t) + x_2(t) = 0. \end{cases} \quad (11)$$

Solving equations in Eq. (11), we obtain four equilibrium states of dynamics (Eq. (10)), which are listed as follows:

$$E_0 = (0, 0, 0, 0),$$

$$E_1 = \left(0, \frac{(1-\theta)(aB_2 - 1 - B_2c_2)}{2bB_2^2}, 0, \frac{aB_2 - 1 - B_2c_2}{2bB_2^2}\right),$$

$$E_2 = \left(\frac{(1-\theta)(aB_1 - 1 - B_1c_1)}{2bB_1^2}, 0, \frac{aB_1 - 1 - B_1c_1}{2bB_1^2}, 0\right),$$

$$E^* = (x_1^*, x_2^*, I_1^*, I_2^*),$$

where

$$x_1^* = \frac{(1-\theta)(B_1B_2(a + c_2 - 2c_1) + B_1 - 2B_2)}{3bB_1^2B_2},$$

$$x_2^* = \frac{(1-\theta)(B_1B_2(a + c_1 - 2c_2) + B_2 - 2B_1)}{3bB_1B_2^2},$$

$$I_1^* = \frac{B_1B_2(a + c_2 - 2c_1) + B_1 - 2B_2}{3bB_1^2B_2},$$

$$I_2^* = \frac{B_1B_2(a + c_1 - 2c_2) + B_2 - 2B_1}{3bB_1B_2^2}.$$

E_0,E_1 and E_2 are all boundary equilibriums and E^* is a unique interior equilibrium. In order to make these equilibrium points have economic meaning, we only consider the nonnegative cases. Since b, B_1, B_2 and θ are positive parameters, E_1,E_2 and E^* are all positive provided that

$$aB_1 - 1 - B_1c_1 > 0, \quad (12a)$$

$$aB_2 - 1 - B_2c_2 > 0, \quad (12b)$$

$$B_1B_2(a + c_2 - 2c_1) + B_1 - 2B_2 > 0, \quad (12c)$$

$$B_1B_2(a + c_1 - 2c_2) + B_2 - 2B_1 > 0. \quad (12d)$$

In the following, all the nonnegativity conditions (12a)–(12d) are assumed.

3.1. Stability of the boundary equilibriums

To investigate the local stability of an equilibrium (x_1,x_2,I_1,I_2) of system (10), we work out its Jacobian matrix J:

$$J(x_1, x_2, I_1, I_2) = \begin{pmatrix} A_1 & -bB_1B_2\alpha_1x_1 & -2\theta bB_1^2\alpha_1x_1 & -\theta bB_1B_2\alpha_1x_1 \\ -bB_1B_2\alpha_2x_2 & A_2 & -\theta bB_1B_2\alpha_2x_2 & -2\theta bB_2^2\alpha_2x_2 \\ 1 & 0 & \theta & 0 \\ 0 & 1 & 0 & \theta \end{pmatrix}, \quad (13)$$

where

$$A_1 = 1 + \alpha_1(aB_1 - 1 - B_1c_1) - 2\alpha_1bB_1^2(\theta I_1 + 2x_1) - b\alpha_1B_1B_2(\theta I_2 + x_2),$$

$$A_2 = 1 + \alpha_2(aB_2 - 1 - B_2c_2) - 2\alpha_2bB_2^2(\theta I_2 + 2x_2) - b\alpha_2B_1B_2(\theta I_1 + x_1).$$

An equilibrium (x_1,x_2,I_1,I_2) will be locally asymptotically stable if all the eigenvalues (real or complex) of the Jacobian matrix $J(x_1,x_2,I_1,I_2)$ lie inside the unit disk, i.e. $|\lambda| < 1$ holds for any eigenvalue λ of $J(x_1,x_2,I_1,I_2)$. An equilibrium (x_1,x_2,I_1,I_2) will be unstable if there is an eigenvalue λ of $J(x_1,x_2,I_1,I_2)$ such that $|\lambda| > 1$.

Proposition 1. *The boundary equilibrium E_0 is an unstable equilibrium.*

Proof. Taking the expression of the equilibrium E_0 into Eq. (13), we get the Jacobian matrix at E_0 as the following:

$$J(E_0) = \begin{pmatrix} 1 + \alpha_1(aB_1 - 1 - B_1c_1) & 0 & 0 & 0 \\ 0 & 1 + \alpha_2(aB_2 - 1 - B_2c_2) & 0 & 0 \\ 1 & 0 & \theta & 0 \\ 0 & 1 & 0 & \theta \end{pmatrix},$$

which has four eigenvalues:

$$\lambda_1 = \lambda_2 = \theta, \lambda_3 = 1 + \alpha_1(aB_1 - 1 - B_1c_1), \lambda_4 = 1 + \alpha_2(aB_2 - 1 - B_2c_2).$$

From the nonnegativity conditions (12a)–(12b) and the positivity of the parameter α_i, it follows that $|\lambda_{3,4}| > 1$. So the equilibrium E_0 is unstable.

Proposition 2. *The boundary equilibriums E_1 and E_2 are both unstable.*

Proof. At the boundary equilibrium point E_1, the Jacobian matrix (Eq. (13)) is given by

$$J(E_1) = \begin{pmatrix} 1 + \frac{U}{2B_2} & 0 & 0 & 0 \\ \frac{(\theta - 1)\alpha_2B_1V}{2B_2} & 1 + (\theta - 1)\alpha_2V & \frac{\theta(\theta - 1)\alpha_2B_1V}{2B_2} & \theta(\theta - 1)\alpha_2V \\ 1 & 0 & \theta & 0 \\ 0 & 1 & 0 & \theta \end{pmatrix},$$

where $U = \alpha_1(B_1B_2(a + c_2 - 2c_1) + B_1 - 2B_2)$,$V = aB_2 - 1 - B_2c_2$. By simple calculation, we get four eigenvalues of the matrix $J(E_1)$:

$$\lambda_1 = \theta,$$
$$\lambda_2 = 1 + \frac{\alpha_1(B_1B_2(a + c_2 - 2c_1) + B_1 - 2B_2)}{2B_2},$$

$$\lambda_{3,4} = \frac{1}{2}(1 + \theta + \alpha_2(\theta - 1)(aB_2 - 1 - B_2c_2) \pm \sqrt{-4\theta + (1 + \theta + \alpha_2(\theta - 1)(aB_2 - 1 - B_2c_2))^2}).$$

According to the inequality (Eq. (12c)) and the condition that B_2 and α_1 are both positive, we see that $\lambda_2 > 1$ and hence conclude that the equilibrium E_1 is unstable. A similar approach shows that E_2 is unstable too.

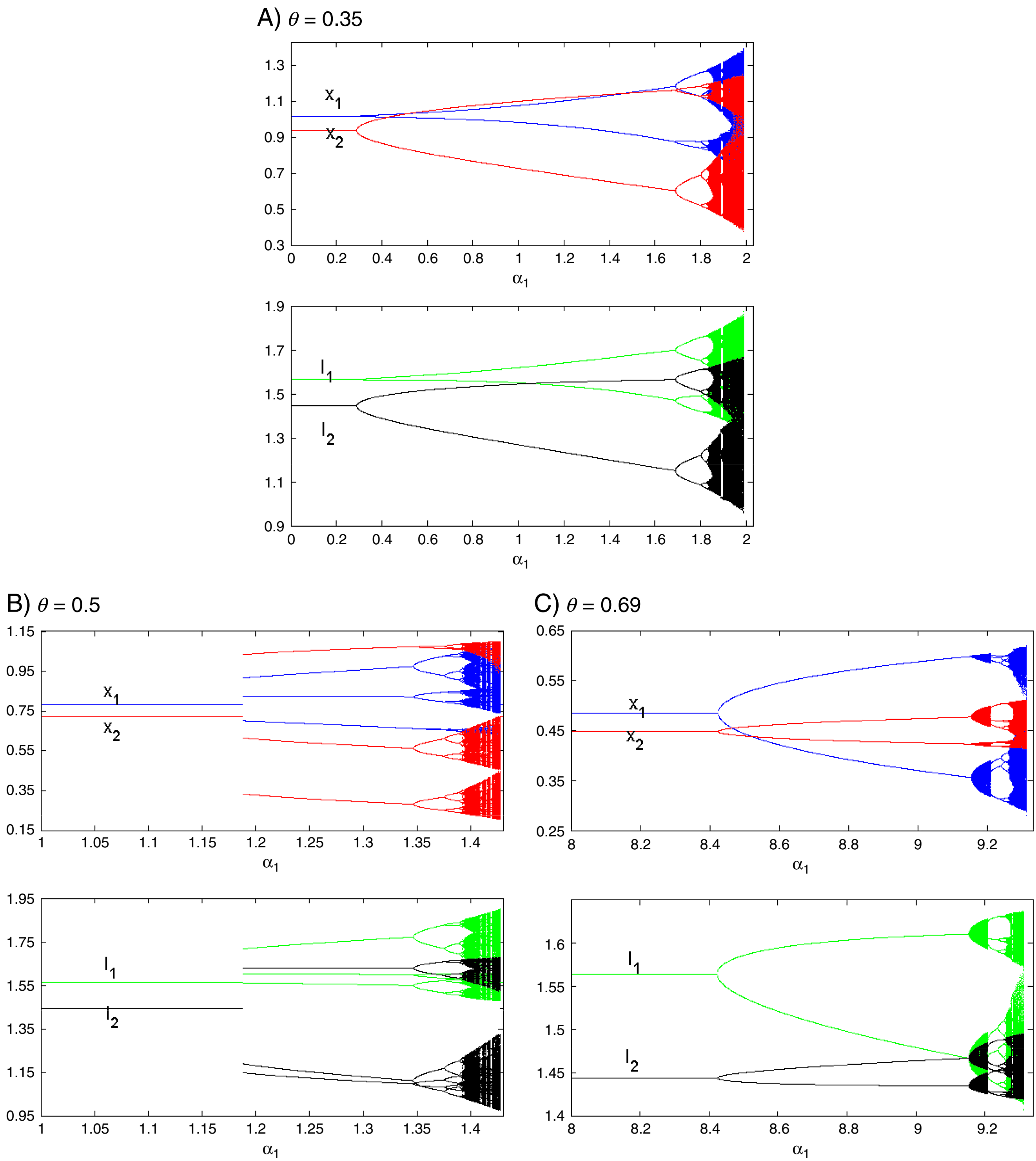

Fig. 1. Bifurcation diagrams with respect to the adjustment rate $\boldsymbol{\alpha}_1$.

3.2. Stability of the interior equilibrium

Now we consider the asymptotical stability of the interior equilibrium E^*. The Jacobian matrix J at $E^* = (x_1^*, x_2^*, I_1^*, I_2^*)$ takes its form as

$$J(E^*) = \begin{pmatrix} 1 + \frac{2(\theta-1)U}{3B_2} & \frac{(\theta-1)U}{3B_1} & \frac{2\theta(\theta-1)U}{3B_2} & \frac{\theta(\theta-1)U}{3B_1} \\ \frac{(\theta-1)W}{3B_2} & 1 + \frac{2(\theta-1)W}{3B_1} & \frac{\theta(\theta-1)W}{3B_2} & \frac{2\theta(\theta-1)W}{3B_1} \\ 1 & 0 & \theta & 0 \\ 0 & 1 & 0 & \theta \end{pmatrix},$$

where U is the same one denoted in $J(E_1)$ and $W = \alpha_2(B_1B_2(a + c_1 - 2c_2) - 2B_1 + B_2)$. If $P(\lambda)$ denotes the characteristic polynomial of the Jacobian matrix $J(E^*)$, then

$$P(\lambda) = \lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4.$$

By calculation, we get

$$p_1 = \frac{2}{3B_1B_2}[B_1^2\alpha_1(1-\theta)X + B_2^2\alpha_2(1-\theta) + B_1B_2(-2\alpha_1(1-\theta) + \alpha_2(1-\theta)Y - 3(\theta+1))],$$

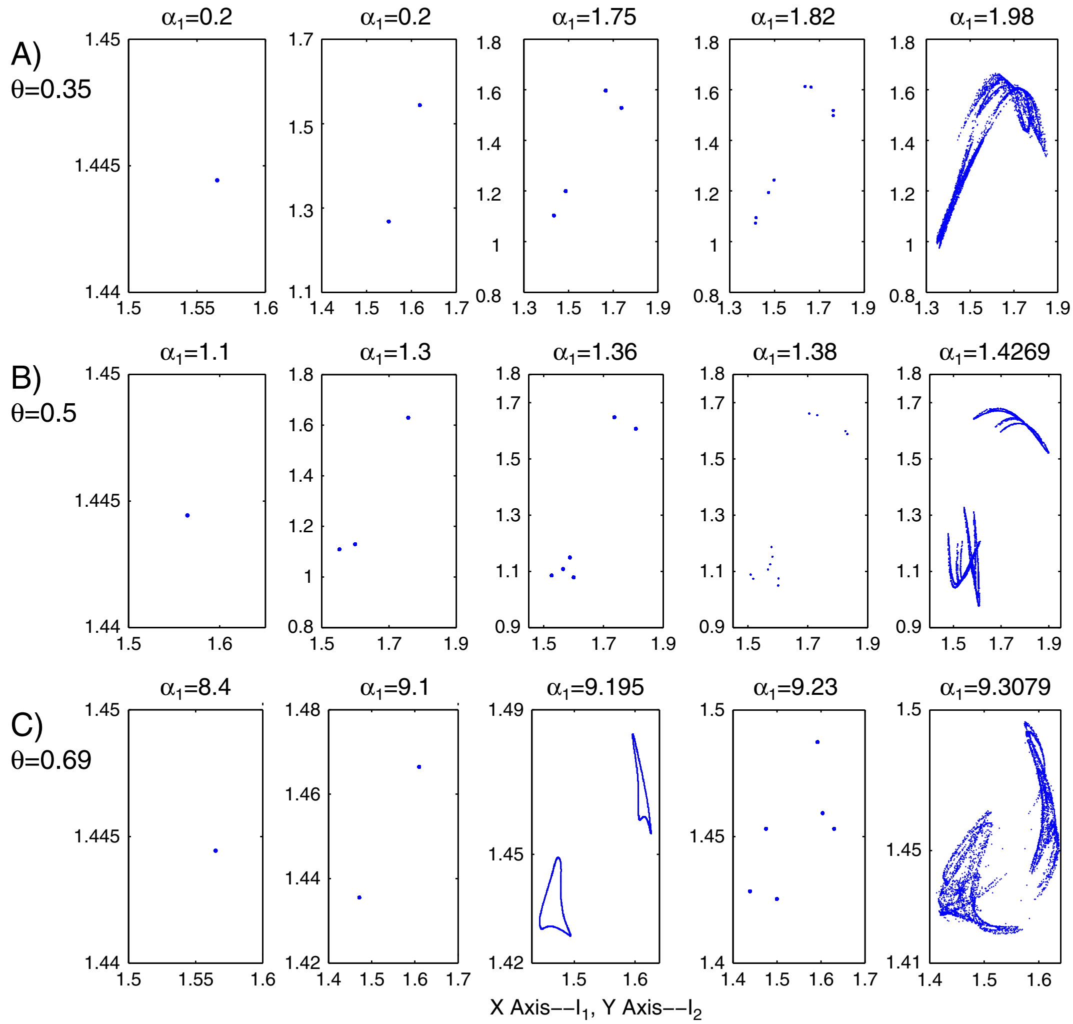

Fig. 2. Phase portraits for Fig. 1(A, B, C) with various values of θ and α_1.

$$p_2 = \frac{1}{3B_1B_2}[2B_2^2\alpha_2(\alpha_1(\theta-1)-1-\theta)(1-\theta) + B_1^2\alpha_1(1-\theta)X(\alpha_2(1-\theta)Y-2(\theta+1)) + B_1B_2\left(2\alpha_2\left(\theta^2-1\right)Y+3\left(1+4\theta+\theta^2\right)\right) + \alpha_1(1-\theta)(\alpha_2(\theta-1)(aB_2+4B_2c_1-5B_2c_2-5)+4(1+\theta)))],$$

$$p_3 = \frac{2\theta}{3B_1B_2}[B_1^2\alpha_1(1-\theta)X+B_2^2\alpha_2(1-\theta) + B_1B_2(-2\alpha_1(1-\theta)+\alpha_2(1-\theta)Y-3(1+\theta))],$$

$$p_4 = \theta^2,$$

where $X = aB_2 + B_2c_2 - 2B_2c_1 + 1$, $Y = aB_2 + B_2c_1 - 2B_2c_2 - 2$.

For all the roots of the polynomial $P(\lambda)$ (the eigenvalues of the Jacobian matrix $J(E^*)$) to lie inside the unit disk, Schur–Cohn Criterion [e.g., (Elaydi, 2005)] gives the necessary and sufficient conditions as:

(i) $P(1) > 0$;
(ii) $(-1)^4P(-1) > 0$;
(iii) The determinants of the 1×1 matrices $M_1^{\pm}$ and the 3×3 matrices $M_3^{\pm}$ are all positive, where

$$M_1^{\pm} = 1 \pm p_4, \quad M_3^{\pm} = \begin{pmatrix} 1 & 0 & 0 \\ p_1 & 1 & 0 \\ p_2 & p_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & p_4 \\ 0 & p_4 & p_3 \\ p_4 & p_3 & p_2 \end{pmatrix}.$$

In our model, we have

$$P(1) = 1+p_1+p_2+p_3+p_4 = \frac{\alpha_1\alpha_2(\theta-1)^2(B_1B_2(a+c_1-2c_2)-2B_1+B_2)(B_1B_2(a+c_2-2c_1)+B_1-2B_2)}{3B_1B_2}.$$

Then from the nonnegativity conditions (12c)–(12d), we know that the first condition $P(1) > 0$ must hold. In addition, $Det(M_1^{\pm}) = 1 \pm p_4 = 1 \pm \theta^2 > 0$ also holds since $0 < \theta < 1$. Consequently, we conclude that the interior equilibrium point E^* of system (10) is asymptotically locally stable if it meets the following conditions

$$P(-1) = 1-p_1+p_2-p_3+p_4>0, \tag{14a}$$

$$Det\left(M_3^+\right)>0, Det(M_3^-)>0, \tag{14b}$$

where $Det(M)$ represents the determinant of the matrix M.

4. Numerical simulation

In this section, we show by numerical simulations how the system evolves under different levels of parameters, especially of the capital residual rate θ and the adjustment speed α. In all the numerical simulations, the other parameters are fixed: $a = 5$, $b = 1$, $c_1 = 0.3$, $c_2 = 0.5$, $B_1 = 0.6$ and $B_2 = 0.8$.

For three cases of the capital residual rate θ, Fig. 1 is about bifurcation diagrams of system (10) with respect to the adjustment rate α_1 while

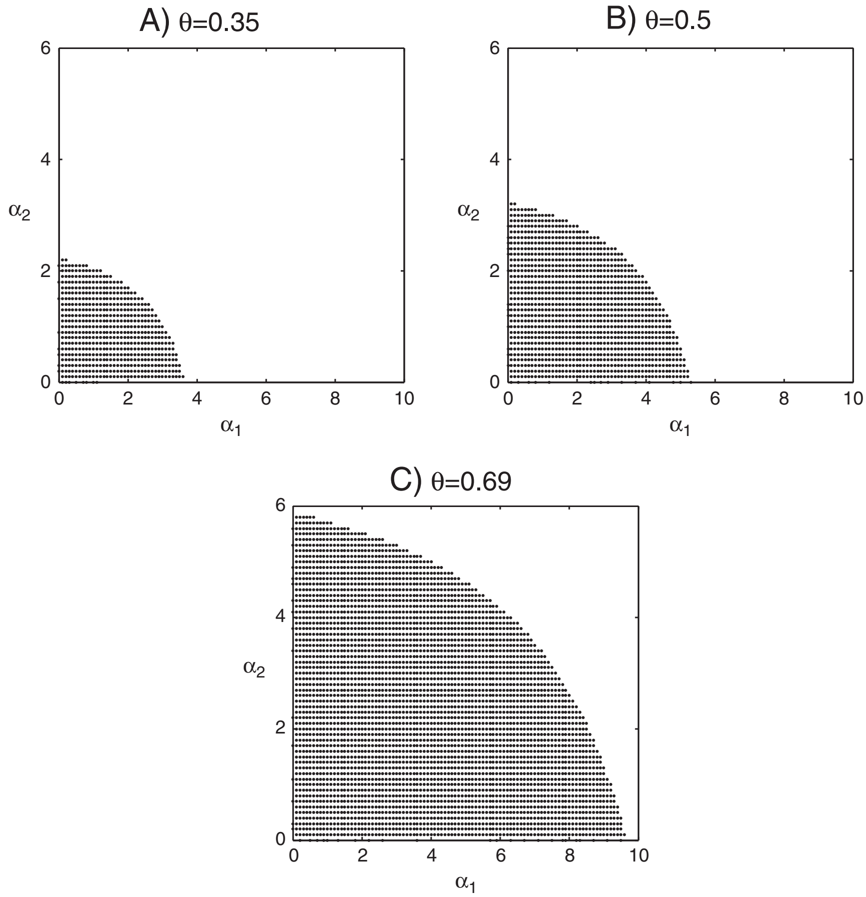

Fig. 3. Stability region in (α_1, α_2)-plane for different levels of θ.

the other one is fixed as $\alpha_2 = 2.2$. Fig. 1(A) shows the bifurcation diagram for $\theta = 0.35$: period doubling bifurcation comes up at about $\alpha_1 = 0.285$ and chaos occurs while α_1 increases to about 1.85. Fig. 1(B) is about the case $\theta = 0.5$: period trebling bifurcation occurs at about $\alpha_1 = 1.18$ and period doubling bifurcation follows when α_1 increases, and chaos occurs in the end of the bifurcation process. In Fig. 1(C) for $\theta = 0.69$, more complicated dynamical behaviors can be observed. The equilibrium is stable for α_1 taking its value up to about 8.42, then it becomes unstable and bifurcates into two stable fixed points; Neimark–Sacher bifurcation occurs at about $\alpha_1 = 9.15$, then a period window follows after about $\alpha_1 = 9.21$; and such a complicated process (period doubling bifurcation, Neimark–Sacher bifurcation, periodic window, etc.) continues leading the system to chaos.

The observations from Fig. 1(A) and (B) tell that the Nash equilibrium of the system can loose stability via flip bifurcations, and Fig. 1(C) shows that the stability loss may also be due to Neimark–Sacker bifurcations. So the system can drive to chaos either in a flip bifurcation process or in a Neimark–Sacher bifurcation process. About the three cases in Fig. 1, two-dimension phase portraits for some values of α_1 are shown in Fig. 2, which gives a more detailed description of the orbits of the system. The phase portraits in Fig. 2(A) ($\theta = 0.35$) are obviously related to a period doubling process leading to chaos. Fig. 2(B) ($\theta = 0.5$) also shows a flip bifurcation process, where chaos occurs in the end after periods 3,6,12,⋯. Fig. 2(C) ($\theta = 0.69$) shows that there exits a Neimark–Sacker bifurcation so that a couple of closed invariant curves is observed when the system loses stability, and it also shows that multi-period orbit follows after the Neimark–Sacker bifurcation and chaotic attractors occur in the end.

Besides different bifurcation processes leading to chaos, Fig. 1 also shows that the point that the system begins to lose stability will be much different when the residual rate θ changes. In Fig. 1(A), (B) and (C) for $\theta = 0.35$, $\theta = 0.5$ and $\theta = 0.69$, the equilibrium stability loss begins about at, respectively, $\alpha_1 = 0.285$, 1.18 and 8.42. It tells that if the residual rate θ is higher (i.e. the depreciation rate is lower), the equilibrium instability will be delayed much more and the system will be led to chaos much later. That is, a larger residual rate (or a smaller depreciation rate) has a stronger stabilization effect on the system's dynamical evolution. This conclusion can be also drawn from another kind of simulation as shown in Fig. 3. By computer work on the stability conditions (18a)–(18b) for three cases ($\theta = 0.35$, 0.5, 0.69), stable regions in the (α_1,α_2)-plane are numerically obtained and are plotted in Fig. 3. Comparing Fig. 3(A), (B) and (C), we see that an increasing θ expands the stability region rapidly.

Fig. 4 shows the sensitivity of system (when loosing stability) to initial conditions, with $\theta = 0.5$, $\alpha_1 = 1.426$, $\alpha_2 = 2.2$, $x_2(1) = 0.72$, $I_1(1) = 1.56$ and $I_2(1) = 1.44$. Fig. 4(A) plots the orbits of firms' investment: the blue ones take an initial value $x_1(1) = 0.78$ and the red ones take a slightly deviated value $x_{11}(1) = 0.78001$. Fig. 4(B) is about the orbits of firms' capital stock under the same initial conditions as Fig. 4(A). Fig. 4(A) and (B) shows that the difference between the orbits with slightly deviated initial values builds up rapidly after a number of iterations, although their initial states are indistinguishable.

In Fig. 5(A) and (B), the bifurcation diagrams are plotted for the capital residual rate θ when α_2 is fixed as 2.2. Fig. 5(A) is for $\alpha_1 = 1.98$ and Fig. 5(B) is for $\alpha_1 = 9.305$. Fig. 5(A) shows a reverse period-doubling bifurcation, while Fig. 5(B) combines reverse period-doubling bifurcation and reverse Neimark–Sacher bifurcation. The two figures show that the system trends towards stability with the residual rate increased, which also tells that a high residual rate has a positive effect on the system stability.

5. Chaos control

From the numerical simulations above, we see that the adjustment rate and the capital residual rate have great influence on the stability

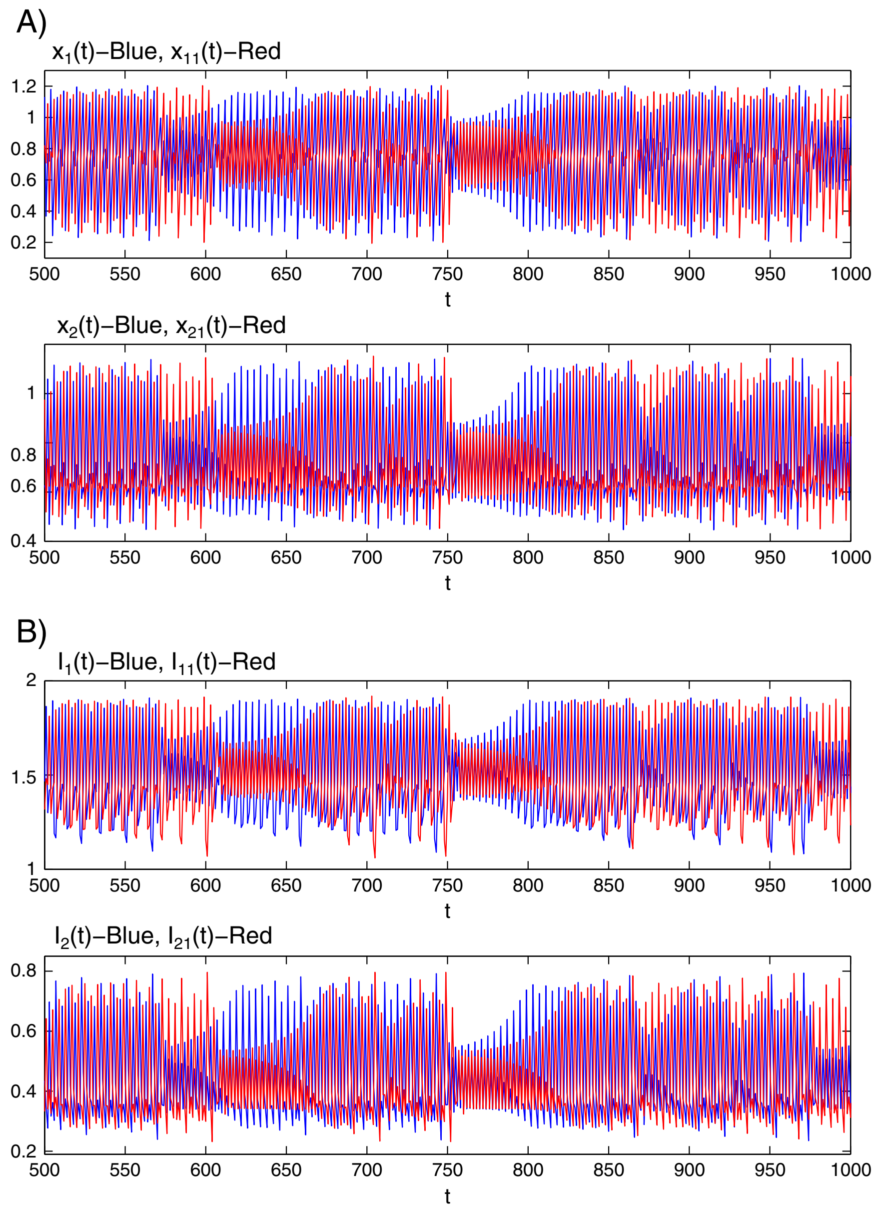

Fig. 4. Sensitivity of the system to initial conditions when losing stability.

of system (10). If the model parameters fail to locate into the stable region required, the behaviors of the dynamics will be much complicated. In a real economic system, chaos is not desirable and will be not expected, and it is needed to be avoided or controlled so that the dynamic system would work better. In this section, we use the time-delayed feedback control [e.g., (Elabbasy et al., 2009; Ding et al., 2011; Holyst & Urbanowicz, 2000)] to control system chaos. We modify the first equation of system (10) by intercalating a controller $k(x_t - x_{t+1})$ as a small perturbation, where $k > 0$ is a controlling coefficient. Then the controlled system is given by

$$\begin{cases} x_1(t+1) = x_1(t) + \alpha_1 x_1(t)(aB_1 - 1 - B_1c_1 - 2bB_1^2(\theta I_1(t) + x_1(t)) \\ \qquad - bB_1B_2(\theta I_2(t) + x_2(t))) + k(x(t) - x(t+1)), \\ x_2(t+1) = x_2(t) + \alpha_2 x_2(t)(aB_2 - 1 - B_2c_2 - 2bB_2^2(\theta I_2(t) + x_2(t)) \\ \qquad - bB_1B_2(\theta I_1(t) + x_1(t))), \\ I_1(t+1) = \theta I_1(t) + x_1(t), \\ I_2(t+1) = \theta I_2(t) + x_2(t). \end{cases} \tag{15}$$

It is easy to see that the new system (15) has the same equilibriums as system (10) and it takes the following equivalent form:

$$\begin{cases} x_1(t+1) = x_1(t) + \dfrac{1}{1+k}\alpha_1 x_1(t)(aB_1 - 1 - B_1c_1 - 2bB_1^2(\theta I_1(t) + x_1(t)) \\ \qquad - bB_1B_2(\theta I_2(t) + x_2(t))), \\ x_2(t+1) = x_2(t) + \alpha_2 x_2(t)(aB_2 - 1 - B_2c_2 - 2bB_2^2(\theta I_2(t) + x_2(t)) \\ \qquad - bB_1B_2(\theta I_1(t) + x_1(t))), \\ I_1(t+1) = \theta I_1(t) + x_1(t), \\ I_2(t+1) = \theta I_2(t) + x_2(t). \end{cases} \tag{16}$$

The Jacobian matrix of the controlled system (16) is given by

$$J(x_1, x_2, I_1, I_2) = \begin{pmatrix} 1 + \dfrac{\alpha_1}{1+k}A_1 & -\dfrac{bB_1B_2\alpha_1x_1}{1+k} & -\dfrac{2\theta bB_1^2\alpha_1x_1}{1+k} & -\dfrac{\theta bB_1B_2\alpha_1x_1}{1+k} \\ -bB_1B_2\alpha_2x_2 & 1+\alpha_2A_2 & -\theta bB_1B_2\alpha_2x_2 & -2\theta bB_2^2\alpha_2x_2 \\ 1 & 0 & \theta & 0 \\ 0 & 1 & 0 & \theta \end{pmatrix}, \tag{17}$$

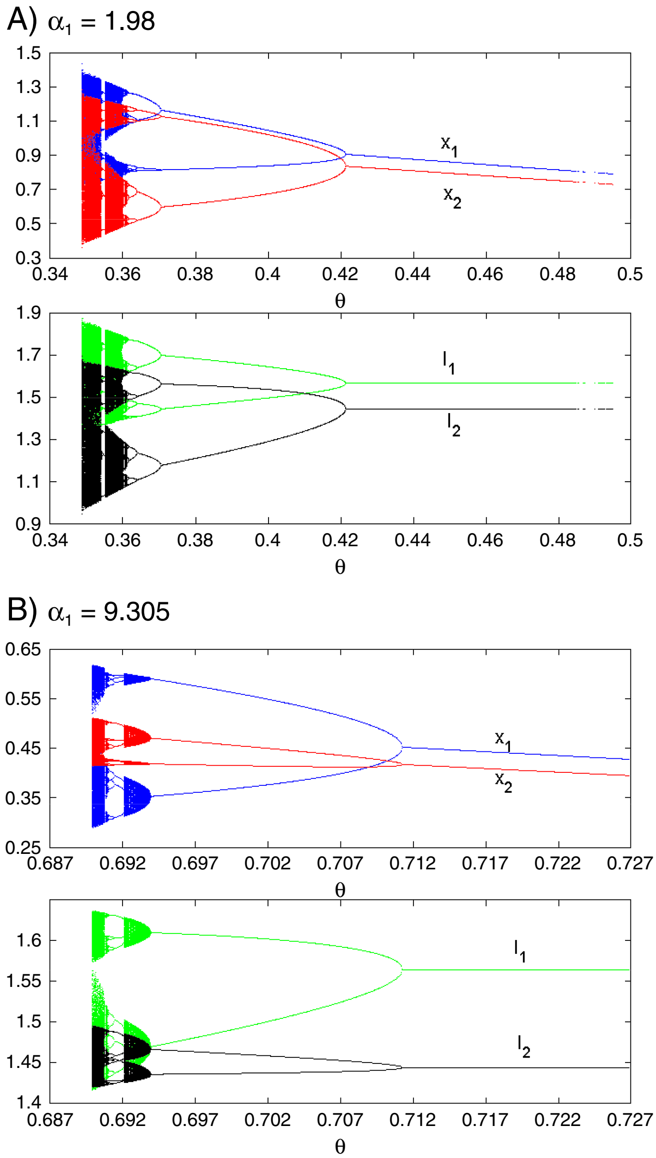

Fig. 5. Bifurcation diagrams with respect to the residual rate θ.

where $A_1 = aB_1 - 1 - B_1c_1 - 2bB_1^2(\theta I_1 + 2x_1) - bB_1B_2(\theta I_2 + x_2)$ and $A_2 = aB_2 - 1 - B_2c_2 - 2bB_2^2(\theta I_2 + 2x_2) - bB_1B_2(\theta I_1 + x_1)$.

As has been shown in Fig. 1(B), chaotic behavior of system (10) occurs when all the model parameters take their values as

$$(a, b, B_1, B_2, c_1, c_2, \alpha_1, \alpha_2, \theta) = (5, 1, 0.6, 0.8, 0.3, 0.5, 1.426, 2.2, 0.5).$$

Using this group of parameters values, we obtain the Jacobian matrix (Eq. (17)) at the interior equilibrium as the following

$$J(x_1, x_2, I_1, I_2) = \begin{pmatrix} 1-\dfrac{1.22408}{1+k} & -\dfrac{0.643411}{1+k} & -\dfrac{0.482558}{1+k} & -\dfrac{0.321706}{1+k} \\ -0.91872 & -2.02368 & -0.45936 & -1.22496 \\ 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0.5 \end{pmatrix}. \quad (18)$$

From the stability conditions (14a)–(14b), we get that all the eigenvalues of the matrix (18) will lie inside the unit disk provided that $k > 0.2005$. That is, when $k > 0.2005$ the controlled system (16) will be asymptotically locally stable.

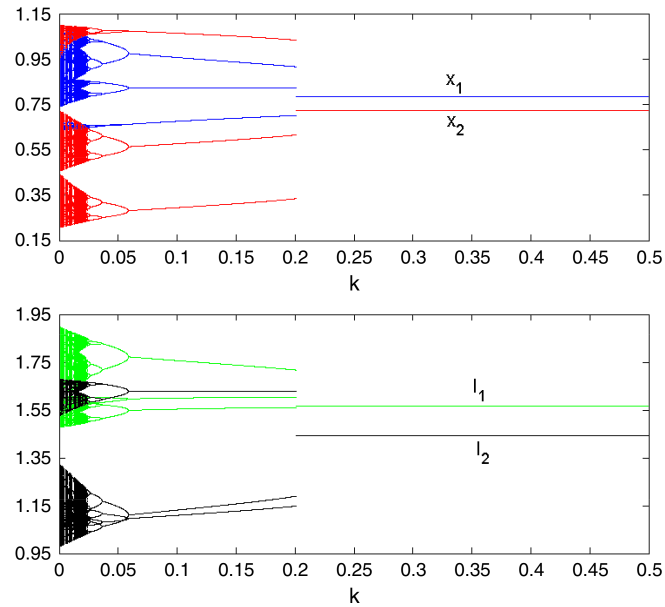

Fig. 6. Bifurcation diagram with respect to the controlling factor k.

In Fig. 6, it is actually observed that with the control coefficient k increasing, the system gradually gets out of chaos and periodic windows and achieves to stability when $k > 0.2005$. For $k = 0.25$, Fig. 7 shows the stable behaviors of the orbits of the controlled system beginning from the initial state $(x_1(0), x_2(0), I_1(0), I_2(0)) = (0.94, 0.87, 1.56.1.44)$.

6. Conclusion

In this work we have taken into consideration firms' investment decisions as substitute for the output choices considered by the existing work on classic Cournot games. We have formulated a novel Cournot form of investment game played by two players with bounded rationality. The main idea in our model is that each firm's decision is to choose its investment in each period according to the marginal profit observed from the previous period. We have established a corresponding

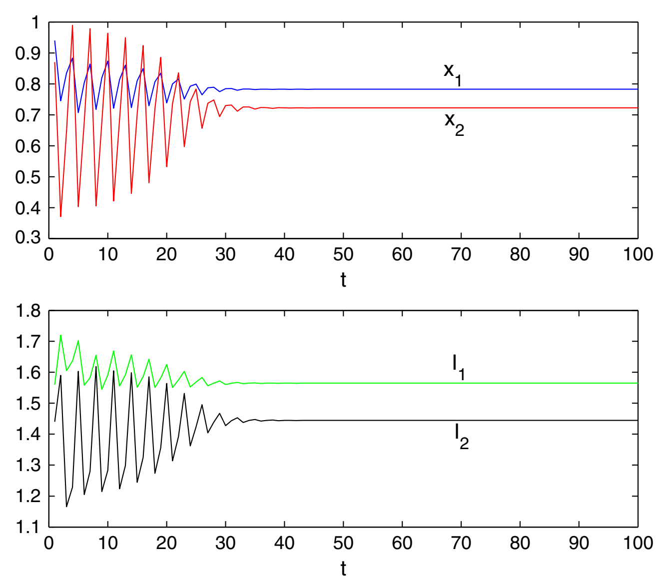

Fig. 7. Stability of equilibrium with $k = 0.25$.

dynamics of players' investment adjustment and done a detailed dynamic analysis for it. There are three boundary equilibriums and a unique interior equilibrium in this system. We have shown the instability of the boundary equilibriums and found the conditions for local stability of the interior equilibrium by Schur–Cohn Criterion. We have made similar numerical simulations for the system evolution as done in other existing work on classic Cournot games for output competition, including bifurcation diagrams, phase portraits, stable regions and sensitivity to initial state. It is shown that a relatively high residual rate (or low depreciation rate) can strengthen the system stability. It is observed that the equilibrium of the system may loose stability via different bifurcations, either flip bifurcation or Neimark–Sacker bifurcation. It is also shown that time-delayed feedback control can be used to stabilize the chaotic behaviors of the system.

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