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Research Article

Nonlinear Dynamics in the Solow Model with Bounded Population Growth and Time-to-Build Technology

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The aim of this paper is to show that the Solow model equipped by realistic assumptions on technology and population dynamics is capable of explaining a well-known stylised fact of growth, that is, the presence of persistent oscillations of demoeconomic variables. In particular, our analysis shows that the coexistence of delays between (i) new investment and production and (ii) the birth date and the recruitment in the labour force is a source of cyclical behaviour in capital accumulation.

1. Introduction

Following a long tradition, the theory of economic growth is nevertheless mostly concerned with constant population growth rates. There are only a few theoretical papers that consider more complex and realistic assumption on population dynamics. For example, Guerrini [1] generalizes the result of existence and global stability of the (stationary) equilibrium in the Solow model by assuming a bounded, but not necessarily constant, population growth rate; Fanti and Manfredi [2] show in a Solow-type model that the existence of distributed delays (age structure) that influence the labour force and a Malthusian mechanism that relates fertility and wage rates are sources of fluctuations in growth paths; and Fabião and Borges [3] study the effects of fluctuating population size on capital accumulation. Other cases of nontrivial population dynamics may be observed in models where agents allocate their resources between childbearing and consumption (e.g., [4–6]).

The idea of the present paper is to study a Solow-type model in which one of the main characteristics is the introduction of reproductive mechanism that prevents unbounded population growth in the long run. In particular, we assume that (i) population dynamics is described by a delayed logistic equation [7], which presents a constant carrying capacity and (ii) a time-to-build technology characterizes the productive sector. Namely, we introduce two different sources of time-delays: with regard to the first assumption, it implies that there exists a lag between the birth date and the recruitment in the labour force; with regard to the second one, it means that production occurs with a delay while new capital is installed (investment gestation lag). Both processes affect the capital accumulation path by means of production function.

It is interesting to notice that the theoretical question whether lags in productive activity induce cycles is a topic that dates back to early economists such as Jevons [8] and Kalecki [9], and the introduction of new mathematical tools for the study of functional differential equations (e.g., the influential book of Hassard et al. [10]) has revitalized this issue. Nonetheless, the large part of the literature is focused on investment lags (e.g., [11–14]), and the role of other kinds of lags in economic growth models or the linkages between population dynamics and lags in productive sector has received little attention. Exceptions are the papers of Fanti and Manfredi [2] described before; Bianca and Guerrini [15], who embed the problem of population dynamics in a network structure; Fanti et al. [16] who study problems related to demographic transition and the emergence of low-fertility traps in a model in which a system of differential equations with distributed delays is introduced to describe the age distribution of the population; Guerrini and Sodini [17] who analyze the role of the assumption of a positive or a negative constant population growth rate in a model with a time-to-build technology; and Ballestra et al. [18] who consider a Kaleckian type model of business cycle where a negative effect of the capital stock increases disproportionally as the capital stock builds up.

In the present paper, two alternative hypotheses on capital depreciation are explored. In the first hypothesis, it is assumed that the *productive capital* (i.e., the capital that actually is used in the productive process) depreciates at a constant rate (e.g., [11]). In the second one, it the case in which depreciation affects directly the *investments* even if they have not yet created *productive capital* is explored. The second case represents a plausible assumption if we consider capital in a broad sense that includes intangible capitals such as new inventions, prototypes, and patents. It is worth noting that, differently by Hongliang and Wenzao [19], business cycles phenomena may be observed even in the second case.

The paper is organised as follows. In Section 2, the models are formally introduced. Section 3 is focused on the study of the first model specification. In Section 4, the second model is analyzed. In Section 5, the global properties of the models are investigated via numerical simulations. Section 5 concludes.

2. The Models

2.1. First Specification. We introduce a Solow [20] model with the following characteristics. We assume that at time t identical and competitive firms produce a homogeneous good, Y, by combining capital and labour K and L, respectively, through the constant returns to scale Cobb-Douglas technology. New investments display a delay of period τ before they can be used for production (time-to-build technology). Thus, the evolution of capital accumulation is governed by the following equation:

$$\dot{K} = sAK_d^{\alpha}L^{1-\alpha} - \delta K_d, \tag{1}$$

where A is a positive parameter representing (exogenous) technological progress, $\alpha \in (0,1)$ measures the productivity of capital, $s \in (0,1)$ is the exogenous constant saving rate, $K_d := K(t-\tau)$ is the state value of K at time $t-\tau$, and δ is the constant rate of productive capital depreciation (in this specification, obsolescence only affects capital if and only if this last is used in a productive process).

We assume that population of workers evolves according to the delayed logistic equation introduced by Hutchinson (see Arino et al. [21] for an alternative formulation):

$$\dot{L} = L(a - bL_d), \tag{2}$$

where $D = a/b$ is the constant carrying capacity and $L_d := L(t-\tau)$ is the state value of L at time $t-\tau$. Notice that the presence of the delayed variable L_d captures the lag between the birth date and the recruitment in the labour force while D defines a limit to population growth. Other specifications (see [16]) that use delayed differential equations are mostly concerned with the description age-specific fertility and mortality rates.

Setting $k = K/L$, we have

$$\begin{aligned}\dot{k} &= \frac{\dot{K}}{L} - \frac{\dot{L}}{L}k = \frac{s\left(AK_d^{\alpha}L^{1-\alpha}\right) - \delta K_d}{L} - \frac{\dot{L}}{L}k \\ &= sA\left(L^{-1}L_d\right)^{\alpha}k_d^{\alpha} - \delta\left(L^{-1}L_d\right)k_d - \frac{\dot{L}}{L}k.\end{aligned} \tag{3}$$

Hence, the model is described by

$$\begin{aligned}\dot{k} &= \left[sA\left(L^{-1}L_d\right)^{\alpha}\right]k_d^{\alpha} - \left[\delta\left(L^{-1}L_d\right)\right]k_d - (a - bL_d)k, \\ \dot{L} &= L(a - bL_d).\end{aligned} \tag{4}$$

2.2. Second Specification. The second model differs from the first one in the fact that obsolescence affects capital regardless of its utilization in the productive process. Thus, (4) is replaced by the following:

$$\dot{K} = sAK_d^{\alpha}L^{1-\alpha} - \delta K. \tag{5}$$

Setting $k = K/L$, we obtain

$$\begin{aligned}\dot{k} &= \frac{\dot{K}}{L} - \frac{\dot{L}}{L}k = \frac{s\left(AK_d^{\alpha}L^{1-\alpha}\right) - \delta K}{L} - \frac{\dot{L}}{L}k \\ &= sA\left(L^{-1}L_d\right)^{\alpha}k_d^{\alpha} - \delta k - \frac{\dot{L}}{L}k.\end{aligned} \tag{6}$$

As a result, the model is now represented by

$$\begin{aligned}\dot{k} &= \left[sA\left(L^{-1}L_d\right)^{\alpha}\right]k_d^{\alpha} - \left[\delta + (a - bL_d)\right]k, \\ \dot{L} &= L(a - bL_d).\end{aligned} \tag{7}$$

It is simple to verify that systems (4) and (7) have the same (nontrivial) equilibria. By setting $\dot{k} = 0$ and $k_d = k = k_*$, $\dot{L} = 0$, and $L_d = L = L_*$ for all t, we find that there exists a unique nontrivial equilibrium (k_*, L_*), where

$$k_* = \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)}, \qquad L_* = \frac{a}{b}. \tag{8}$$

Notice that because the population dynamics is bounded from above and the level of technological progress is fixed, in the present work, differently from Guerrini and Sodini [17], a balanced growth path is not a feasible solution and (k_*, L_*) identifies the stationary solution of the model, for which $K = k_*L_*$.

3. Local Analysis of System (4)

In order to study the local properties of the system around nontrivial equilibrium, we linearize system (4) at (k_*, L_*). This gives

$$\begin{aligned}\begin{bmatrix}\dot{k} \\ \dot{L}\end{bmatrix} &= \begin{bmatrix}0 & \delta L_*^{-1}\left(-\alpha L_*^{-1} + k_*\right) \\ 0 & 0\end{bmatrix}\begin{bmatrix}k - k_* \\ L - L_*\end{bmatrix} \\ &\quad + \begin{bmatrix}(\alpha - 1)\delta & (\alpha - 1)\delta L_*^{-1}k_* + bk_* \\ 0 & -bL_*\end{bmatrix}\begin{bmatrix}k_d - k_* \\ L_d - L_*\end{bmatrix}.\end{aligned} \tag{9}$$

The characteristic equation can be derived from

$$\det\left\{\begin{bmatrix}0 & \delta L_*^{-1}k_*\left(-\alpha L_*^{-1}+1\right)\\ 0 & 0\end{bmatrix}-\lambda I + e^{-\lambda\tau}\begin{bmatrix}(\alpha-1)\delta & (\alpha-1)\delta L_*^{-1}k_*+bk_*\\ 0 & -bL_*\end{bmatrix}\right\}=0, \tag{10}$$

where I is the identity matrix. A direct calculation shows that the above equation can be rewritten as

$$P(\lambda,\tau)=P_1(\lambda,\tau)\cdot P_2(\lambda,\tau)=0, \tag{11}$$

where

$$P_1(\lambda,\tau)=\lambda+(1-\alpha)\delta e^{-\lambda\tau}, \qquad P_2(\lambda,\tau)=\lambda+ae^{-\lambda\tau}. \tag{12}$$

By putting $\tau=0$ in (11), we obtain $[\lambda+(1-\alpha)\delta](\lambda+a)=0$. We derive that the two roots of this equation are negative. Hence, the equilibrium (k_*,L_*) is stable in case there is no delay. As τ increases, the stability of the equilibrium point will change when (11) under consideration has zero or a pair of purely imaginary eigenvalues. The former occurs when $\lambda=0$, and this is not possible. The latter deals with the existence of a root $\lambda=i\omega$ for (11). This means we need to investigate when $P_1(i\omega,\tau)=0$ and/or $P_2(i\omega,\tau)=0$. Without loss of generality, since the complex roots of (11) appear as complex conjugate pairs, we assume that $\omega>0$.

Lemma 1. *Equation $P_1(\lambda,\tau)=0$ (resp., $P_2(\lambda,\tau)=0$) with $\tau=\tau_j^{(1)}$ (resp., $\tau=\tau_j^{(2)}$), $j\in\mathbb{N}^0=\mathbb{N}\cup\{0\}$, has a pair of purely imaginary roots $\lambda=\pm i\omega_1$ (resp., $\lambda=\pm i\omega_2$), where*

$$\omega_1=(1-\alpha)\delta\ (\text{resp. } \omega_2=a),$$
$$\tau_j^{(1)}=\frac{1}{\omega_1}\left(\frac{\pi}{2}+2j\pi\right)\left(\text{resp. } \tau_j^{(2)}=\frac{1}{\omega_2}\left(\frac{\pi}{2}+2j\pi\right)\right). \tag{13}$$

Proof. Consider $P_1(i\omega,\tau)=0$; that is, $i\omega+(1-\alpha)\delta e^{-i\omega\tau}=0$. Separating real and imaginary parts of this equation, we obtain

$$\omega=(1-\alpha)\delta\sin\omega\tau, \qquad 0=(1-\alpha)\delta\cos\omega\tau. \tag{14}$$

From $\cos\omega\tau=0$ in (14), and ω and $(1-\alpha)\delta$ being both positive, it follows that $\omega\tau=\pi/2+2j\pi$ $(j=0,1,2,\ldots)$ and $\omega=(1-\alpha)\delta$. We also obtain a sequence of the critical values $\tau_j^{(1)}$ of τ. Next, suppose $P_2(i\omega,\tau)=0$; that is, $i\omega+ae^{-i\omega\tau}=0$. Then

$$\omega=a\sin\omega\tau, \qquad 0=a\cos\omega\tau, \tag{15}$$

which lead to $\omega=a$ and the existence of $\tau_j^{(2)}$. This concludes the proof. □

We easily obtain the following result.

Proposition 2. *(1) If $a\neq(1-\alpha)\delta$, then characteristic equation* (11) *has a pair of purely imaginary roots $\lambda=\pm i\omega_m$ at $\tau=\tau_j^{(m)}$, $m=1,2$, $j\in\mathbb{N}^0$.*

(2) If $a=(1-\alpha)\delta$, then characteristic equation (11) *has a multiple root with the multiplicity of two.*

Remark 3. When $a=(1-\alpha)\delta$, the system (4) is a degenerated case and it is difficult to determine the crossing direction of the characteristic roots through the imaginary axis. Moreover, the criteria of stability switches fail to analyze the stability switches of system (4).

Henceforth, we assume $a\neq(1-\alpha)\delta$, so that (11) has no repeated roots.

Lemma 4. *For $\tau=\tau_j^m$, $j\in\mathbb{N}^0$, $\pm i\omega_m$, $m=1,2$, are simple roots of* (11) *with*

$$\left.\frac{d(\operatorname{Re}\lambda)}{d\tau}\right|_{\lambda=i\omega_m}>0. \tag{16}$$

Proof. Both $\lambda=i\omega_1$ and $\lambda=i\omega_2$ are simple. If we suppose, by contradiction, that, for example, $\lambda=i\omega_1$ is a repeated root, then $P(i\omega_1,\tau)=0$ and $dP(i\omega_1,\tau)/d\lambda=0$ lead to a contradiction. Let $\lambda(\tau)=\mu(\tau)+i\omega(\tau)$ be the branch of characteristic roots of (11) such that $\mu(\tau_j^{(1)})=0$ and $\omega(\tau_j^{(1)})=\omega_1$ $(j=0,1,2,\ldots)$. Taking the derivative of λ with respect to τ in (11), it follows that

$$\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda=i\omega_1}=-\frac{1}{\lambda^2}-\frac{\tau}{\lambda}. \tag{17}$$

Then, we get

$$\begin{aligned}\operatorname{sign}\left\{\left.\frac{d(\operatorname{Re}\lambda)}{d\tau}\right|_{\lambda=i\omega_1}\right\}&=\operatorname{sign}\left\{\operatorname{Re}\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda=i\omega_1}\right\}\\&=\operatorname{sign}\left\{\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda=i\omega_1}\right\}=\operatorname{sign}\left\{\frac{1}{\omega_1^2}\right\}.\end{aligned} \tag{18}$$

Similarly for $\lambda=i\omega_2$, the statement holds. □

Therefore, a pair of pure imaginary roots will cross the imaginary axis from left to right as τ increases. Furthermore, this also yields to the existence of a Hopf bifurcation at the equilibrium when $\tau=\tau_j^m$. Using this last lemma and the previous analysis about the existence of purely imaginary roots of (11), we can state the following theorem.

Theorem 5. *Let $\tau_j^{(m)}$, $m=1,2$, $j\in\mathbb{N}^0$, be defined as in* (13), *and let $\tau_0=\min\{\tau_0^{(1)},\tau_0^{(2)}\}$, where $\tau_0^{(1)}=\pi/[(1-\alpha)]\delta$, $\tau_0^{(2)}=\pi/(2a)$.*

1. *The equilibrium point (k_*,L_*) of* (4) *is locally asymptotically stable for $\tau\in[0,\tau_0)$ and unstable for $\tau>\tau_0$.*
2. *System* (4) *undergoes a Hopf bifurcation at the equilibrium (k_*,L_*) when $\tau=\tau_j^{(m)}$, $m=1,2$, $j\in\mathbb{N}^0$.*

4. Local Analysis of System (7)

In this section, we consider the model described by system (7). It is known that the modification of the Zak model [11] with the assumption that capital obsolescence affects the present capital makes the stationary equilibrium globally asymptotically stable for any value of time-delay (no Hopf bifurcation may occur) (see [19]). We will see that the introduction of population dynamics drastically changes the results.

As in the previous model, in order to study the local dynamics, we linearize system (7) at (k_*, L_*). We have

$$\begin{bmatrix} \dot{k} \\ \dot{L} \end{bmatrix} = \begin{bmatrix} -\delta & -\alpha\delta L_*^{-2} k_* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k - k_* \\ L - L_* \end{bmatrix} + \begin{bmatrix} \alpha\delta & \alpha\delta L_*^{-1} k_* + b k_* \\ 0 & -b L_* \end{bmatrix} \begin{bmatrix} k_d - k_* \\ L_d - L_* \end{bmatrix}. \quad (19)$$

The associated characteristic equation of system (19) is

$$\det \left\{ \begin{bmatrix} -\delta & -\alpha\delta L_*^{-2} k_* \\ 0 & 0 \end{bmatrix} - \lambda I + e^{-\lambda\tau} \begin{bmatrix} \alpha\delta & \alpha\delta L_*^{-1} k_* + b k_* \\ 0 & -b L_* \end{bmatrix} \right\} = 0 \quad (20)$$

and takes the form

$$Q(\lambda, \tau) = Q_1(\lambda, \tau) \cdot Q_2(\lambda, \tau) = 0, \quad (21)$$

where

$$Q_1(\lambda, \tau) = \lambda + \delta - \alpha\delta e^{-\lambda\tau}, \qquad Q_2(\lambda, \tau) = \lambda + a e^{-\lambda\tau}. \quad (22)$$

When $\tau = 0$, all roots of (21) are negative, so that the system is stable. We will let τ vary and investigate possible stability switches and bifurcations. For (k_*, L_*) to become unstable, characteristic roots have to cross the imaginary axis to the right when τ increases. Let $\lambda = i\omega$ $(\omega > 0)$ be a purely imaginary root of (21). Then, it satisfies $Q_1(i\omega, \tau) = 0$ and/or $Q_2(i\omega, \tau) = 0$. Let $Q_1(i\omega, \tau) = 0$. This can be rewritten as the following two equations:

$$\omega = -\alpha\delta \sin \omega\tau, \qquad \delta = \alpha\delta \cos \omega\tau, \quad (23)$$

which imply

$$\omega^2 = \left(\alpha^2 - 1\right) \delta^2 < 0. \quad (24)$$

Therefore, $Q_1(\lambda, \tau)$ has no purely imaginary roots. Noticing that $Q_2(\lambda, \tau) = P_2(\lambda, \tau)$, with $P_2(\lambda, \tau)$ defined as in (12), it follows that $Q_2(\lambda, \tau)$ has a unique pair of simple purely imaginary roots $\pm i\omega_2$ at a sequence of critical values $\tau_j^{(2)}$, where ω_2 and $\tau_j^{(2)}$ are defined by (13). Let $\lambda(\tau) = \mu(\tau) + i\omega(\tau)$ denote the roots of (21) near $\tau = \tau_j^{(2)}$ satisfying $\mu(\tau_j^{(2)}) = 0$ and $\omega(\tau_j^{(2)}) = \omega_2$ $(j \in \mathbb{N}^0)$. Then, the following transversality condition $d[\operatorname{Re} \lambda(\tau_j^{(2)})]/d\tau > 0$ holds and so the root crosses the imaginary axis from left to right (at $\tau = \tau_j^{(2)}$) as τ increases. Since (k_*, L_*) is locally asymptotically stable when $\tau = 0$, then it becomes unstable at the smallest value of τ for which an imaginary root exists. Thus, we can get the following results.

Theorem 6. *Let $\tau_j^{(2)}$, $j \in \mathbb{N}^0$, be defined as in* (13)*, where $\tau_0^{(2)} = \pi/(2a)$.*

(1) *The equilibrium point (k_*, L_*) of* (7) *is locally asymptotically stable for $\tau \in [0, \tau_0^{(2)})$ and unstable for $\tau > \tau_0^{(2)}$.*

(2) *System* (7) *undergoes a Hopf bifurcation at the equilibrium (k_*, L_*) when $\tau = \tau_j^{(2)}$ for $j \in \mathbb{N}^0$.*

Remark 7. Notice that bifurcation value of both the models is the same if $\alpha > \delta/(2 + \delta)$.

5. Numerical Simulations

In order to understand the dynamical properties of the model, we consider a numerical specification and let τ vary. In particular, we set

$$A = 1; \quad \delta = 0.09; \quad \alpha = 0.43; \quad a = 0.6; \quad b = 0.4; \quad s = 0.8. \quad (25)$$

For this parameter specification $(k^*, L^*) = (1.5, 46.2)$ and from Theorem 5, we have $\tau_1 < \tau_0$. Thus, the bifurcation values for the two models are the same and we have verified by means of several numerical experiments that the associated dynamics look very similar. Therefore, we focus on the simulations for model 1. Figure 1 shows the evolution of long run dynamics when τ is varied. It is interesting to note that after bifurcation, differently from Guerrini and Sodini [17], both the state variables oscillate permanently along the unstable equilibrium. If τ is increased, the enlargement of the limit cycle induces larger and larger oscillations. However, by means of several numerical experiments, it seems that only nonchaotic dynamics arise.

6. Conclusions

In this paper, we have analyzed the dynamics in Solow-type model with investment lags and nonstationary but bounded population. We have shown that the interaction between the population dynamics and time-to-build technology may be an engine of cyclical behaviour of capital accumulation. Formally, by applying the techniques developed by Hassard et al. [10] for delayed differential equations, we have proved that a Hopf bifurcation occurs once lag τ passes through a critical value and a family of periodic orbits bifurcates from the stationary equilibrium. Numerical simulations confirm the existence of oscillating dynamics in the demoeconomic variables. More complex structures of time-delays and microeconomic foundation of the model are left for future research.

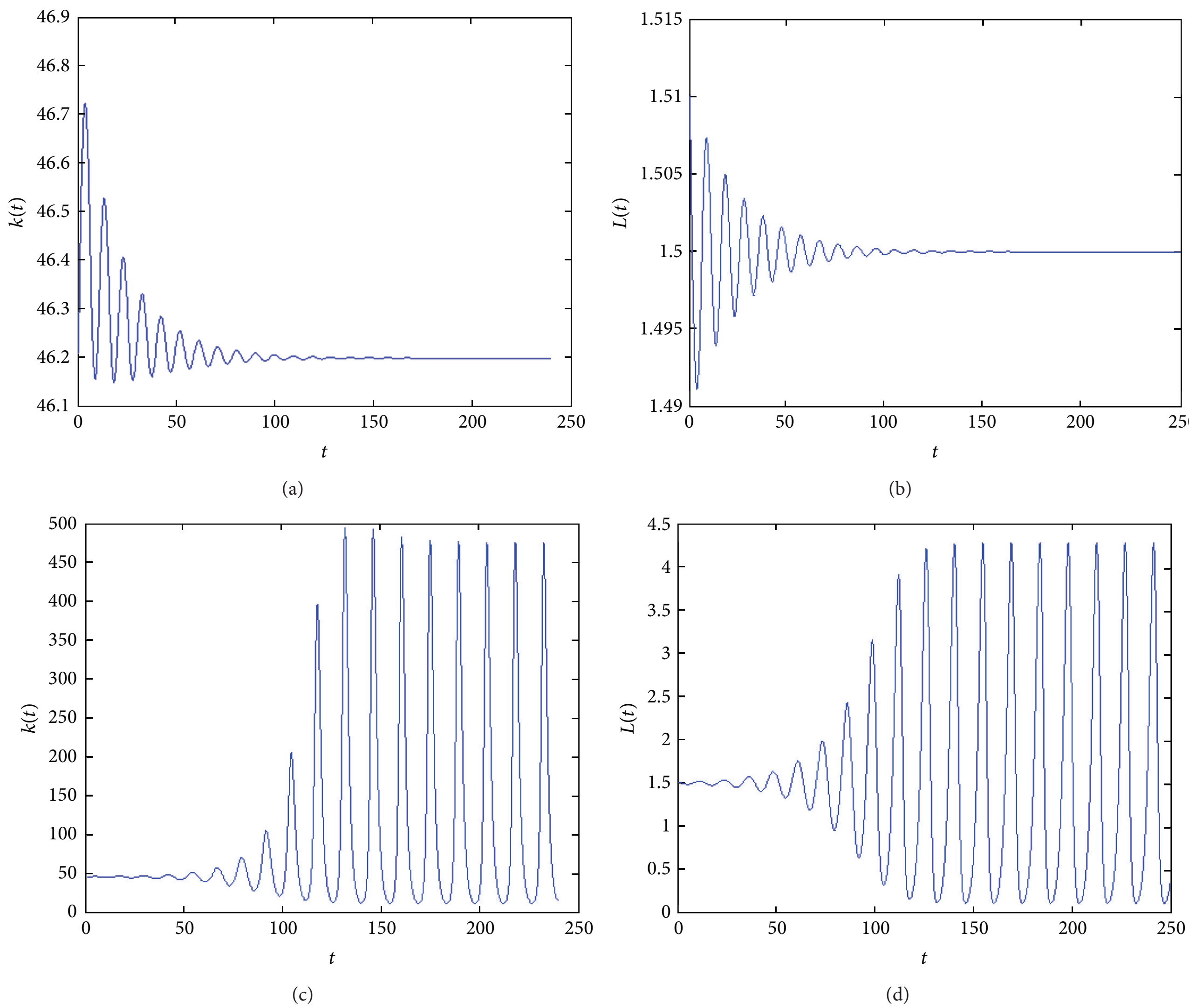

FIGURE 1: (a) Time evolution of size capital per worker ($\tau = 2.3$); (b) time evolution of population ($\tau = 2.3$); (c) time evolution of capital per worker ($\tau = 3.3$); and (d) time evolution of population ($\tau = 3.3$).

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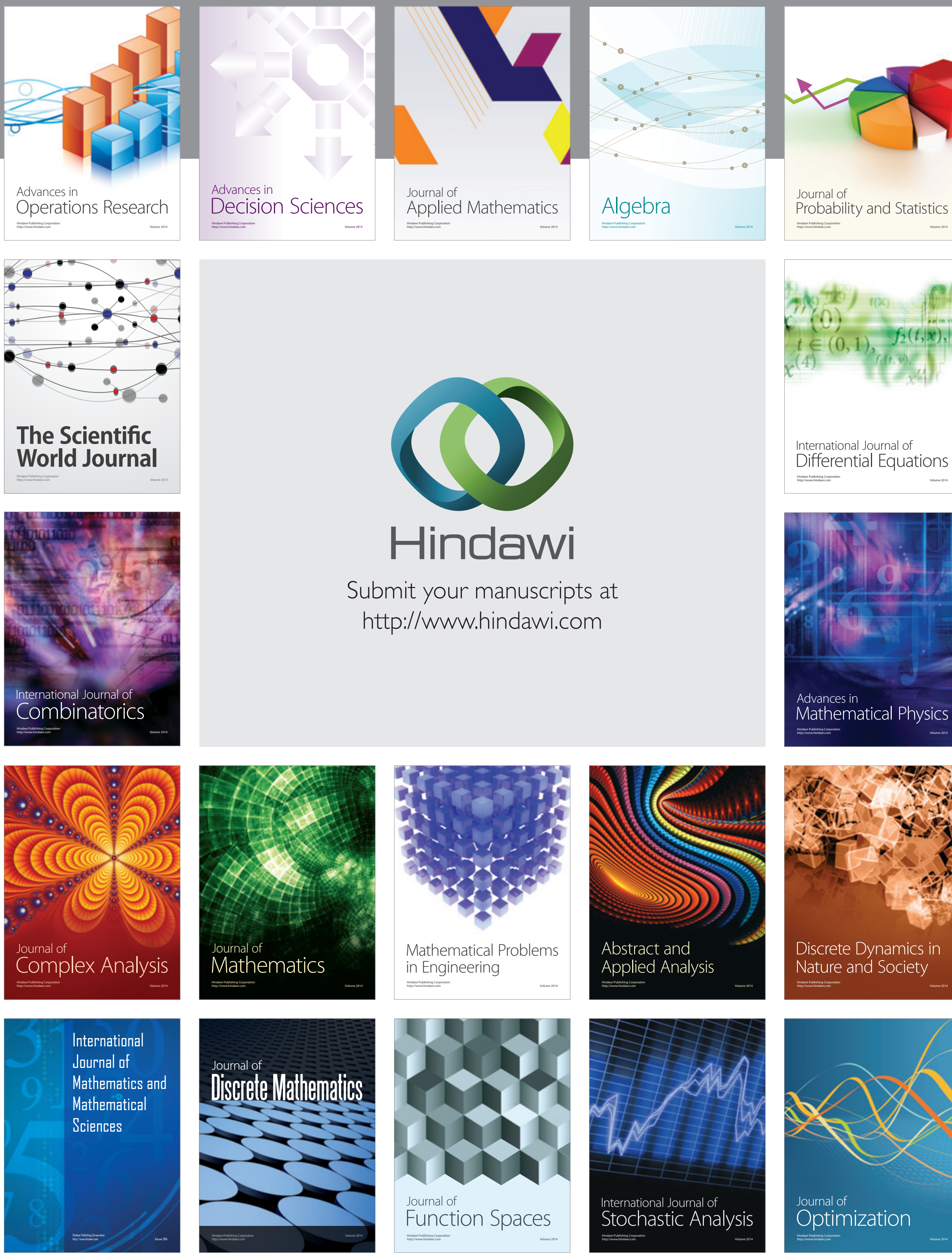
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