International Journal of Bifurcation and Chaos, Vol. 5, No. 1 (1995) 255–263

CHAOTIC BEHAVIOR IN AN ADVERTISING DIFFUSION MODEL

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Received December 18, 1993; Revised April 25, 1994

The paper is concerned with an advertising diffusion model where a firm devotes a fixed proportion of sales to advertising, while customers go through a two-stage adoption process. The model takes the form of a second-order, time-invariant, nonlinear dynamic system, in which a homoclinic bifurcation to infinity is shown to exist. Allowance is subsequently made for a seasonal fluctuation in the firm's advertising rate. A bifurcation study of the periodic solutions is then accomplished by means of a continuation procedure. Emphasis is placed on the emergence of chaos. Only the chaotic solutions stemming from a cascade of period doublings appear to be economically meaningful.

1. Introduction

A consumer cannot purchase a new product if he/she is not aware of it. Thus, for a new product to be successful, there must be an effective spread of information about it. New product diffusion models, first conceived in the 1960s, represent this dissemination process by considering the effects of word-of-mouth, advertising, as well as other communication forms. The simplest models (see, e.g., Bass [1969], Lekvall & Wahlbin [1973], and Dodson & Muller [1978]) focus on demand, disregarding supply and, hence, such questions as price, capacity restrictions, and marketing strategies. The choice optimization problems of buyers and sellers are also neglected. Nonetheless, there is a bulk of empirical evidence supporting these models (see, for instance, the survey by Mahajan *et al.* [1990]), which have also proven to be the most effective in forecasting the diffusion of a new product. As a consequence, they have been used by many companies, as reported by Bass [1986].

The model by Feichtinger [1992a] (see also Feichtinger & Novak [1994]) retains the above-mentioned attitute. The setting consists of a firm selling a new product at a constant price and devoting a fixed proportion of sales to advertising. Customers go through a two-stage adoption process, turning from potential into actual ones under the influence of word-of-mouth and advertising. This results in a second-order, time-invariant, nonlinear dynamic system.

Further examination of this model is given in the present work, where the existence of a homoclinic bifurcation to infinity is pointed out by using

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both geometrical and numerical arguments. Owing to this bifurcation, business failure proves to be unavoidable for some parameter values. The notion of homoclinic bifurcation to infinity was first introduced by Brons & Sturis [1991, 1992], who showed it to occur in a model of the economic long wave proposed by Rasmussen *et al.* [1985].

It is subsequently recognized that both advertising and its effects are likely to possess a seasonal component. For instance, since television watching is more intense in the cold seasons than in the warm ones, television spots generally behave in a similar manner. Moreover, when a product's sales display a seasonal pattern, the advertising policy is usually seasonal, yet not necessarily synchronized with sales [Kotler, 1984; p. 650]. Allowance is accordingly made for a periodic change in the firm's advertising rate, thus giving rise to a periodically forced, second-order, nonlinear dynamic system. A bifurcation study of the periodic solutions is then numerically accomplished by means of a continuation procedure. This eventually uncovers a wide set of asymptotic behaviors, ranging from periodic and quasi-periodic to chaotic solutions. Emphasis is placed on the bifurcation sequences leading to chaos. In the light of the findings about the time-invariant case, it is readily realized why a cascade of period doublings is the only meaningful path to chaos.

In writing the paper, both the economic and the mathematical detail have been kept to a minimum, so as to make the work appealing to a large audience, ranging, in principle, from marketing scholars to applied mathematicians. The paper is organized as follows. The model is briefly introduced in Sec. 2 (a detailed account can be found in Feichtinger [1992a]). The time-invariant case is briefly resumed and further investigated in the same section, while the periodically forced case is dealt with in Sec. 3. Final remarks and directions for future research appear in Sec. 4.

2. An Advertising Diffusion Model

2.1. *Analytical setting and Hopf bifurcation*

Consider a firm launching a new product, a low-priced and frequently repurchased one. Let $x(t)$ and $y(t)$ be the number of potential and actual buyers at time t, respectively. Contacts between potential and actual buyers, essentially through word-of-mouth [Lilien & Kotler, 1983; Chap. 19], can turn the former into the latter. If contacts are random, the rate of generation of actual buyers is $\hat{\alpha}xy$, where the term $\hat{\alpha}$ reflects the word-of-mouth effectiveness. Such an analytical form, originally borrowed from the epidemic field [Kermack & McKendrick, 1927], is standard in the literature about new product diffusion. Mahajan *et al.* [1984], among others, provide empirical evidence in support of it.

Moreover, Simon & Sebastian [1987], among others, provide empirical evidence suggesting that advertising enhances the word-of-mouth effectiveness. Thus, it is also assumed that $\hat{\alpha} = \alpha y$, where the constant α reflects the advertising effectiveness. Such a linear relationship makes sense if actual buyers (re)purchase the product at a constant rate and at a constant price, and the firm devotes a fixed proportion of sales to advertising. The latter, which is consistent with the celebrated rule by Dorfman & Steiner [1954], is also the optimal advertising policy for dynamic optimization problems where both the advertising and the price elasticity of demand are constant [Schmalensee, 1972].

Therefore, $x(t)$ and $y(t)$ are governed by the following differential equations:

$$\dot{x}(t) = k - \alpha x(t)y^2(t) + \beta y(t)\,, \tag{1}$$

$$\dot{y}(t) = \alpha x(t)y^2(t) - (\beta + \varepsilon)y(t)\,. \tag{2}$$

Notice that there is an external source of new potential customers, who enter the market at a constant rate k. Moreover, actual buyers are supposed to leave the market at a rate $(\beta+\varepsilon)y$, where $(\beta+\varepsilon)$ is the removal constant. Some of them are lost forever but some others are not, since they just switch to other brands, thus flowing back at a rate βy into the class of potential buyers.

Remark 2.1. It can be ascertained that (see Feichtinger [1992a, 1992b] for a formal proof):

(a) The system (1),(2) is positive, i.e., $x(0)$, $y(0) \geq 0$ implies that $x(t), y(t) \geq 0$ for all $t \geq 0$.
(b) The system (1),(2) has a unique equilibrium, i.e.,

$$(\overline{x}, \overline{y}) = \left(\frac{(\beta+\varepsilon)\varepsilon}{\alpha\kappa}, \frac{\kappa}{\varepsilon}\right),$$

which is stable for $\alpha > \alpha_{\rm HP}$ and unstable for $\alpha < \alpha_{\rm HP}$, with $\alpha_{\rm HP} = (\beta+\varepsilon)\varepsilon^2/k^2$.
(c) $\alpha = \alpha_{\rm HP}$ marks a *supercritical Hopf bifurcation*. As a matter of fact, the Jacobian matrix

evaluated at $(\overline{x}, \overline{y})$ has a pair of pure imaginary eigenvalues $\lambda = \pm i\omega$, with $\omega = \sqrt{(\beta + \varepsilon)\varepsilon}$. Moreover, the computation of the normal form coefficient a [Guckenheimer & Holmes, 1986; p. 152] yields $a = -1/8(1 + \omega^2) < 0$, so that the Hopf bifurcation is always supercritical.

It is straightforward to check that the equilibrium $(\overline{x}, \overline{y})$ cannot be involved in any other local bifurcation.

2.2. *Further analytical results*

Proposition 2.1. *For all values of α, there are unbounded trajectories with $(x(t), y(y)) \to (+\infty, 0)$.*

Proof. The phase diagram for system (1),(2) is portrayed in Fig. 1. The region Γ has the $y = 0$ axis, the line segments OA and AB, and the line γ as boundary, where $O = (0, 0)$, $A = (0, \beta\varepsilon/\alpha k)$, $B = (k/\varepsilon, \beta\varepsilon/\alpha k)$ and γ is the curve $y = \gamma(x) = \beta/\alpha x$. Since $\dot{x} > 0$ in all Γ, showing that Γ is a trapping region amounts to proving the existence of unbounded trajectories.

Owing to the phase diagram features, all trajectories passing through the line segments OA and AB enter Γ. Moreover, the $y = 0$ axis is itself a trajectory. Thus, it only remains to ascertain that all trajectories passing through the line γ also enter Γ. In other words, if P is a point of γ, i.e., $P = (x, \beta/\alpha x)$, then the slope of γ at P must be greater than, or equal to, the slope of the trajectory passing through P, namely

$$\left(\frac{d\gamma}{dx}\right)_P \geq \left(\frac{dy}{dx}\right)_P .$$

Through simple analytical steps, the above inequality can be rewritten as

$$x \geq \frac{k}{\varepsilon},$$

which holds true, given that $B = (k/\varepsilon, \beta\varepsilon/\alpha k)$. ■

Therefore, neither the equilibrium $(\overline{x}, \overline{y})$, which is stable for $\alpha > \alpha_{\mathrm{HP}}$, nor the stable limit cycle, which exists for $\alpha < \alpha_{\mathrm{HP}}$, are globally attracting in the first quadrant. More precisely, Proposition 2.1 implies the existence of an invariant line S, separating the trajectories approaching the equilibrium (or the limit cycle) from those whose x grows without bound. As a consequence, S is part of the boundary of the domain of attraction of infinity.

Proposition 2.2. *There exists no cycle for $\alpha \leq \alpha_{\mathrm{NC}}$, with $\alpha_{\mathrm{NC}} = (\beta + \varepsilon)^2\varepsilon^2/2(\beta + 3\varepsilon)k^2 < \alpha_{\mathrm{HP}}$.*

Proof. The region Γ, portrayed in Fig. 2, has the $y = 0$ axis, the line segment AB, and the line γ

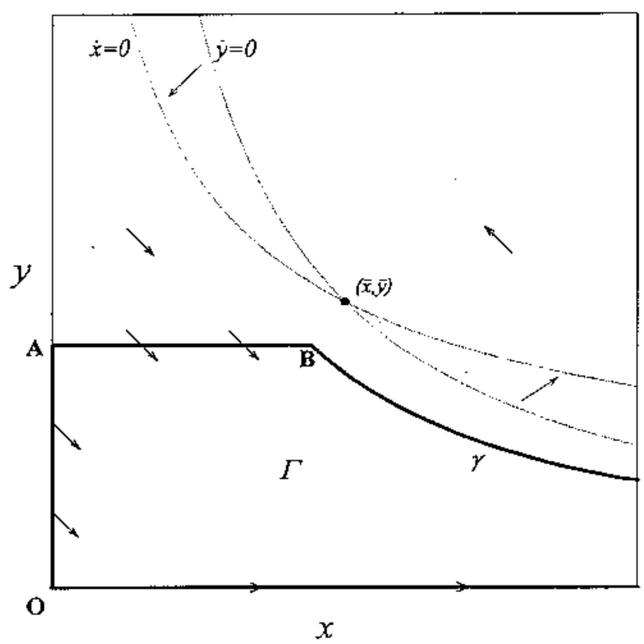

Fig. 1. Phase diagram of system (1),(2).

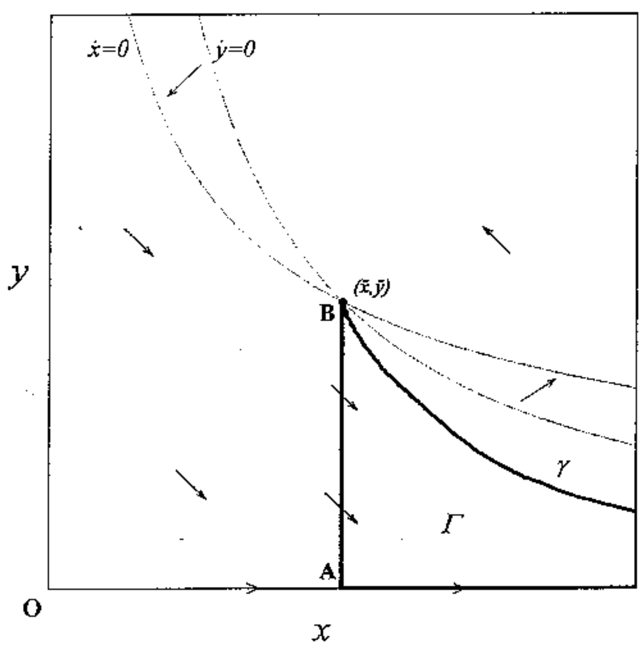

Fig. 2. Phase diagram of system (1),(2) for $\alpha \leq \alpha_{\mathrm{NC}}$.

as boundary, where $A = (\overline{x}, 0)$, $B = (\overline{x}, \overline{y})$, and γ is the curve $y = \gamma(x) = (\beta+\varepsilon)^2\varepsilon/\alpha^2kx^2$. Showing that Γ is a trapping region for $\alpha \le \alpha_{\rm NC}$ amounts to proving that no cycle can exist for $\alpha \le \alpha_{\rm NC}$. This can be accomplished by using the same arguments as in the proof of Proposition 2.1.

In particular, it must be shown that if P is a point of γ, i.e., $P = (x, (\beta+\varepsilon)^2\varepsilon/\alpha^2kx^2)$ with $x > \overline{x}$, then the slope of γ at P must be greater than, or equal to, the slope of the trajectory passing through P, namely

$$\left(\frac{d\gamma}{dx}\right)_P \ge \left(\frac{dy}{dx}\right)_P .$$

Through simple analytical steps, the above inequality can be first rewritten as

$$(\beta+\varepsilon)x^4 - \frac{2\alpha k^2 + (\beta+\varepsilon)^2\varepsilon}{\alpha k}x^3 - \frac{2(\beta+\varepsilon)\beta\varepsilon}{\alpha^2 k}x + \frac{2(\beta+\varepsilon)^4\varepsilon^2}{\alpha^3k^2} \ge 0,$$

and subsequently as

$$(\beta+\varepsilon)z^4 + \frac{3(\beta+\varepsilon)^2\varepsilon - 2\alpha k^2}{\alpha k}z^3 + \frac{3(\beta+\varepsilon)[(\beta+\varepsilon)^2\varepsilon - 2\alpha k^2]\varepsilon}{\alpha^2k^2}z^2 + \frac{(\beta+\varepsilon)^2[(\beta+\varepsilon)^2\varepsilon^2 - 2(\beta+3\varepsilon)\alpha k^2]\varepsilon}{\alpha^3k^3}z \ge 0,$$

where $z = x - \overline{x} = x - (\beta+\varepsilon)\varepsilon/\alpha k$. The resulting inequality has positive coefficients for $\alpha \le \alpha_{\rm NC}$, and hence holds true $\forall z > 0$. ■

Proposition 2.2 entails that system (1),(2) must be involved in a bifurcation other than Hopf, taking place at some $\alpha = \alpha_{\rm HM}$, with $\alpha_{\rm NC} < \alpha_{\rm HM} < \alpha_{\rm HP}$, and marking the disappearence of the stable limit cycle. Some numerical results about this bifurcation are presented and interpreted in the next section.

2.3. *Homoclinic bifurcation to infinity*

It can be checked by simulation that when α is decreased below the Hopf bifurcation value $\alpha_{\rm HP}$, the limit cycle grows to infinity in both amplitude and period (see Fig. 3), and then disappears at $\alpha = \alpha_{\rm HM} < \alpha_{\rm HP}$. Thus, the limit cycle undergoes a *homoclinic bifurcation to infinity*, as defined by Brons & Sturis [1992]. It can also be numerically ascertained that, for $\alpha < \alpha_{\rm HM}$, every trajectory in the first quadrant is unbounded with $(x(t), y(t)) \to (+\infty, 0)$. The entire one parameter bifurcation analysis is tentatively summarized in Remark 2.3.

Remark 2.3. As α is varied, the following five occurrences can be distinguished:

(a) For $\alpha > \alpha_{\rm HP}$, the equilibrium $(\overline{x}, \overline{y})$ is stable. The invariant line S is part of the boundary of the domain of attraction of infinity: all trajectories starting above S converge to $(\overline{x}, \overline{y})$, whereas the remaining ones are unbounded with $(x(t), y(t)) \to (+\infty, 0)$ [see Fig. 4(a)]. A decrease in α moves S upward.
(b) For $\alpha = \alpha_{\rm HP}$, the equilibrium $(\overline{x}, \overline{y})$ undergoes a supercritical Hopf bifurcation, as explained in Remark 2.1.
(c) For $\alpha_{\rm HM} < \alpha < \alpha_{\rm HP}$, the equilibrium $(\overline{x}, \overline{y})$ is unstable and is surrounded by a stable limit cycle. All trajectories starting above S converge to the limit cycle, whereas the remaining ones are unbounded with $(x(t), y(t)) \to (+\infty, 0)$ [see Fig. 4(b)]. A decrease in α gives rise to an increase in both the period and amplitude of the stable limit cycle.
(d) For $\alpha = \alpha_{\rm HM}$, there occurs a homoclinic bifurcation to infinity [Brons & Sturis, 1992]: the limit cycle has infinite period and amplitude.
(e) For $\alpha < \alpha_{\rm HM}$, the limit cycle has disappeared, so that any trajectory is unbounded with $(x(t), y(t)) \to (+\infty, 0)$ [see Fig. 4(c)].

The existence of unbounded trajectories has a sound economic interpretation, as it reflects the possibility of a business failure. Under this circumstance, potential buyers $x(t)$ grow without bound but actual buyers $y(t)$ are doomed to extinction. A decrease in α, i.e., the advertising effectiveness, increases the chance of business failure, as it shrinks the domain of attraction of either the equilibrium $(\overline{x}, \overline{y})$ $(\alpha > \alpha_{\rm HP})$ or the limit cycle $(\alpha_{\rm HM} < \alpha < \alpha_{\rm HP})$. Business failure is certain for $\alpha < \alpha_{\rm HM}$.

Needless to say, the homoclinic bifurcation to infinity deserves further analysis, which is however beyond the scope of the present work. A still unclear question concerns the character of infinity. Consider the mapping from the plane onto a sphere, taking the circle with infinite radius into the North Pole, the origin into the South Pole, and all the points at finite distance from the origin into the remaining points of the sphere (see, e.g., Arnol'd

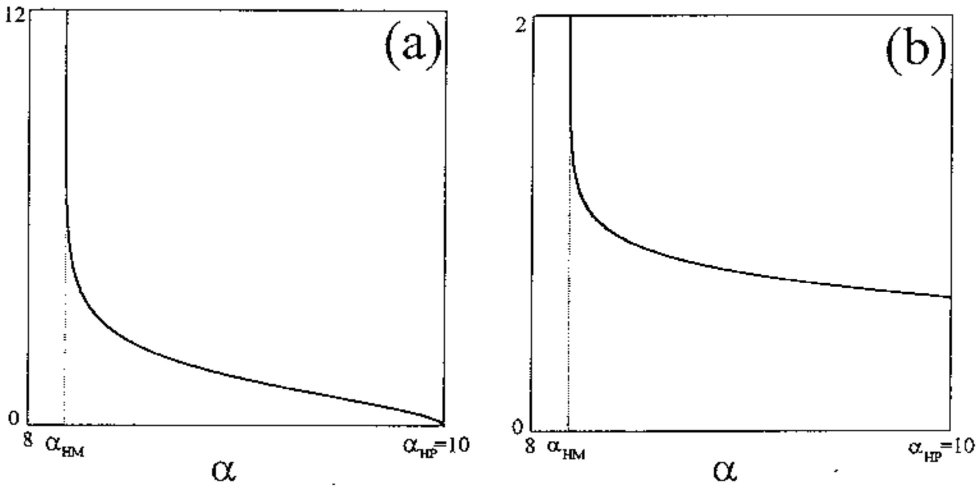

Fig. 3. (a) Amplitude (max $x(t) - \bar{x}$) and (b) period of the limit cycle as a function of α [$\beta = 0.5$, $\varepsilon = k = 9.5$].

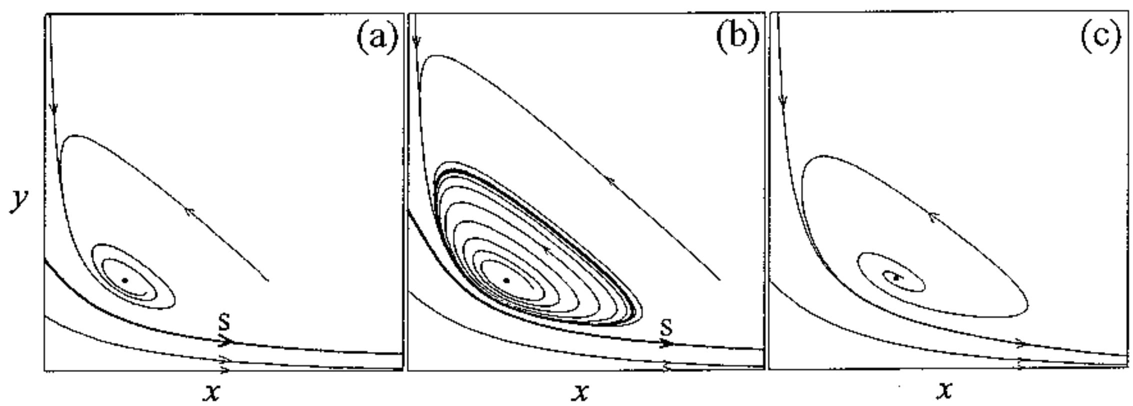

Fig. 4. Trajectories of system (1),(2): (a) $\alpha > \alpha_{HP}$; (b) $\alpha_{HP} > \alpha > \alpha_{HM}$; (c) $\alpha_{HM} > \alpha$.

[1992], Chap. 5). The resulting dynamic system must have two equilibria, with the former corresponding to $(\bar{x}, \bar{y})$ and the latter, corresponding to infinity, lying at the North Pole. It cannot be otherwise, because the sum of the indices of the equilibria on a sphere must be 2: $(\bar{x}, \bar{y})$, a node or a focus, has index 1, so that the North Pole must have index 1 too. In our opinion, the latter acts as a *tangle* [Brunella & Miari, 1990], a rather involved nonhyperbolic equilibrium that has index 1 and fits well the phase diagram of system (1),(2).

3. Seasonal Advertising and Chaos

3.1. *Analytical setting and investigation procedure*

As made clear in the introduction, both advertising and its effects are likely to be periodic with a period of one year. Reference is accordingly made to a firm that seasonally changes its advertising to sales ratio. As a consequence, the advertising effectiveness α is now periodic with a period of one year, i.e.,

$$\alpha = \alpha(t) = \alpha_0(1 + \delta \sin 2\pi t)\,, \qquad (3)$$

where α_0 is the average value of α, and $\alpha_0\delta$ is the amplitude of the seasonal perturbation ($0 \le \delta \le 1$).

System (1)–(3) is periodically forced, so that chaotic behavior can arise for some values of the parameters (δ, α_0). To analyze such an occurrence, resort is made to an approach in which theoretical arguments and numerical methods are suitably blended. More specifically, the bifurcations involving the periodic solutions of (1)-(3) are numerically analyzed by means of a continuation procedure (see Doedel *et al.* [1991]), interactively supported by the package LOCBIF [Khibnik *et al.*, 1993]. The aim

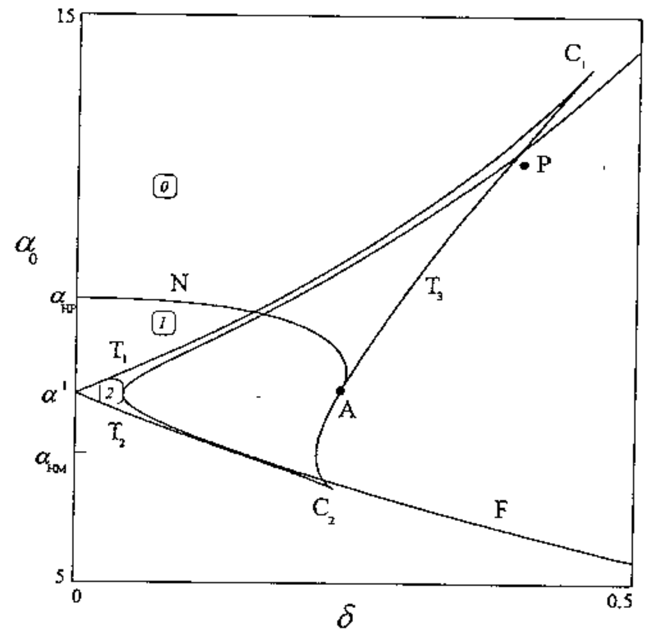

Fig. 5. Bifurcation diagram of the periodically forced system (1)–(3).

is to obtain a *bifurcation diagram*, namely a partition of the parameter space (δ, α_0) into regions of structural stability for system (1)–(3). An educated interpretation of this diagram allows one to detect the regions in the parameter space (δ, α_0) where a stable chaotic solution is likely to exist.

3.2. *Bifurcation diagram*

Reference is made in the sequel to the case $\alpha = 10$, $\beta = 0.5$, $k = \varepsilon = 9.5$. The remainder of this subsection is devoted to an explanation of Fig. 5, where a bifurcation diagram of system (1)–(3) is portrayed in the (δ, α_0) parameter space. The bifurcation diagram contains three curves, each marking a codimension-1 bifurcation of a period-1 solution, namely a solution with the same period of the forcing term. More precisely, there are a *flip (period doubling)* curve F, a *tangent (saddle-node)* curve T (formed by three branches T_i, $i = 1, 2, 3$), and a *Naimark–Sacker (Hopf)* curve N (see, for instance, Wiggins [1990], Chap. 3). The bifurcation diagram also includes three codimension-2 bifurcation points, with C_1 and C_2 denoting a *cusp*, and A representing a *strong resonance 1:1* [Arnol'd, 1988; Chap. 6].

The equilibrium ($\bar{x}$, $\bar{y}$) of the unforced system ($\delta = 0$) is stable for $\alpha_0 > \alpha_{\rm HP}$. By continuity, the forced system ($\delta > 0$) has a stable period-1 solution in region *0*. Moreover, since the equilibrium ($\bar{x}$, $\bar{y}$) of the unforced system ($\delta = 0$) undergoes a Hopf bifurcation at $\alpha_0 = \alpha_{\rm HP}$, a curve N is rooted on the α_0-axis at $\alpha_{\rm HP}$. On crossing N toward region 1, the period-1 solution bifurcates into an unstable period-1 solution and a stable quasi-periodic solution.

As we know from Sec. 2, the unforced system ($\delta = 0$) has a stable limit cycle for $\alpha_{\rm HM} < \alpha_0 < \alpha_{\rm HP}$, with period $\tau = 2\pi/\sqrt{\varepsilon(\beta + \varepsilon)} \cong 0.645$ at the Hopf bifurcation and period $\tau \to +\infty$ as the homoclinic bifurcation is approached. Therefore, as α_0 decreases toward $\alpha_{\rm HM}$, the period τ passes through the integer values 1, 2, 3, ... Let α^1 be the value yielding $\tau = 1$. A pair of tangent bifurcation curves T_i ($i = 1, 2$), enclosing an *Arnol'd tongue* (see, for instance, Wiggins [1990], p. 414), is rooted on the α_0-axis at α^1. On crossing T_1 from region 1 to 2, two period-1 solutions, a stable and a saddle one, appear through tangent bifurcation, and subsequently disappear through tangent bifurcation as T_2 is crossed. It is worthwhile to remark that there exist analogous pairs of tangent bifurcation curves $T^i_{1,2}$ for $i = 2, 3, \ldots$ (not shown in Fig. 5). The pair $T^i_{1,2}$ is rooted on the α_0-axis at α^i, where the unforced cycle has period $\tau = i$, while the sequence α^1, α^2, $\alpha^3, \ldots$ accumulates on $\alpha_{\rm HM}$. Each Arnol'd tongue contains two period-i solutions, a stable and a saddle one.

On moving rightward through the curve F, a period-1 stable solution bifurcates into a period-2 stable solution and a period-1 unstable solution. A flip curve F^2 (not shown in Fig. 5) lies close to F. On crossing F^2, a period-2 stable solution is replaced with a period-4 stable solution and a period-2 unstable solution.

It is clear from the previous description that there are regions in the parameter space (δ, α_0) marking the coexistence of several stable solutions. This occurs, for instance, in the strip between the curves T_1 and F.

3.3. *Chaotic behavior*

In the light of the bifurcation diagram of Fig. 5, the chaotic regions in the parameter space (δ, α_0) can be tentatively located, as made clear in Remark 3.2 below. Remark 3.1 is a helpful premise, as it points out some empirical evidence about the existence of unbounded trajectories.

Remark 3.1. It can be checked by simulation that, for all pairs (δ, α_0), there are unbounded trajectories with $(x(t), y(t)) \to (+\infty, 0)$. Actually, Proposition 2.1 could be readily extended to cover the periodically forced case. Moreover, it can be numerically ascertained that, for any δ, if α_0 is small enough, every trajectory in the first quadrant is unbounded with $(x(t), y(t)) \to (+\infty, 0)$. Also this bears some resemblance with the time-invariant case, where the loss of any stable solution is brought about by a homoclinic bifurcation to infinity (see Remarks 2.2 and 2.3).

Remark 3.2. Two routes to chaos are present in the periodically forced system (1)–(3):

(a) *Period doubling cascade.* Stable chaotic solutions can arise from a period-doubling cascade [Guckenheimer & Holmes, 1986; Chap. 6] on the right of the curve F. They can be easily observed by simulation, as shown in Fig. 6, where $(\delta, \alpha_0) = (0.4, 12.4)$ (see point P in Fig. 5). Figure 7 portrays two time patterns of $y(t)$, corresponding to two slightly different initial states: the divergence between them is a typical hallmark of chaos. According to simulation evidence, the domain of attraction of a chaotic solution gets narrower and narrower as α_0 is decreased. Notably, this is in accordance with Remark 3.1 as well as the results about the unforced system ($\delta = 0$) obtained in Sec. 2 (recall that a decrease in α moves S upward, thus shrinking the domain of attraction of either the equilibrium or the limit cycle).

(b) *Torus destruction.* Stable chaotic solutions can also arise from a complex sequence of bifurcations involving a quasi-periodic solution (torus) and a saddle type periodic solution [Arnol'd *et al.*, 1993]. Although this can occur, in principle, below the curve N, where there are Arnol'd tongues, we have not succeeded in detecting such solutions by simulation, since α_0 is close to the critical value α_{HM}, so that any kind of stable solution must possess a very small domain of attraction.

Therefore, only the chaotic solutions arising from a period-doubling cascade are of economic interest. Moreover, the larger α_0, the larger their domain of attraction. In view of this, both α_0 and δ must be relatively large for chaos to unfold.

Other bifurcation diagrams, unreported for brevity, have been computed with reference to different parameter values. As far as chaos is concerned, the previous qualitative conclusions can be drawn in all instances.

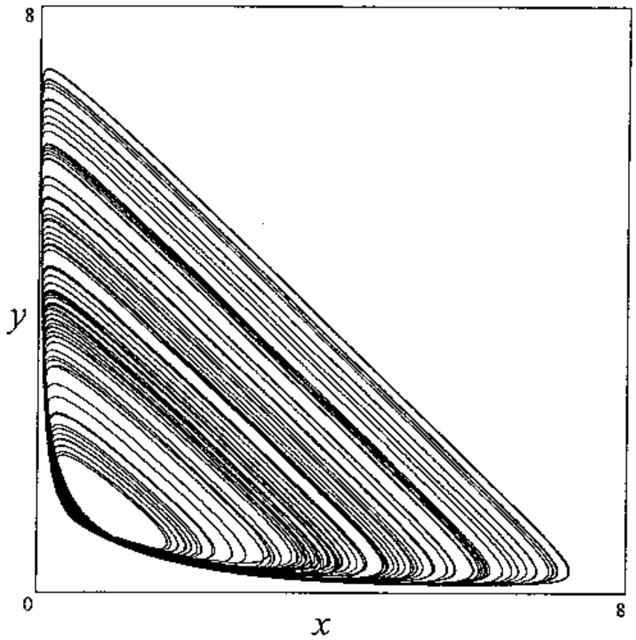

Fig. 6. A chaotic trajectory of system (1)–(3).

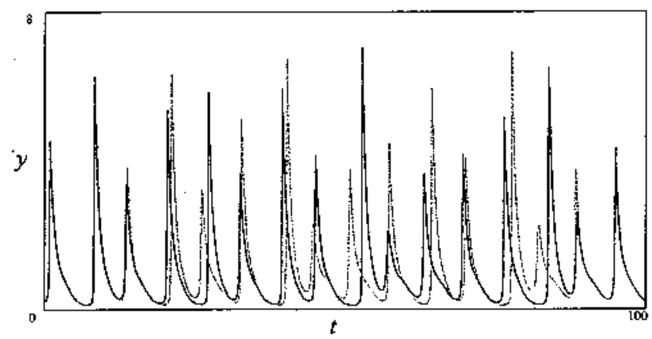

Fig. 7. Time patterns of $y(t)$ for two slightly different initial states.

4. Discussion and Conclusions

The diffusion of a new product and the underlying adoption process have been studied by means of a second-order, nonlinear dynamic system, which may include a periodically varying parameter. According to our analysis, if the firm has a poor advertising effectiveness, the new product is doomed to failure. When actual customers at the very beginning are too few, such a conclusion cannot be escaped even if the firm has a good advertising effectiveness. More precisely, the larger the average value of the advertising effectiveness, the larger the domain of attraction of any stable solution, the fewer are the chances of a failure. All the above results do hold whether advertising is periodically forced or not. All of them seem to make economic sense.

As for the periodically forced case, if the average advertising effectiveness is large enough there can exist two basic kinds of stable asymptotic behavior. If the seasonal change in the firm's advertising-to-sales ratio is slight, a periodic solution is the most likely occurrence. In contrast, if the seasonal change is substantial, there can be a chaotic solution. The shift from a periodic solution to a chaotic one takes place through a cascade of period doublings.

Unfortunately, the above-mentioned results cannot be compared with any empirical evidence, since little attention has been paid to chaos in the marketing literature. This is somehow surprising, since, as previously mentioned, the advertising diffusion models borrow a lot from the epidemic ones, where the existence of chaos has been ascertained both on a theoretical and on an empirical ground [Schaffer *et al.*, 1988; Olsen *et al.*, 1988; Kuznetsov & Piccardi, 1994]. Thus, empirical work on this subject could disclose interesting results.

Other possible lines of research are closely connected to the model investigated in this work. First of all, emphasis could be placed on highly seasonal products, such as, for instance, drinks, swimming units, lawn mowers, and ski sets. In view of this, allowance should be made for a seasonal change in the rate k at which new potential customers enter the market.

Moreover, and perhaps more interestingly, the time-invariant model could be further analyzed. Although the homoclinic bifurcation to infinity has already been observed and described in detail [Brons & Sturis, 1991, 1992], no complete analytic proof has been obtained yet in a specific set of equations. As made clear in Sec. 2, a first step in this direction could be that of understanding the character of infinity.

Acknowledgments

This work was supported by the Austrian Science Foundation under contract P7783 and the Italian Ministry for University (MURST) under the project *Teoria dei sistemi e del controllo*. The authors would like to thank M. Miari, S. Rinaldi and U. Wagner for their helpful comments and suggestions.

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