or

$$\mathcal{N}\big(h(x)\big) = h\big(Ax + f\big(x, h(x)\big)\big) - Bh(x) - g\big(x, h(x)\big) = 0. \qquad (18.4.5)$$

(Note: the reader should compare (18.4.5) with (18.1.10).) The next theorem justifies the approximate solution of (18.4.5) via power series expansions.

Theorem 18.4.3 (Approximation) *Let $\phi : \mathbb{R}^c \to \mathbb{R}^s$ be a $\mathbf{C}^1$ map with $\phi(0) = 0, \phi'(0) = 0$, and $\mathcal{N}\big(\phi(x)\big) = \mathcal{O}(|x|^q)$ as $x \to 0$ for some $q > 1$. Then*

$$h(x) = \phi(x) + \mathcal{O}(|x|^q) \qquad \text{as } x \to 0.$$

Proof: See Carr [1981]. □

We now give an example.

Example 18.4.1. Consider the map

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} vw \\ u^2 \\ -uv \end{pmatrix}, \quad (u, v, w) \in \mathbb{R}^3. \qquad (18.4.6)$$

It should be clear that $(u, v, w) = (0, 0, 0)$ is a fixed point of (18.4.6), and the eigenvalues associated with the map linearized about this fixed point are -1, $-\frac{1}{2}$, $\frac{1}{2}$. Thus, the linear approximation does not suffice to determine the stability or instability. We will apply center manifold theory to this problem.

The center manifold can locally be represented as follows

$$W^c(0) = \Big\{(u, v, w) \in \mathbb{R}^3 \mid v = h_1(u), w = h_2(u), h_i(0) = 0, \\ Dh_i(0) = 0, i = 1, 2\Big\} \qquad (18.4.7)$$

for u sufficiently small. Recall that the center manifold must satisfy the following equation

$$\mathcal{N}\Big(h(x)\Big) = h\Big(Ax + f\Big(x, h(x)\Big)\Big) - Bh(x) - g\Big(x, h(x)\Big) = 0, \qquad (18.4.8)$$

where, in this example,

$$\begin{aligned} x = u, \quad y &\equiv (v, w), \quad h = (h_1, h_2), \\ A &= -1, \\ B &= \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \\ f(u, v, w) &= vw, \\ g(u, v, w) &= \begin{pmatrix} u^2 \\ -uv \end{pmatrix}. \end{aligned} \qquad (18.4.9)$$

We assume a center manifold of the form

$$h(u) = \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix} = \begin{pmatrix} a_1 u^2 + b_1 u^3 + \mathcal{O}(u^4) \\ a_2 u^2 + b_2 u^3 + \mathcal{O}(u^4) \end{pmatrix}. \qquad (18.4.10)$$

Substituting (18.4.10) into (18.4.8) and using (18.4.9) yields

$$\begin{aligned} \mathcal{N}\Big(h(u)\Big) &= \begin{pmatrix} a_1 u^2 - b_1 u^3 + \mathcal{O}(u^5) \\ a_2 u^2 - b_2 u^3 + \mathcal{O}(u^5) \end{pmatrix} \\ &\quad - \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} a_1 u^2 + b_1 u^3 + \cdots \\ a_2 u^2 + b_2 u^3 + \cdots \end{pmatrix} - \begin{pmatrix} u^2 \\ -u h_1(u) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \qquad (18.4.11)$$

Balancing powers of coefficients for each component gives

$$u^2 : \begin{pmatrix} a_1 + \frac{1}{2}a_1 - 1 \\ a_2 - \frac{1}{2}a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} a_1 & = & \frac{2}{3}, \\ a_2 & = & 0 \end{matrix} \qquad (18.4.12)$$

$$u^3 : \begin{pmatrix} -b_1 + \frac{1}{2}b_1 \\ -b_2 - \frac{1}{2}b_2 + a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} b_1 & = & 0 & & \\ b_2 & = & a_1 \frac{2}{3} & = & \frac{4}{9} \end{matrix};$$

hence, the center manifold is given by the graph of $\Big(h_1(u), h_2(u)\Big)$, where

$$\begin{aligned} h_1(u) &= \frac{2}{3}u^2 + \mathcal{O}(u^4), \\ h_2(u) &= \frac{4}{9}u^3 + \mathcal{O}(u^4). \end{aligned} \qquad (18.4.13)$$

The map on the center manifold is given by

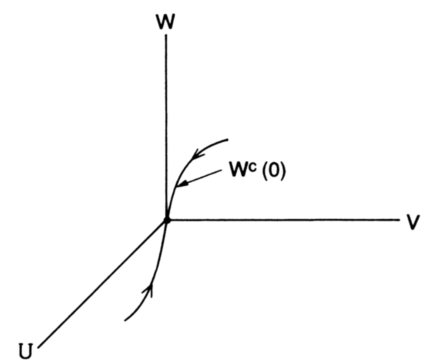

FIGURE 18.4.1.

$$u \longmapsto -u + \frac{8}{27}u^5 + \mathcal{O}(u^6); \qquad (18.4.14)$$

thus, the origin is attracting; see Figure 18.4.1.

End of Example 18.4.1

Example 18.4.2. Consider the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -y^3 \end{pmatrix}, \qquad (x,y) \in \mathbb{R}^2. \tag{18.4.15}$$

The origin is a fixed point of the map. Computing the eigenvalues of the map linearized about the origin gives

$$\lambda_{1,2} = 1, \frac{1}{2}.$$

Therefore, there is a one-dimensional center manifold and a one-dimensional stable manifold with the orbit structure in a neighborhood of $(0,0)$ determined by the orbit structure on the center manifold.

We wish to compute the center manifold, but first we must put the linear part in block diagonal form as given in (18.4.1). The matrix associated with the linear transformation has columns consisting of the eigenvectors of the linearized map and is easily calculated. It is given by

$$T = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{with } T^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}. \tag{18.4.16}$$

Thus, letting

$$\begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix},$$

our map becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -2(u+v)^3 \\ (u+v)^3 \end{pmatrix}. \tag{18.4.17}$$

We seek a center manifold

$$W^c(0) = \{ (u,v) \mid v = h(u); h(0) = Dh(0) = 0 \} \tag{18.4.18}$$

for u sufficiently small. The next step is to assume $h(u)$ of the form

$$h(u) = au^2 + bu^3 + \mathcal{O}(u^4) \tag{18.4.19}$$

and substitute (18.4.19) into the center manifold equation

$$\mathcal{N}\Big(h(u)\Big) = h\Big(Au + f\big(u, h(u)\big)\Big) - Bh(u) - g\Big(u, h(u)\Big) = 0, \tag{18.4.20}$$

where, in this example, we have

$$\begin{aligned} A &= 1, \\ B &= \frac{1}{2}, \\ f(u,v) &= -2(u+v)^3, \\ g(u,v) &= (u+v)^3, \end{aligned} \tag{18.4.21}$$

and (18.4.20) becomes

$$\begin{aligned} & a\Big(u - 2\big(u + au^2 + bu^3 + \mathcal{O}(u^4)\big)^3\Big)^2 \\ & + \; b\Big(u - 2\big(u + au^2 + bu^3 + \mathcal{O}(u^4)\big)^3\Big)^3 \\ & + \; \cdots - \frac{1}{2}\Big(au^2 + bu^3 + \mathcal{O}(u^4)\Big) - \Big(u + au^2 + bu^3 + \mathcal{O}(u^4)\Big)^3 = 0. \end{aligned} \tag{18.4.22}$$

or

$$au^2 + bu^3 - \frac{1}{2}au^2 - \frac{1}{2}bu^3 - u^3 + \mathcal{O}(u^4) = 0. \tag{18.4.23}$$

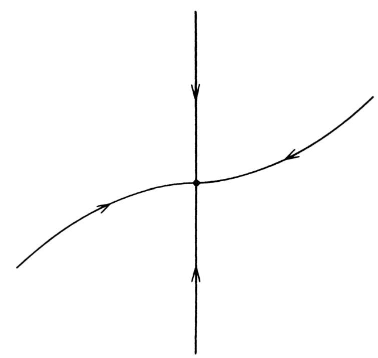

FIGURE 18.4.2.

Equating coefficients of like powers to zero gives

$$\begin{aligned} u^2 &: a - \frac{1}{2}a = 0 \Rightarrow a = 0, \\ u^3 &: b - \frac{1}{2}b - 1 = 0 \Rightarrow b = 2. \end{aligned} \tag{18.4.24}$$

Thus, the center manifold is given by the graph of

$$h(u) = 2u^3 + \mathcal{O}(u^4), \tag{18.4.25}$$

and the map restricted to the center manifold is given by

$$u \mapsto u - 2\Big(u + 2u^3 + \mathcal{O}(u^4)\Big)^3 \tag{18.4.26}$$

or

$$u \mapsto u - 2u^3 + \mathcal{O}(u^4). \tag{18.4.27}$$

Therefore, the orbit structure in the neighborhood of $(0,0)$ appears as in Figure 18.4.2 and $(0,0)$ is stable.

End of Example 18.4.2