- For any $c \in \mathbb{R}$, V is invariant with respect to cA.
- For any integer $n > 1$, V is invariant with respect to A^n.
- Suppose A_1 and A_2 are linear maps on $\mathbb{R}^n$ and V is invariant with respect to both A_1 and A_2. Then V is invariant with respect to $A_1 + A_2$. From this result it also follows that for any finite number of linear maps A_i, $i = 1, \ldots, n$, with V invariant under each, V is also invariant under $\sum_{i=1}^n A_i$.

Using each of these facts one can easily conclude that V is invariant under the linear map

$$L_n(t) \equiv \text{id} + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n = \sum_{i=0}^{n} \frac{1}{i!}A^it^i,$$

for any n, where id is the $n \times n$ identity matrix (and $0! \equiv 1$). Now using the fact that V is closed, and that $L_n(t)$ converges to e^{At} uniformly, we conclude that V is invariant with respect to e^{At}.

3.1b Some Examples

We now illustrate these ideas with three examples where for simplicity and easier visualization we will work in $\mathbb{R}^3$.

Example 3.1.1. Suppose the three eigenvalues of A are real and distinct and denoted by λ_1, $\lambda_2 < 0$, $\lambda_3 > 0$. Then A has three linearly independent eigenvectors e_1, e_2, and e_3 corresponding to λ_1, λ_2, and λ_3, respectively. If we form the 3×3 matrix T by taking as columns the eigenvectors e_1, e_2, and e_3, which we write as

$$T \equiv \begin{pmatrix} \vdots & \vdots & \vdots \\ e_1 & e_2 & e_3 \\ \vdots & \vdots & \vdots \end{pmatrix}, \tag{3.1.9}$$

then we have

$$\Lambda \equiv \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = T^{-1}AT. \tag{3.1.10}$$

Recall that the solution of (3.1.2) through $y_0 \in \mathbb{R}^3$ at $t = 0$ is given by

$$y(t) = e^{At}y_0 = e^{T\Lambda T^{-1}t}y_0. \tag{3.1.11}$$

Using (3.1.4), it is easy to see that (3.1.11) is the same as

$$\begin{aligned} y(t) &= Te^{\Lambda t}T^{-1}y_0 \\ &= T\begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} T^{-1}y_0 \end{aligned}$$

$$= \begin{pmatrix} \vdots & \vdots & \vdots \\ e_1e^{\lambda_1 t} & e_2e^{\lambda_2 t} & e_3e^{\lambda_3 t} \\ \vdots & \vdots & \vdots \end{pmatrix} T^{-1}y_0. \tag{3.1.12}$$

Now we want to give a geometric interpretation to (3.1.12). Recall from (3.1.5) that we have

$$\begin{aligned} E^s &= \text{span}\{e_1, e_2\}, \\ E^u &= \text{span}\{e_3\}. \end{aligned}$$

Invariance

Choose any point $y_0 \in \mathbb{R}^3$. Then T^{-1} is the transformation matrix which changes the coordinates of y_0 with respect to the standard basis on $\mathbb{R}^3$ (i.e., $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$) into coordinates with respect to the basis e_1, e_2, and e_3. Thus, for $y_0 \in E^s$, $T^{-1}y_0$ has the form

$$T^{-1}y_0 = \begin{pmatrix} \tilde{y}_{01} \\ \tilde{y}_{02} \\ 0 \end{pmatrix}, \tag{3.1.13}$$

and, for $y_0 \in E^u$, $T^{-1}y_0$ has the form

$$T^{-1}y_0 = \begin{pmatrix} 0 \\ 0 \\ \tilde{y}_{03} \end{pmatrix}. \tag{3.1.14}$$

Therefore, by substituting (3.1.13) (resp., (3.1.14)) into (3.1.12), it is easy to see that $y_0 \in E^s$ (resp., E^u) implies $e^{At}y_0 \in E^s$ (resp., E^u). Thus, E^s and E^u are *invariant* manifolds.

Asymptotic Behavior

Using (3.1.13) and (3.1.12), we can see that, for any $y_0 \in E^s$, we have $e^{At}y_0 \to 0$ as $t \to +\infty$ and, for any $y_0 \in E^u$, we have $e^{At}y_0 \to 0$ as $t \to -\infty$ (hence the reason behind the names stable and unstable manifolds).

See Figure 3.1.1 for an illustration of the geometry of E^s and E^u.

End of Example 3.1.1

Example 3.1.2. Suppose A has two complex conjugate eigenvalues $\rho \pm i\omega$, $\rho < 0$, $\omega \neq 0$ and one real eigenvalue $\lambda > 0$. Then A has three real generalized eigenvectors e_1, e_2, and e_3, which can be used as the columns of a matrix T in order to transform A as follows

$$\Lambda \equiv \begin{pmatrix} \rho & \omega & 0 \\ -\omega & \rho & 0 \\ 0 & 0 & \lambda \end{pmatrix} = T^{-1}AT. \tag{3.1.15}$$

From Example 3.1.1 it is easy to see that in this example we have

$$\begin{aligned} y(t) &= Te^{\Lambda t}T^{-1}y_0 \\ &= T\begin{pmatrix} e^{\rho t}\cos\omega t & e^{\rho t}\sin\omega t & 0 \\ -e^{\rho t}\sin\omega t & e^{\rho t}\cos\omega t & 0 \\ 0 & 0 & e^{\lambda t} \end{pmatrix} T^{-1}y_0. \end{aligned} \tag{3.1.16}$$

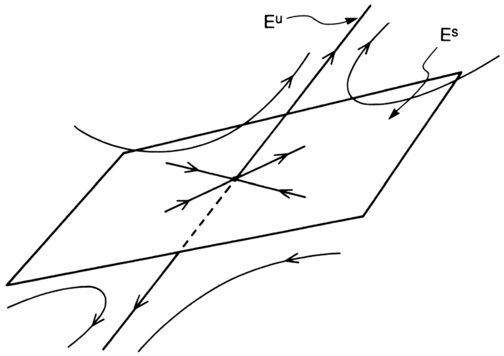

FIGURE 3.1.1. The geometry of E^s and E^u for Example 3.1.1.

Using the same arguments given in Example 3.1.1 it should be clear that $E^s = \text{span}\{e_1, e_2\}$ is an invariant manifold of solutions that decay exponentially to zero as $t \to +\infty$, and $E^u = \text{span}\{e_3\}$ is an invariant manifold of solutions that decay exponentially to zero as $t \to -\infty$ (see Figure 3.1.2).

End of Example 3.1.2

Example 3.1.3. Suppose A has two real repeated eigenvalues, $\lambda < 0$, and a third distinct eigenvalue $\gamma > 0$ such that there exist generalized eigenvectors e_1, e_2, and e_3 which can be used to form the columns of a matrix T so that A is transformed as follows

$$\Lambda = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \gamma \end{pmatrix} = T^{-1}AT. \tag{3.1.17}$$

Following Examples 3.1.1 and 3.1.2, in this example the solution through the point $y_0 \in \mathbb{R}^3$ at $t = 0$ is given by

$$\begin{aligned} y(t) &= Te^{\Lambda t}T^{-1}y_0 \\ &= T \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & e^{\gamma t} \end{pmatrix} T^{-1}y_0. \end{aligned} \tag{3.1.18}$$

Using the same arguments as in Example 3.1.1, it is easy to see that $E^s = \text{span}\{e_1, e_2\}$ is an invariant manifold of solutions that decay to $y = 0$ as $t \to +\infty$, and $E^u = \text{span}\{e_3\}$ is an invariant manifold of solutions that decay to $y = 0$ as $t \to -\infty$ (see Figure 3.1.3).

End of Example 3.1.3

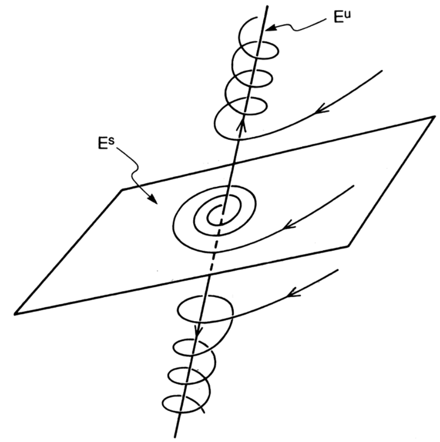

FIGURE 3.1.2. The geometry of E^s and E^u for Example 3.1.2(for $\omega < 0$).

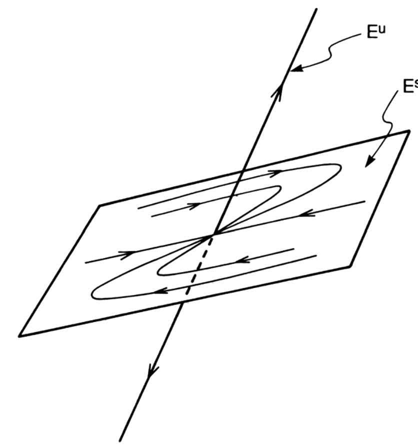

FIGURE 3.1.3. The geometry of E^s and E^u for Example 3.1.3

The reader should review enough linear algebra so that he or she can justify each step in the arguments given in these examples. We remark that we have not considered an example of a linear vector field having a center subspace. The reader can construct his or her own examples from Example 3.1.2 by setting $\rho = 0$ or from Example 3.1.3 by setting $\lambda = 0$; we leave these as exercises and now turn to the nonlinear system.

iii. For vector field b), discuss the cases $\lambda < \mu$, $\lambda = \mu$, $\lambda > \mu$. What are the qualitative and quantitative differences in the dynamics for these three cases? Describe all zero- and one-dimensional invariant manifolds for this vector field. Describe the nature of the trajectories at the origin. In particular, which trajectories are tangent to either the x_1 or x_2 axis?

iv. In vector field c), describe how the trajectories depend on the relative magnitudes of λ and ω. What happens when $\lambda = 0$? When $\omega = 0$?

v. Describe the effect of linear perturbations on each of the vector fields.

vi. Describe the effect *near the origin* of nonlinear perturbations on each of the vector fields. Can you say anything about the effects of nonlinear perturbations on the dynamics outside of a neighborhood of the origin?

We remark that vi) is a difficult problem for the nonhyperbolic fixed points. We will study this situation in great detail when we develop center manifold theory and bifurcation theory.

23. Give a characterization of the stable, unstable, and center subspaces for linear maps in terms of generalized eigenspaces along the same lines as we did for linear vector fields according to the formulae (3.1.6), (3.1.7), and (3.1.8).

24. For the following linear vector fields find the general solution, and compute the stable, unstable, and center subspaces and plot them in the phase space.

a) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

b) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -5 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

c) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

d) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

e) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

f) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

g) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 15 \\ 1 & 0 & -17 \\ 0 & 1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

h) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

25. Consider the following linear maps on $\mathbb{R}^2$.

a) $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \begin{matrix} |\lambda| < 1 \\ |\mu| > 1 \end{matrix}.$$

b) $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \begin{matrix} |\lambda| < 1 \\ |\mu| < 1 \end{matrix}.$$

c) $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \omega > 0.$$

d) $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad |\lambda| < 1.$$

e) $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \lambda > 0.$$

f) $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

i. For each map compute all the orbits and illustrate them graphically on the phase plane. Describe the stable, unstable, and center manifolds of the origin.

ii. For map a), discuss the cases $\lambda, \mu > 0$; $\lambda = 0$, $\mu > 0$; $\lambda, \mu < 0$; and $\lambda < 0$, $\mu > 0$. What are the qualitative differences in the dynamics for these four cases? Discuss how the orbits depend on the relative magnitudes of the eigenvalues. Discuss the attracting nature of the unstable manifold of the origin and its dependence on the relative magnitudes of the eigenvalues.

iii. For map b), discuss the cases $\lambda, \mu > 0$; $\lambda = 0$, $\mu > 0$; $\lambda, \mu < 0$; and $\lambda < 0$, $\mu > 0$. What are the qualitative differences in the dynamics for these four cases? Describe all zero- and one-dimensional invariant manifolds for this map. Do all orbits lie on invariant manifolds?

iv. For map c), consider the cases $\lambda^2 + \omega^2 < 1$, $\lambda^2 + \omega^2 > 1$, and $\lambda + i\omega = e^{i\alpha}$ for α rational and α irrational. Describe the qualitative differences in the dynamics for these four cases.

v. Describe the effect of linear perturbations on each of the maps.

vi. Describe the effect *near the origin* of nonlinear perturbations on each of the maps. Can you say anything about the effects of nonlinear perturbations on the dynamics outside of a neighborhood of the origin?

We remark that vi) is very difficult for nonhyperbolic fixed points (more so than the analogous case for vector fields in the previous exercise) and will be treated in great detail when we develop center manifold theory and bifurcation theory.

26. Consider the following vector fields.

a) $$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\delta y - \mu x, \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$$

b) $$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\delta y - \mu x - x^2, \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$$

c) $$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\delta y - \mu x - x^3, \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$$

d) $$\begin{aligned} \dot{x} &= -\delta x - \mu y + xy, \\ \dot{y} &= \mu x - \delta y + \tfrac{1}{2}(x^2 - y^2), \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$$

e) $$\begin{aligned} \dot{x} &= -x + x^3, \\ \dot{y} &= x + y, \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$$

f) $$\begin{aligned} \dot{r} &= r(1 - r^2), \\ \dot{\theta} &= \cos 4\theta, \end{aligned} \qquad (r, \theta) \in \mathbb{R}^+ \times S^1.$$

g) $$\begin{aligned} \dot{r} &= r(\delta + \mu r^2 - r^4), \\ \dot{\theta} &= 1 - r^2, \end{aligned} \qquad (r, \theta) \in \mathbb{R}^+ \times S^1.$$

h) $$\begin{aligned} \dot{\theta} &= v, \\ \dot{v} &= -\sin\theta - \delta v + \mu, \end{aligned} \qquad (\theta, v) \in S^1 \times \mathbb{R}.$$

i) $$\begin{aligned} \dot{\theta}_1 &= \omega_1, \\ \dot{\theta}_2 &= \omega_2 + \theta_1^n, \quad n \geq 1, \end{aligned} \qquad (\theta_1, \theta_2) \in S^1 \times S^1.$$

j) $$\begin{aligned} \dot{\theta}_1 &= \theta_2 - \sin\theta_1, \\ \dot{\theta}_2 &= -\theta_2, \end{aligned} \qquad (\theta_1, \theta_2) \in S^1 \times S^1.$$

k) $$\begin{aligned} \dot{\theta}_1 &= \theta_1^2, \\ \dot{\theta}_2 &= \omega_2, \end{aligned} \qquad (\theta_1, \theta_2) \in S^1 \times S^1.$$

Describe the nature of the stable and unstable manifolds of the fixed points by drawing phase portraits. Can you determine anything about the global behavior of the manifolds?

In a), b), c), d), g), and h) consider the cases $\delta < 0$, $\delta = 0$, $\delta > 0$, $\mu < 0$, $\mu = 0$, and $\mu > 0$. In i) and k) consider $\omega_1 > 0$ and $\omega_2 > 0$.