Thus, we see that (18.2.7) is very similar to (18.1.10).

Before considering a specific example we want to point out an important fact. By considering ε as a new dependent variable, terms such as

$$x_i \varepsilon_j, \qquad 1 \le i \le c, \quad 1 \le j \le p,$$

or

$$y_i \varepsilon_j, \qquad 1 \le i \le s, \quad 1 \le j \le p,$$

become *nonlinear terms*. In this case, returning to a question asked at the beginning of this section, the parts of the matrices A and B depending on ε are now viewed as nonlinear terms and are included in the f and g terms of (18.2.2), respectively. We remark that in applying center manifold theory to a given system, it must first be transformed into the standard form (either (18.1.1) or (18.2.2)).

Example 18.2.1 (The Lorenz Equations). Consider the Lorenz equations

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= \bar{\rho} x + x - y - xz, \qquad (x, y, z) \in \mathbb{R}^3, \\ \dot{z} &= -\beta z + xy, \end{aligned} \tag{18.2.8}$$

where σ and β are viewed as fixed positive constants and $\bar{\rho}$ is a parameter (note: in the standard version of the Lorenz equations it is traditional to put $\bar{\rho} = \rho - 1$). It should be clear that $(x, y, z) = (0, 0, 0)$ is a fixed point of (18.2.9). Linearizing (18.2.9) about this fixed point, we obtain the associated matrix

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}. \tag{18.2.9}$$

(Note: recall, $\bar{\rho} x$ is a nonlinear term.)

Since (18.2.9) is in block form, the eigenvalues are particularly easy to compute and are given by

$$0, -\sigma - 1, -\beta, \tag{18.2.10}$$

with eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{18.2.11}$$

Our goal is to determine the nature of the stability of $(x, y, z) = (0, 0, 0)$ for $\bar{\rho}$ near zero. First, we must put (18.2.9) into the standard form (18.2.2). Using the eigenbasis (18.2.11), we obtain the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{18.2.12}$$

with inverse

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{1+\sigma} \begin{pmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1+\sigma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{18.2.13}$$

which transforms (18.2.9) into

$$\begin{aligned} \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(1+\sigma) & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ &\quad + \frac{1}{1+\sigma} \begin{pmatrix} \sigma\bar{\rho}(u + \sigma v) - \sigma w(u + \sigma v) \\ -\bar{\rho}(u + \sigma v) + w(u + \sigma v) \\ (1+\sigma)(u + \sigma v)(u - v) \end{pmatrix}, \\ \dot{\bar{\rho}} &= 0. \end{aligned} \tag{18.2.14}$$

Thus, from center manifold theory, the stability of $(x, y, z) = (0, 0, 0)$ near $\bar{\rho} = 0$ can be determined by studying a one-parameter family of first-order ordinary differential equations on a center manifold, which can be represented as a graph over the u and $\bar{\rho}$ variables, i.e.,

$$\begin{aligned} W^c(0) = \Big\{ &(u, v, w, \bar{\rho}) \in \mathbb{R}^4 \mid v = h_1(u, \bar{\rho}), w = h_2(u, \bar{\rho}), \\ &h_i(0, 0) = 0, Dh_i(0, 0) = 0, i = 1, 2 \Big\} \end{aligned} \tag{18.2.15}$$

for u and $\bar{\rho}$ sufficiently small.

We now want to compute the center manifold and derive the vector field on the center manifold. Using Theorem 18.1.4, we assume

$$\begin{aligned} h_1(u, \bar{\rho}) &= a_1 u^2 + a_2 u \bar{\rho} + a_3 \bar{\rho}^2 + \cdots, \\ h_2(u, \bar{\rho}) &= b_1 u^2 + b_2 u \bar{\rho} + b_3 \bar{\rho}^2 + \cdots. \end{aligned} \tag{18.2.16}$$

Recall from (2.1.27) that the center manifold must satisfy

$$\begin{aligned} \mathcal{N}\Big(h(x, \varepsilon)\Big) = D_x h(x, \varepsilon) &\left[Ax + f(x, h(x, \varepsilon), \varepsilon)\right] \\ &- Bh(x, \varepsilon) - g(x, h(x, \varepsilon), \varepsilon) = 0, \end{aligned} \tag{18.2.17}$$

where, in this example,

$$\begin{aligned} &x \equiv u, \qquad y \equiv (v, w), \qquad \varepsilon \equiv \bar{\rho}, \qquad h = (h_1, h_2), \\ &A = 0, \\ &B = \begin{pmatrix} -(1+\sigma) & 0 \\ 0 & -\beta \end{pmatrix}, \\ &f(x, y, \varepsilon) = \frac{1}{1+\sigma}[\sigma\bar{\rho}(u + \sigma v) - \sigma w(u + \sigma v)], \\ &g(x, y, \varepsilon) = \frac{1}{1+\sigma} \begin{pmatrix} -\bar{\rho}(u + \sigma v) + w(u + \sigma v) \\ (1+\sigma)(u + \sigma v)(u - v) \end{pmatrix}. \end{aligned} \tag{18.2.18}$$

Substituting (18.2.16) into (18.2.17) and using (18.2.19) gives the two components of the equation for the center manifold.

$$\begin{aligned} (2a_1 u + a_2 \bar{\rho} + \cdots) &\left[\frac{\sigma}{1+\sigma}\Big(\bar{\rho}(u + \sigma h_1) - h_2(u + \sigma h_1)\Big)\right] \\ &+ (1+\sigma)h_1 + \frac{\bar{\rho}}{1+\sigma}(u + \sigma h_1) - \frac{h_2}{1+\sigma}(u + \sigma h_1) = 0, \end{aligned}$$

$$(2b_1u + b_2\bar{\rho} + \cdots)\left[\frac{\sigma}{1+\sigma}\Big(\bar{\rho}(u+\sigma h_1) - h_2(u+\sigma h_1)\Big)\right]$$
$$+ \beta h_2 - (u+\sigma h_1)(u-h_1) = 0. \tag{18.2.19}$$

Equating terms of like powers to zero gives

$$\begin{aligned} u^2 &: a_1(1+\sigma) = 0 \Rightarrow a_1 = 0, \\ &\beta b_1 - 1 = 0 \Rightarrow b_1 = \frac{1}{\beta}, \\ u\bar{\rho} &: (1+\sigma)a_2 + \frac{1}{1+\sigma} = 0 \Rightarrow a_2 = \frac{-1}{(1+\sigma)^2}, \\ &\beta b_2 = 0 \Rightarrow b_2 = 0. \end{aligned} \tag{18.2.20}$$

Then, using (18.2.21) and (18.2.16), we obtain

$$\begin{aligned} h_1(u,\bar{\rho}) &= -\frac{1}{(1+\sigma)^2}u\bar{\rho} + \cdots, \\ h_2(u,\bar{\rho}) &= \frac{1}{\beta}u^2 + \cdots. \end{aligned} \tag{18.2.21}$$

Finally, substituting (18.2.21) into (18.2.14) we obtain the vector field reduced to the center manifold

$$\begin{aligned} \dot{u} &= \frac{\sigma}{1+\sigma}u\left(\bar{\rho} - \frac{1}{\beta}u^2 + \cdots\right), \\ \dot{\bar{\rho}} &= 0. \end{aligned} \tag{18.2.22}$$

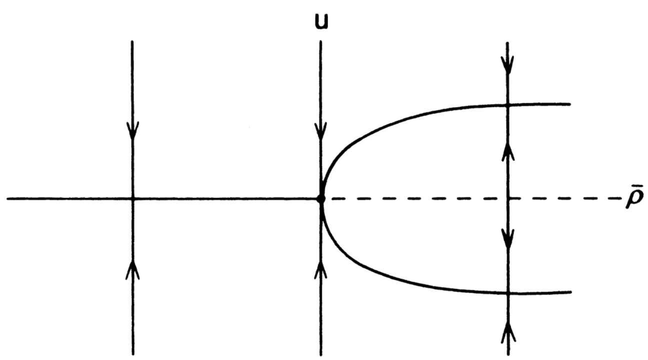

FIGURE 18.2.1.

In Figure 18.2.1 we plot the fixed points of (18.2.22) *neglecting* higher order terms such as $\mathcal{O}(\bar{\rho}^2)$, $\mathcal{O}(u\bar{\rho}^2)$, $\mathcal{O}(u^3)$, etc. It should be clear that $u = 0$ is always a fixed point and is stable for $\bar{\rho} < 0$ and unstable for $\bar{\rho} > 0$. At the point of exchange of stability (i.e., $\bar{\rho} = 0$) two new stable fixed points are created and are given by

$$\bar{\rho} = \frac{1}{\beta}u^2. \tag{18.2.23}$$

A simple calculation shows that these fixed points are stable. In Chapter 20 we will see that this is an example of a *pitchfork bifurcation.*

Before leaving this example two comments are in order.

1. Figure 18.2.1 shows the advantage of introducing the parameter as a new dependent variable. In a full neighborhood in parameter space new solutions are "captured" on the center manifold. In Figure 18.2.1, for each fixed $\bar{\rho}$ we have a flow in the u direction; this is represented by the vertical lines with arrows.
2. We have not considered the effects of the higher order terms in (18.2.22) on Figure 18.2.1. In Chapter 20 we will show that they do not qualitatively change the figure (i.e., they do not create, destroy, or change the stability of any of the fixed points) near the origin.

End of Example 18.2.1

18.3 The Inclusion of Linearly Unstable Directions

Suppose we consider the system

$$\begin{aligned} \dot{x} &= Ax + f(x,y,z), \\ \dot{y} &= By + g(x,y,z), \\ \dot{z} &= Cz + h(x,y,z), \qquad (x,y,z) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u, \end{aligned} \tag{18.3.1}$$

where

$$\begin{aligned} f(0,0,0) &= 0, \qquad Df(0,0,0) = 0, \\ g(0,0,0) &= 0, \qquad Dg(0,0,0) = 0, \\ h(0,0,0) &= 0, \qquad Dh(0,0,0) = 0, \end{aligned}$$

and f, g, and h are $\mathbf{C}^r (r \geq 2)$ in some neighborhood of the origin, A is a $c \times c$ matrix having eigenvalues with zero real parts, B is an $s \times s$ matrix having eigenvalues with negative real parts, and C is a $u \times u$ matrix having eigenvalues with positive real parts.

In this case $(x,y,z) = (0,0,0)$ is unstable due to the existence of a u-dimensional unstable manifold. However, much of the center manifold theory still applies, in particular Theorem 18.1.2 concerning existence, with the center manifold being locally represented by

$$\begin{aligned} W^c(0) = \big\{(x,y,z) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u \mid y = h_1(x), z = h_2(x), \\ h_i(0) = 0, Dh_i(0) = 0, i = 1,2\big\} \end{aligned} \tag{18.3.2}$$

for x sufficiently small. The vector field restricted to the center manifold is given by

$$\dot{u} = Au + f\big(u, h_1(u), h_2(u)\big), \qquad u \in \mathbb{R}^c. \tag{18.3.3}$$