where

$$A(t) \equiv Df(\bar{x}(t)).$$

In applying the method of normal forms to (19.13.2), the fact that $A(t)$ is time dependent causes problems. If $\bar{x}(t)$ is periodic in t, then $A(t)$ is periodic. Hence, Floquet theory can be used to transform (19.13.2) to a vector field where the linear part is constant (this is described in Arnold [1983]). In this case the method of normal forms as developed in this chapter can then be applied. Recently, Floquet theory has been generalized to the quasiperiodic case by Johnson [1986, 1987]; using these ideas, the normal form theory can be applied in this case also. Recent results along these lines have also been obtained by Jorba and Simo [1992], Treshchev [1995], and Jorba et al [1997]. Their results also apply to linear systems with quasiperiodically varying coefficients, and are computationally very efficient.

Concerning center manifold theory, Sell [1978] has proved existence theorems for stable, unstable, and center manifolds in nonautonomous systems.

Some very interesting work of Siegmund [2002] has recently appeared that develops the method of normal forms for very general nonautonomous vector fields.

Smooth Linearization. There exists a number of results concerning differentiable coordinate changes that linearize a dynamical system (vector field or diffeomorphism) in the neighborhood of an invariant manifold. A recent review of these results can be found in Bronstein and Kopanskii [1994].

Real Normal Forms and Complex Coordinates. We have seen numerous examples in this chapter where the use of complex coordinates simplifies normal form calculations. A systematic development of this approach can be found in Menck [1993].

Normal Forms for Stochastic Systems. Normal form theory for stochastic dynamical systems has been worked out in Namachchivaya and Leng [1990] and Namachchivaya and Lin [1991].

20

Bifurcation of Fixed Points of Vector Fields

Consider the parameterized vector field

$$\dot{y} = g(y, \lambda), \qquad y \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^p, \tag{20.0.1}$$

where g is a $\mathbf{C}^r$ function on some open set in $\mathbb{R}^n \times \mathbb{R}^p$. The degree of differentiability will be determined by our need to Taylor expand (20.0.1). Usually $\mathbf{C}^5$ will be sufficient.

Suppose (20.0.1) has a fixed point at $(y, \lambda) = (y_0, \lambda_0)$, i.e.,

$$g(y_0, \lambda_0) = 0. \tag{20.0.2}$$

Two questions immediately arise.

1. Is the fixed point stable or unstable?
2. How is the stability or instability affected as λ is varied?

To answer Question 1, the first step to take is to examine the linear vector field obtained by linearizing (20.0.1) about the fixed point $(y, \lambda) = (y_0, \lambda_0)$. This linear vector field is given by

$$\dot{\xi} = D_y g(y_0, \lambda_0)\xi, \qquad \xi \in \mathbb{R}^n. \tag{20.0.3}$$

If the fixed point is hyperbolic (i.e., none of the eigenvalues of $D_y g(y_0, \lambda_0)$ lie on the imaginary axis), we know that the stability of (y_0, λ_0) in (20.0.1) is determined by the linear equation (20.0.3) (cf. Chapter 1). This also enables us to answer Question 2, because since hyperbolic fixed points are structurally stable (cf. Chapter 12), varying λ slightly does not change the nature of the stability of the fixed point. This should be clear intuitively, but let us belabor the point slightly.

We know that

$$g(y_0, \lambda_0) = 0, \tag{20.0.4}$$

and that

$$D_y g(y_0, \lambda_0) \tag{20.0.5}$$

has no eigenvalues on the imaginary axis. Therefore, $D_y g(y_0, \lambda_0)$ is invertible. By the implicit function theorem, there thus exists a *unique* $\mathbf{C}^r$ function, $y(\lambda)$, such that

$$g(y(\lambda), \lambda) = 0 \tag{20.0.6}$$

for λ sufficiently close to λ_0 with

$$y(\lambda_0) = y_0. \tag{20.0.7}$$

Now, by continuity of the eigenvalues with respect to parameters, for λ sufficiently close to λ_0,

$$D_y g(y(\lambda), \lambda) \tag{20.0.8}$$

has no eigenvalues on the imaginary axis. Therefore, for λ sufficiently close to λ_0, the hyperbolic fixed point (y_0, λ_0) of (20.0.1) persists and its stability type remains unchanged. To summarize, in a neighborhood of λ_0 an isolated fixed point of (20.0.1) persists and always has the same stability type.

The real fun starts when the fixed point (y_0, λ_0) of (20.0.1) is not hyperbolic, i.e., when $D_y g(y_0, \lambda_0)$ has some eigenvalues on the imaginary axis. In this case, for λ very close to λ_0 (and for y close to y_0), radically new dynamical behavior can occur. For example, fixed points can be created or destroyed and time-dependent behavior such as periodic, quasiperiodic, or even chaotic dynamics can be created. In a certain sense (to be clarified later), the more eigenvalues on the imaginary axis, the more exotic the dynamics will be.

We will begin our study by considering the simplest way in which $D_y g(y_0, \lambda_0)$ can be nonhyperbolic. This is the case where $D_y g(y_0, \lambda_0)$ has a single zero eigenvalue with the remaining eigenvalues having nonzero real parts. The question we ask in this situation is what is the nature of this nonhyperbolic fixed point for λ close to λ_0? It is under these circumstances where the real power of the center manifold theory becomes apparent, since we know that this question can be answered by studying the vector field (20.0.1) restricted to the associated center manifold (cf. Section 18.2). In this case the vector field on the center manifold will be a p-parameter family of one-dimensional vector fields. This represents a vast simplification of (20.0.1).

20.1 A Zero Eigenvalue

Suppose that $D_y g(y_0, \lambda_0)$ has a single zero eigenvalue with the remaining eigenvalues having nonzero real parts; then the orbit structure near (y_0, λ_0) is determined by the associated center manifold equation, which we write as

$$\dot{x} = f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^p, \tag{20.1.1}$$

where $\mu = \lambda - \lambda_0$. Furthermore, we know that (20.1.1) must satisfy

$$f(0,0) = 0, \tag{20.1.2}$$

$$\frac{\partial f}{\partial x}(0,0) = 0. \tag{20.1.3}$$

Equation (20.1.2) is simply the fixed point condition and (20.1.3) is the zero eigenvalue condition. We remark that (20.1.1) is $\mathbf{C}^r$ if (20.0.1) is $\mathbf{C}^r$. Let us begin by studying a few specific examples. In these examples we will assume

$$\mu \in \mathbb{R}^1.$$

If there are more parameters in the problem (i.e., $\mu \in \mathbb{R}^p$, $p > 1$), we will consider all, except one, as fixed. Later we will consider more carefully the role played by the number of parameters in the problem. We remark also that we have not yet precisely defined what we mean by the term "bifurcation." We will consider this after the following series of examples.

20.1A Examples

Example 20.1.1. Consider the vector field

$$\dot{x} = f(x, \mu) = \mu - x^2, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{20.1.4}$$

It is easy to verify that

$$f(0,0) = 0 \tag{20.1.5}$$

and

$$\frac{\partial f}{\partial x}(0,0) = 0, \tag{20.1.6}$$

but in this example we can determine much more. The set of all fixed points of (20.1.4) is given by

$$\mu - x^2 = 0$$

or

$$\mu = x^2. \tag{20.1.7}$$

This represents a parabola in the $\mu - x$ plane as shown in Figure 20.1.1.

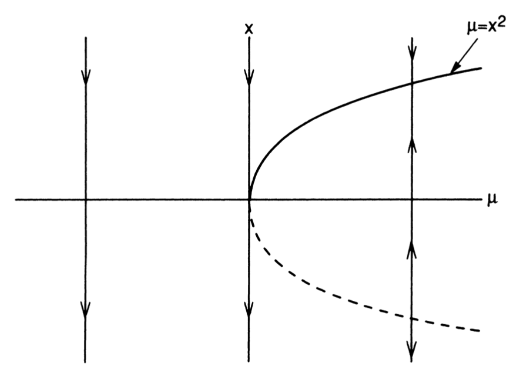

FIGURE 20.1.1.

In the figure the arrows along the vertical lines represent the flow generated by (20.1.4) along the x-direction. Thus, for $\mu < 0$, (20.1.4) has no fixed points, and the vector field is decreasing in x. For $\mu > 0$, (20.1.4) has two fixed points. A simple linear stability analysis shows that one of the fixed points is stable (represented by the solid branch of the parabola), and the other fixed point is unstable (represented by the broken branch of the parabola). However, we hope that it is obvious to the reader that, given a $\mathbf{C}^r$ $(r \geq 1)$ vector field on $\mathbb{R}^1$ having only two *hyperbolic* fixed points, one must be stable and the other unstable.

This is an example of *bifurcation*. We refer to $(x, \mu) = (0, 0)$ as a *bifurcation point* and the parameter value $\mu = 0$ as a *bifurcation value*.

Figure 20.1.1 is referred to as a *bifurcation diagram*. This particular type of bifurcation (i.e., where on one side of a parameter value there are no fixed points and on the other side there are two fixed points) is referred to as a *saddle-node bifurcation*. Later on we will worry about seeking precise conditions on the vector field on the center manifold that define the saddle-node bifurcation unambiguously.

End of Example 20.1.1

Example 20.1.2. Consider the vector field

$$\dot{x} = f(x, \mu) = \mu x - x^2, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{20.1.8}$$

It is easy to verify that

$$f(0, 0) = 0 \tag{20.1.9}$$

and

$$\frac{\partial f(0, 0)}{\partial x} = 0. \tag{20.1.10}$$

Moreover, the fixed points of (20.1.8) are given by

$$x = 0 \tag{20.1.11}$$

and

$$x = \mu \tag{20.1.12}$$

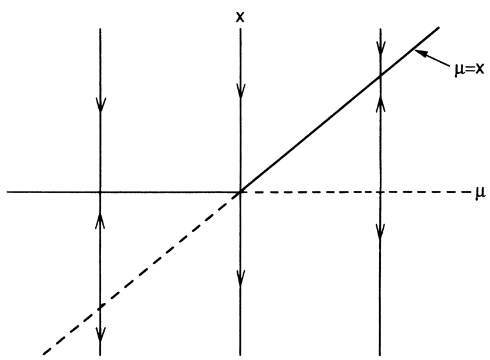

FIGURE 20.1.2.

and are plotted in Figure 20.1.2. Hence, for $\mu < 0$, there are two fixed points; $x = 0$ is stable and $x = \mu$ is unstable. These two fixed points coalesce at $\mu = 0$ and, for $\mu > 0$, $x = 0$ is unstable and $x = \mu$ is stable. Thus, an exchange of stability has occurred at $\mu = 0$. This type of bifurcation is called a *transcritical bifurcation*.

End of Example 20.1.2

Example 20.1.3. Consider the vector field

$$\dot{x} = f(x, \mu) = \mu x - x^3, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{20.1.13}$$

It is clear that we have

$$f(0, 0) = 0, \tag{20.1.14}$$

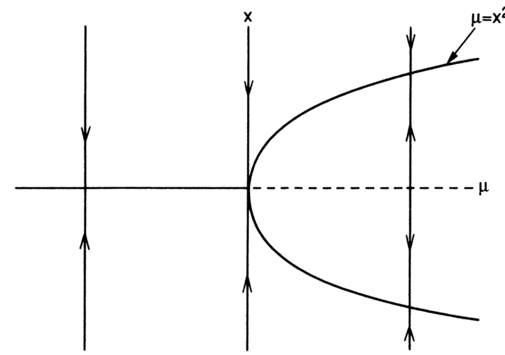

FIGURE 20.1.3.

$$\frac{\partial f}{\partial x}(0, 0) = 0. \tag{20.1.15}$$

Moreover, the fixed points of (20.1.13) are given by

$$x = 0 \tag{20.1.16}$$

and

$$x^2 = \mu \tag{20.1.17}$$

and are plotted in Figure 20.1.3.

Hence, for $\mu < 0$, there is one fixed point, $x = 0$, which is stable. For $\mu > 0$, $x = 0$ is still a fixed point, but two new fixed points have been created at $\mu = 0$ and are given by $x^2 = \mu$. In the process, $x = 0$ has become unstable for $\mu > 0$, with the other two fixed points stable. This type of bifurcation is called a *pitchfork bifurcation*.

End of Example 20.1.3

Example 20.1.4. Consider the vector field

$$\dot{x} = f(x, \mu) = \mu - x^3, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{20.1.18}$$

It is trivial to verify that

$$f(0,0) = 0 \tag{20.1.19}$$

and

$$\frac{\partial f}{\partial x}(0,0) = 0. \tag{20.1.20}$$

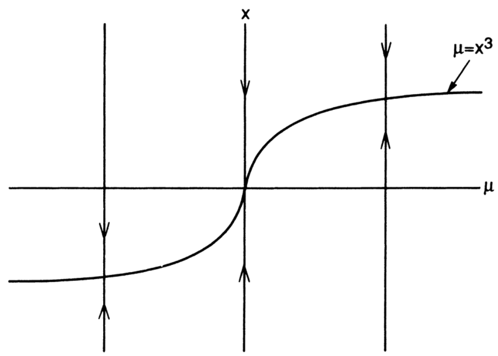

FIGURE 20.1.4.

Moreover, all fixed points of (20.1.18) are given by

$$\mu = x^3 \tag{20.1.21}$$

and are shown in Figure 20.1.4.

However in this example, despite (20.1.19) and (20.1.20), the dynamics of (20.1.18) are qualitatively the same for $\mu > 0$ and $\mu < 0$. Namely, (20.1.18) possesses a unique, stable fixed point.

End of Example 20.1.4

20.1B What Is A "Bifurcation of a Fixed Point"?

The term "bifurcation" is extremely general. We will begin to learn its uses in dynamical systems by understanding its use in describing the orbit structure near nonhyperbolic fixed points. Let us consider what we learned from the previous examples.

In all four examples we had

$$f(0,0) = 0$$

and

$$\frac{\partial f}{\partial x}(0,0) = 0,$$

and yet the orbit structure near $\mu = 0$ was different in all four cases. Hence, knowing that a fixed point has a zero eigenvalue for $\mu = 0$ is not sufficient to determine the orbit structure for λ near zero. Let us consider each example individually.

1. (*Example 20.1.1*). In this example a *unique* curve (or branch) of fixed points passed through the origin. Moreover, the curve lay entirely on one side of $\mu = 0$ in the $\mu - x$ plane.

2. (*Example 20.1.2*). In this example two curves of fixed points intersected at the origin in the $\mu - x$ plane. Both curves existed on either side of $\mu = 0$. However, the stability of the fixed point along a given curve changed on passing through $\mu = 0$.

3. (*Example 20.1.3*). In this example two curves of fixed points intersected at the origin in the $\mu - x$ plane. Only one curve ($x = 0$) existed on both sides of $\mu = 0$; however, its stability changed on passing through $\mu = 0$. The other curve of fixed points lay entirely to one side of $\mu = 0$ and had a stability type that was the opposite of $x = 0$ for $\mu > 0$.

4. (*Example 20.1.4*). This example had a unique curve of fixed points passing through the origin in the $\mu - x$ plane and existing on both sides of $\mu = 0$. Moreover, all fixed points along the curve had the same stability type. Hence, despite the fact that the fixed point $(x, \mu) = (0, 0)$ was nonhyperbolic, the orbit structure was qualitatively the same for all μ.

We want to apply the term "bifurcation" to Examples 20.1.1, 20.1.2, and 20.1.3 but not to Example 20.1.4 to describe the change in orbit structure as μ passes through zero. We are therefore led to the following definition.

Definition 20.1.1 (Bifurcation of a Fixed Point) *A fixed point* $(x, \mu) = (0, 0)$ *of a one-parameter family of one-dimensional vector fields is said to undergo a* bifurcation *at* $\mu = 0$ *if the flow for* μ *near zero and* x *near zero is* not *qualitatively the same as the flow near* $x = 0$ *at* $\mu = 0$.

Several remarks are now in order concerning this definition.

Remark 1. The phrase "qualitatively the same" is a bit vague. It can be made precise by substituting the term "$\mathbf{C}^0$-equivalent" (cf. Section 19.12), and this is perfectly adequate for the study of the bifurcation of fixed points of *one-dimensional* vector fields. However, we will see that as we explore higher dimensional phase spaces and global bifurcations, how to make mathematically precise the statement "two dynamical systems have qualitatively the same dynamics" becomes more and more ambiguous.

Remark 2. Practically speaking, a fixed point (x_0, μ_0) of a one-dimensional vector field is a bifurcation point if either more than one curve of fixed points passes through (x_0, μ_0) in the $\mu - x$ plane or if only one curve of fixed points passes (x_0, μ_0) in the $\mu - x$ plane; then it (locally) lies entirely on one side of the line $\mu = \mu_0$ in the $\mu - x$ plane.

Remark 3. It should be clear from Example 20.1.4 that the condition that a fixed point is nonhyperbolic is a necessary but not sufficient condition for bifurcation to occur in one-parameter families of vector fields.

We next turn to deriving general conditions on one-parameter families of one-dimensional vector fields which exhibit bifurcations exactly as in Examples 20.1.1, 20.1.2, and 20.1.3.

20.1c The Saddle-Node Bifurcation

We now want to derive conditions under which a general one-parameter family of one-dimensional vector fields will undergo a saddle-node bifurcation exactly as in Example 20.1.1. These conditions will involve derivatives of the vector field evaluated at the bifurcation point and are obtained by a consideration of the geometry of the curve of fixed points in the $\mu - x$ plane in a neighborhood of the bifurcation point.

Let us recall Example 20.1.1. In this example a *unique* curve of fixed points, parameterized by x, passed through $(\mu, x) = (0, 0)$. We denote the curve of fixed points by $\mu(x)$. The curve of fixed points satisfied two properties.

1. It was tangent to the line $\mu = 0$ at $x = 0$, i.e.,
$$\frac{d\mu}{dx}(0) = 0. \tag{20.1.22}$$

2. It lay entirely to one side of $\mu = 0$. Locally, this will be satisfied if we have
$$\frac{d^2\mu}{dx^2}(0) \neq 0. \tag{20.1.23}$$

Now let us consider a general, one-parameter family of one-dimensional vector fields.
$$\dot{x} = f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{20.1.24}$$
Suppose (20.1.24) has a fixed point at $(x, \mu) = (0, 0)$, i.e.,
$$f(0, 0) = 0. \tag{20.1.25}$$

Furthermore, suppose that the fixed point is not hyperbolic, i.e.,
$$\frac{\partial f}{\partial x}(0, 0) = 0. \tag{20.1.26}$$
Now, if we have
$$\frac{\partial f}{\partial \mu}(0, 0) \neq 0, \tag{20.1.27}$$
then, by the implicit function theorem, there exists a unique function
$$\mu = \mu(x), \qquad \mu(0) = 0 \tag{20.1.28}$$
defined for x sufficiently small such that $f(x, \mu(x)) = 0$. (Note: the reader should check that (20.1.27) holds in Example 20.1.1.) Now we want to derive conditions in terms of derivatives of f evaluated at $(\mu, x) = (0, 0)$ so that we have
$$\frac{d\mu}{dx}(0) = 0, \tag{20.1.29}$$
$$\frac{d^2\mu}{dx^2}(0) \neq 0. \tag{20.1.30}$$
Equations (20.1.29) and (20.1.30), along with (20.1.25), (20.1.26), and (20.1.27), imply that $(\mu, x) = (0, 0)$ is a bifurcation point at which a saddle-node bifurcation occurs.

We can derive expressions for (20.1.29) and (20.1.30) in terms of derivatives of f at the bifurcation point by implicitly differentiating f along the curve of fixed points.

Using (20.1.27), we have
$$f(x, \mu(x)) = 0. \tag{20.1.31}$$
Differentiating (20.1.31) with respect to x gives
$$\frac{df}{dx}(x, \mu(x)) = 0 = \frac{\partial f}{\partial x}(x, \mu(x)) + \frac{\partial f}{\partial \mu}(x, \mu(x))\frac{d\mu}{dx}(x). \tag{20.1.32}$$
Evaluating (20.1.32) at $(\mu, x) = (0, 0)$, we obtain
$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial f}{\partial x}(0, 0)}{\frac{\partial f}{\partial \mu}(0, 0)}; \tag{20.1.33}$$
thus we see that (20.1.26) and (20.1.27) imply that
$$\frac{d\mu}{dx}(0) = 0, \tag{20.1.34}$$
i.e., the curve of fixed points is tangent to the line $\mu = 0$ at $x = 0$.

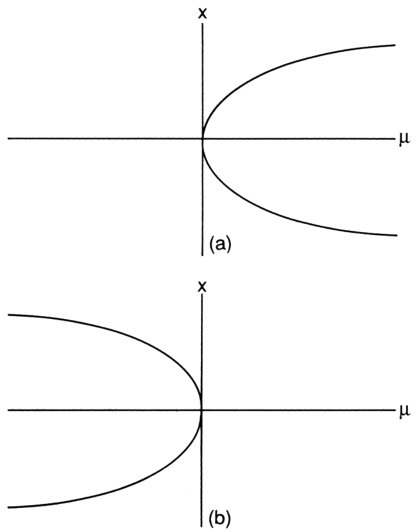

FIGURE 20.1.5.
a) $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial f}{\partial \mu}(0,0)\right) > 0$; b) $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial f}{\partial \mu}(0,0)\right) < 0$.

Next, let us differentiate (20.1.32) once more with respect to x to obtain

$$\begin{aligned}\frac{d^2 f}{dx^2}(x,\mu(x)) = 0 = \frac{\partial^2 f}{\partial x^2}(x,\mu(x)) + 2\frac{\partial^2 f}{\partial x \partial \mu}(x,\mu(x))\frac{d\mu}{dx}(x) \\ + \frac{\partial^2 f}{\partial \mu^2}(x,\mu(x))\left(\frac{d\mu}{dx}(x)\right)^2 \\ + \frac{\partial f}{\partial \mu}(\mu,\mu(x))\frac{d^2\mu}{dx^2}(x).\end{aligned} \tag{20.1.35}$$

Evaluating (20.1.35) at $(\mu, x) = (0,0)$ and using (20.1.33) gives

$$\frac{\partial^2 f}{\partial x^2}(0,0) + \frac{\partial f}{\partial \mu}(0,0)\frac{d^2\mu}{dx^2}(0) = 0$$

or

$$\frac{d^2\mu}{dx^2}(0) = \frac{-\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial f}{\partial \mu}(0,0)}. \tag{20.1.36}$$

Hence, (20.1.36) is nonzero provided we have

$$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0. \tag{20.1.37}$$

Let us summarize. In order for (20.1.24) to undergo a saddle-node bifurcation we must have

$$\left.\begin{aligned} f(0,0) = 0 \\ \frac{\partial f}{\partial x}(0,0) = 0 \end{aligned}\right\} \quad \text{nonhyperbolic fixed point} \tag{20.1.38}$$

and

$$\frac{\partial f}{\partial \mu}(0,0) \neq 0, \tag{20.1.39}$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0. \tag{20.1.40}$$

Equation (20.1.39) implies that a unique curve of fixed points passes through $(\mu, x) = (0,0)$, and (20.1.40) implies that the curve lies locally on one side of $\mu = 0$. It should be clear that the sign of (20.1.36) determines on which side of $\mu = 0$ the curve lies. In Figure 20.1.5 we show both cases without indicating stability and leave it as an exercise for the reader to verify the stability types of the different branches of fixed points emanating from the bifurcation point.

Let us end our discussion of the saddle-node bifurcation with the following remark. Consider a general one-parameter family of one-dimensional vector fields having a nonhyperbolic fixed point at $(x, \mu) = (0,0)$. The Taylor expansion of this vector field is given as follows

$$f(x,\mu) = a_0\mu + a_1 x^2 + a_2 \mu x + a_3 \mu^2 + \mathcal{O}(3). \tag{20.1.41}$$

Our computations show that the dynamics of (20.1.41) near $(\mu, x) = (0,0)$ are qualitatively the same as one of the following vector fields

$$\dot{x} = \mu \pm x^2. \tag{20.1.42}$$

Hence, (20.1.42) can be viewed as the *normal form* for saddle-node bifurcations.

This brings up another important point. In applying the method of normal forms there is always the question of truncation of the normal form; namely, how are the dynamics of the normal form including only the $\mathcal{O}(k)$ terms modified when the higher order terms are included? We see that, in the study of the saddle-node bifurcation, all terms of $\mathcal{O}(3)$ and higher could be neglected and the dynamics would not be qualitatively changed. The implicit function theorem was the tool that enabled us to verify this fact.

20.1D The Transcritical Bifurcation

We want to follow the same strategy as in our discussion and derivation of general conditions for the saddle-node bifurcation given in the previous section, namely, to use the implicit function theorem to characterize the geometry of the curves of fixed points passing through the bifurcation point in terms of derivatives of the vector field evaluated at the bifurcation point.

For the example of transcritical bifurcation discussed in Example 20.1.2, the orbit structure near the bifurcation point was characterized as follows.

1. Two curves of fixed points passed through $(x, \mu) = (0, 0)$, one given by $x = \mu$, the other by $x = 0$.
2. Both curves of fixed points existed on both sides of $\mu = 0$.
3. The stability along each curve of fixed points changed on passing through $\mu = 0$.

Using these three points as a guide, let us consider a general one-parameter family of one-dimensional vector fields

$$\dot{x} = f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{20.1.43}$$

We assume that at $(x, \mu) = (0, 0)$, (20.1.43) has a nonhyperbolic fixed point, i.e.,

$$f(0, 0) = 0 \tag{20.1.44}$$

and

$$\frac{\partial f}{\partial x}(0, 0) = 0. \tag{20.1.45}$$

Now, in Example 20.1.2 we had two curves of fixed points passing through $(\mu, x) = (0, 0)$. In order for this to occur it is necessary to have

$$\frac{\partial f}{\partial \mu}(0, 0) = 0, \tag{20.1.46}$$

or else, by the implicit function theorem, only one curve of fixed points could pass through the origin.

Equation (20.1.46) presents a problem if we wish to proceed as in the case of the saddle-node bifurcation; in that situation we used the condition $\frac{\partial f}{\partial \mu}(0, 0) \neq 0$ in order to conclude that a unique curve of fixed points, $\mu(x)$, passed through the bifurcation point. We then evaluated the vector field on the curve of fixed points and used implicit differentiation to derive local characteristics of the geometry of the curve of fixed points based on properties of the derivatives of the vector field evaluated at the bifurcation point. However, if we use Example 20.1.2 as a guide, we can extricate ourselves from this difficulty.

In Example 20.1.2, $x = 0$ was a curve of fixed points passing through the bifurcation point. We will *require* that to be the case for (20.1.43), so that (20.1.43) has the form

$$\dot{x} = f(x, \mu) = xF(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{20.1.47}$$

where, by definition, we have

$$F(x, \mu) \equiv \left\{ \begin{array}{ll} \frac{f(x,\mu)}{x}, & x \neq 0 \\ \frac{\partial f}{\partial x}(0, \mu), & x = 0 \end{array} \right\}. \tag{20.1.48}$$

Since $x = 0$ is a curve of fixed points for (20.1.47), in order to obtain an additional curve of fixed points passing through $(\mu, x) = (0, 0)$ we need to seek conditions on F whereby F has a curve of zeros passing through $(\mu, x) = (0, 0)$ (that is not given by $x = 0$). These conditions will be in terms of derivatives of F which, using (20.1.48), can be expressed as derivatives of f.

Using (20.1.48), it is easy to verify the following

$$F(0, 0) = 0, \tag{20.1.49}$$

$$\frac{\partial F}{\partial x}(0, 0) = \frac{\partial^2 f}{\partial x^2}(0, 0), \tag{20.1.50}$$

$$\frac{\partial^2 F}{\partial x^2}(0, 0) = \frac{\partial^3 f}{\partial x^3}(0, 0), \tag{20.1.51}$$

and (most importantly)

$$\frac{\partial F}{\partial \mu}(0, 0) = \frac{\partial^2 f}{\partial x \partial \mu}(0, 0). \tag{20.1.52}$$

Now let us assume that (20.1.52) is *not* zero; then by the implicit function theorem there exists a function, $\mu(x)$, defined for x sufficiently small, such that

$$F(x, \mu(x)) = 0. \tag{20.1.53}$$

Clearly, $\mu(x)$ is a curve of fixed points of (20.1.47). In order for $\mu(x)$ to not coincide with $x = 0$ and to exist on both sides of $\mu = 0$, we must require that

$$0 < \left| \frac{d\mu}{dx}(0) \right| < \infty.$$

Implicitly differentiating (20.1.53) exactly as in the case of the saddle-node bifurcation we obtain

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial F}{\partial x}(0, 0)}{\frac{\partial F}{\partial \mu}(0, 0)}. \tag{20.1.54}$$

Using (20.1.49), (20.1.50), (20.1.51), and (20.1.52), (20.1.54) becomes

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial^2 f}{\partial x^2}(0, 0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0, 0)}. \tag{20.1.55}$$

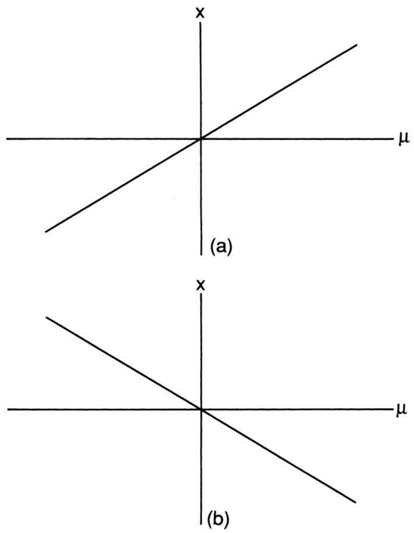

FIGURE 20.1.6.
a) $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) > 0$; b) $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) < 0$.

We now summarize our results. In order for a vector field

$$\dot{x} = f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{20.1.56}$$

to undergo a transcritical bifurcation, we must have

$$\left.\begin{aligned} f(0,0) &= 0 \\ \frac{\partial f}{\partial x}(0,0) &= 0 \end{aligned}\right\} \quad \text{nonhyperbolic fixed point} \tag{20.1.57}$$

and

$$\frac{\partial f}{\partial \mu}(0,0) = 0, \tag{20.1.58}$$

$$\frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0, \tag{20.1.59}$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0. \tag{20.1.60}$$

We note that the slope of the curve of fixed points not equal to $x = 0$ is given by (20.1.55). These two cases are shown in Figure 20.1.6; however, we do not indicate stabilities of the different branches of fixed points. We leave it as an exercise to the reader to verify the stability types of the different curves of fixed points emanating from the bifurcation point.

Thus, (20.1.57), (20.1.58), (20.1.59), and (20.1.60) show that the orbit structure near $(x, \mu) = (0, 0)$ is qualitatively the same as the orbit structure near $(x, \mu) = (0, 0)$ of

$$\dot{x} = \mu x \mp x^2. \tag{20.1.61}$$

Equation (20.1.61) can be viewed as a normal form for the transcritical bifurcation.

20.1e The Pitchfork Bifurcation

The discussion and derivation of conditions under which a general one-parameter family of one-dimensional vector fields will undergo a bifurcation of the type shown in Example 20.1.3 follows very closely our discussion of the transcritical bifurcation.

The geometry of the curves of fixed points associated with the bifurcation in Example 20.1.3 had the following characteristics.

1. Two curves of fixed points passed through $(\mu, x) = (0, 0)$, one given by $x = 0$, the other by $\mu = x^2$.
2. The curve $x = 0$ existed on both sides of $\mu = 0$; the curve $\mu = x^2$ existed on one side of $\mu = 0$.
3. The fixed points on the curve $x = 0$ had different stability types on opposite sides of $\mu = 0$. The fixed points on $\mu = x^2$ all had the same stability type.

Now we want to consider conditions on a general one-parameter family of one-dimensional vector fields having two curves of fixed points passing through the bifurcation point in the $\mu - x$ plane that have the properties given above.

We denote the vector field by

$$\dot{x} = f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{20.1.62}$$

and we suppose

$$f(0,0) = 0, \tag{20.1.63}$$

$$\frac{\partial f}{\partial x}(0,0) = 0. \tag{20.1.64}$$

As in the case of the transcritical bifurcation, in order to have more than one curve of fixed points passing through $(\mu, x) = (0, 0)$ we must have

$$\frac{\partial f}{\partial \mu}(0,0) = 0. \tag{20.1.65}$$

Proceeding further along these lines, we *require* $x = 0$ to be a curve of fixed points for (20.1.62) by assuming the vector field (20.1.62) has the form

$$\dot{x} = x F(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{20.1.66}$$

where

$$F(x,\mu) \equiv \begin{Bmatrix} \frac{f(x,\mu)}{x}, & x \neq 0 \\ \frac{\partial f}{\partial x}(0,\mu), & x = 0 \end{Bmatrix}. \tag{20.1.67}$$

In order to have a second curve of fixed points passing through $(\mu, x) = (0,0)$ we must have

$$F(0,0) = 0 \tag{20.1.68}$$

with

$$\frac{\partial F}{\partial \mu}(0,0) \neq 0. \tag{20.1.69}$$

Equation (20.1.69) insures that only *one* additional curve of fixed points passes through $(\mu, x) = (0,0)$. Also, using (20.1.69), the implicit function theorem implies that for x sufficiently small there exists a unique function $\mu(x)$ such that

$$F(x, \mu(x)) = 0. \tag{20.1.70}$$

In order for the curve of fixed points, $\mu(x)$, to satisfy the above-mentioned characteristics, it is sufficient to have

$$\frac{d\mu}{dx}(0) = 0 \tag{20.1.71}$$

and

$$\frac{d^2\mu}{dx^2}(0) \neq 0. \tag{20.1.72}$$

The conditions for (20.1.71) and (20.1.72) to hold in terms of the derivatives of F evaluated at the bifurcation point can be obtained via implicit differentiation of (20.1.70) along the curve of fixed points exactly as in the case of the saddle-node bifurcation. They are given by

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial F}{\partial x}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)} = 0 \tag{20.1.73}$$

and

$$\frac{d^2\mu}{dx^2}(0) = \frac{-\frac{\partial^2 F}{\partial x^2}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)} \neq 0. \tag{20.1.74}$$

Using (20.1.67), (20.1.73) and (20.1.74) can be expressed in terms of derivatives of f as follows

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0,0)} = 0 \tag{20.1.75}$$

and

$$\frac{d^2\mu}{dx^2}(0) = \frac{-\frac{\partial^3 f}{\partial x^3}(0,0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0,0)} \neq 0. \tag{20.1.76}$$

We summarize as follows. In order for the vector field

$$\dot{x} = f(x,\mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{20.1.77}$$

to undergo a pitchfork bifurcation at $(x, \mu) = (0,0)$, it is sufficient to have

$$\left.\begin{array}{l} f(0,0) = 0 \\ \frac{\partial f}{\partial x}(0,0) = 0 \end{array}\right\} \quad \text{nonhyperbolic fixed point} \tag{20.1.78}$$

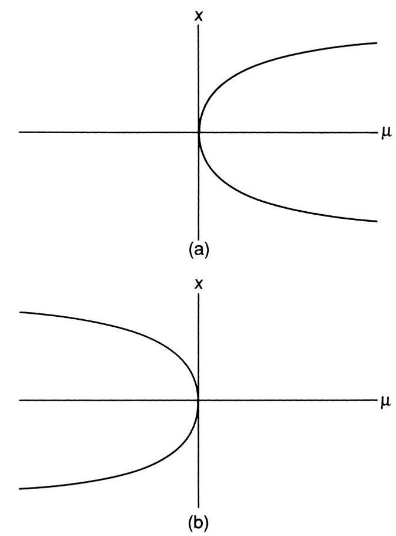

FIGURE 20.1.7.
a) $\left(-\frac{\partial^3 f}{\partial x^3}(0,0) / \frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) > 0$; b) $\left(-\frac{\partial^3 f}{\partial x^3}(0,0) / \frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) < 0$.

with

$$\frac{\partial f}{\partial \mu}(0,0) = 0, \tag{20.1.79}$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 0, \tag{20.1.80}$$

$$\frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0, \tag{20.1.81}$$

$$\frac{\partial^3 f}{\partial x^3}(0,0) \neq 0. \tag{20.1.82}$$

There are two possibilities for the disposition of the two branches of fixed points depending on the sign of (20.1.76). These two possibilities are shown in Figure 20.1.7 without indicating stabilities. We leave it as an exercise for the reader to verify the stability types for the different branches of fixed points emanating from the bifurcation point.

We conclude by noting that (20.1.78), (20.1.79), (20.1.80), (20.1.81), and (20.1.82) imply that the orbit structure near $(x, \mu) = (0, 0)$ is qualitatively the same as the orbit structure near $(x, \mu) = (0, 0)$ in the vector field

$$\dot{x} = \mu x \mp x^3. \tag{20.1.83}$$

Thus, (20.1.83) can be viewed as a normal form for the pitchfork bifurcation.

20.1F Exercises

1. In our development of the transcritical and pitchfork bifurcations we assumed that $x = 0$ was a trivial solution. Was this necessary? In particular, would the conditions for transcritical and pitchfork bifurcations change if this were not the case?

2. Consider a $\mathbf{C}^r$ $(r \geq 1)$ autonomous vector field on $\mathbb{R}^1$ having precisely two *hyperbolic* fixed points. Can you infer the nature of the stability of the two fixed points? How does the situation change if one of the fixed points is not hyperbolic? Can both fixed points be nonhyperbolic? Construct explicit examples illustrating each situation.

3. Consider the saddle-node bifurcation for vector fields and Figure 20.1.5. For the case $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial f}{\partial \mu}(0,0)\right) > 0$, give conditions under which the upper part of the curve of fixed points is stable and the lower part is unstable. Alternatively, give conditions under which the upper part of the curve of fixed points is unstable and the lower part is stable.

   Repeat the exercise for the case $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial f}{\partial \mu}(0,0)\right) < 0$.

4. Consider the transcritical bifurcation for vector fields and Figure 20.1.6. For the case $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) > 0$, give conditions for $x = 0$ to be stable for $\mu > 0$ and unstable for $\mu < 0$. Alternatively, give conditions for $x = 0$ to be unstable for $\mu > 0$ and stable for $\mu < 0$.

   Repeat the exercise for the case $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) < 0$.

5. Consider the pitchfork bifurcation for vector fields and Figure 20.1.7. For the case $\left(-\frac{\partial^3 f}{\partial x^3}(0,0)/\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) > 0$, give conditions for $x = 0$ to be stable for $\mu > 0$ and unstable for $\mu < 0$. Alternatively, give conditions for $x = 0$ to be unstable for $\mu > 0$ and stable for $\mu < 0$.

   Repeat the exercise for the case $\left(-\frac{\partial^3 f}{\partial x^3}(0,0)/\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) < 0$.

6. In Exercise 4 following Chapter 18 we computed center manifolds near the origin for the following one-parameter families of vector fields. Describe the bifurcations of the origin. In, for example, a) and a′) the parameter ε multiplies a linear and nonlinear term, respectively. In terms of bifurcations, is there a qualitative difference in the two cases? What kinds of general statements can you make?

   a) $\begin{aligned} \dot\theta &= -\theta + \varepsilon v + v^2, \\ \dot v &= -\sin\theta, \end{aligned} \qquad (\theta, v) \in S^1 \times \mathbb{R}^1.$

   a′) $\begin{aligned} \dot\theta &= -\theta + v^2 + \varepsilon v^2, \\ \dot v &= -\sin\theta. \end{aligned}$

   b) $\begin{aligned} \dot x &= \frac{1}{2}x + y + x^2 y, \\ \dot y &= x + 2y + \varepsilon y + y^2, \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$

   b′) $\begin{aligned} \dot x &= \frac{1}{2}x + y + x^2 y, \\ \dot y &= x + 2y + y^2 + \varepsilon y^2. \end{aligned}$

   d) $\begin{aligned} \dot x &= 2x + 2y + \varepsilon y, \\ \dot y &= x + y + x^4, \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$

   d′) $\begin{aligned} \dot x &= 2x + 2y, \\ \dot y &= x + y + x^4 + \varepsilon y^2. \end{aligned}$

   f) $\begin{aligned} \dot x &= -2x + 3y + \varepsilon x + y^3, \\ \dot y &= 2x - 3y + x^3, \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$

   f′) $\begin{aligned} \dot x &= -2x + 3y + y^3 + \varepsilon x^2, \\ \dot y &= 2x - 3y + x^3. \end{aligned}$

   h) $\begin{aligned} \dot x &= -x + y, \\ \dot y &= -e^x + e^{-x} + 2x + \varepsilon y, \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$

   h′) $\begin{aligned} \dot x &= -x + y + \varepsilon x^2, \\ \dot y &= -e^x + e^{-x} + 2x. \end{aligned}$

   i) $\begin{aligned} \dot x &= -2x + y + z + \varepsilon x + y^2 z, \\ \dot y &= x - 2y + z + \varepsilon x + x z^2, \\ \dot z &= x + y - 2z + \varepsilon x + x^2 y, \end{aligned} \qquad (x, y, z) \in \mathbb{R}^3.$

   i′) $\begin{aligned} \dot x &= -2x + y + z + \varepsilon x^2 + y^2 z, \\ \dot y &= x - 2y + z + \varepsilon x y + x z^2, \\ \dot z &= x + y - 2z + x^2 y. \end{aligned}$

   j) $\begin{aligned} \dot x &= -x - y + z^2, \\ \dot y &= 2x + y + \varepsilon y - z^2, \\ \dot z &= x + 2y - z, \end{aligned} \qquad (x, y, z) \in \mathbb{R}^3.$

   j′) $\begin{aligned} \dot x &= -x - y + \varepsilon x^2 + z^2, \\ \dot y &= 2x + y - z^2 + \varepsilon y^2, \\ \dot z &= x + 2y - z. \end{aligned}$

   k) $\begin{aligned} \dot x &= -x - y - z + \varepsilon x - yz, \\ \dot y &= -x - y - z - xz, \\ \dot z &= -x - y - z - yz, \end{aligned} \qquad (x, y, z) \in \mathbb{R}^3.$

   k′) $\begin{aligned} \dot x &= -x - y - z - yz + \varepsilon x^2, \\ \dot y &= -x - y - z - xz, \\ \dot z &= -x - y - z - xy. \end{aligned}$

   l) $\begin{aligned} \dot x &= y + x^2 + \varepsilon y, \\ \dot y &= -y - x^2, \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$

   l′) $\begin{aligned} \dot x &= y + x^2 + \varepsilon y^2, \\ \dot y &= -y - x^2. \end{aligned}$

   m) $\begin{aligned} \dot x &= x^2 + \varepsilon y, \\ \dot y &= -y - x^2, \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$

   m′) $\begin{aligned} \dot x &= x^2 + \varepsilon y^2, \\ \dot y &- -y - x^2. \end{aligned}$

7. *Center Manifolds at a Saddle-node Bifurcation Point for Vector Fields*

In developing the center manifold theory for parametrized families of vector fields, we dealt with equations of the following form

$$\begin{aligned}\dot{x} &= Ax + f(x,y,\varepsilon),\\ \dot{y} &= By + g(x,y,\varepsilon),\end{aligned} \qquad (x,y,\varepsilon) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^p, \tag{20.1.84}$$

where A is a $c \times c$ matrix whose eigenvalues all have zero real parts, B is an $s \times s$ matrix whose eigenvalues all have negative real parts, and

$$\begin{aligned} f(0,0,0) &= 0, \qquad & Df(0,0,0) &= 0,\\ g(0,0,0) &= 0, \qquad & Dg(0,0,0) &= 0.\end{aligned} \tag{20.1.85}$$

The conditions $Df(0,0,0) = 0$, $Dg(0,0,0) = 0$ do not allow for terms that are linear in the parameter ε. Clearly, this *may not* be the case at a saddle-node bifurcation point, and we want to consider this issue in this exercise. Although this could have been done in Chapter 18, in that chapter we were introducing only center manifold theory and were not really concerned with bifurcations. In this case the form of the equations given by (20.1.84) and (20.1.85) was the "cleanest and quickest" way to introduce the notion of parametrized families of center manifolds.

We will start at a very basic level. Consider the $\mathbf{C}^r$ (r as large as necessary) vector field

$$\dot{z} = F(z,\varepsilon), \qquad (z,\varepsilon) \in \mathbb{R}^{c+s} \times \mathbb{R}^p. \tag{20.1.86}$$

Suppose that $(z,\varepsilon) = (0,0)$ is a fixed point of (20.1.86) at which the matrix

$$D_z F(0,0) \tag{20.1.87}$$

has c eigenvalues with zero real parts and s eigenvalues with negative real parts. Our goal is to apply the center manifold theory in order to examine the dynamics of (20.1.86) near $(z,\varepsilon) = (0,0)$.

We rewrite Equation (20.1.86) as follows

$$\dot{z} = D_z F(0,0)z + D_\varepsilon F(0,0)\varepsilon + G(z,\varepsilon), \tag{20.1.88}$$

where

$$G(z,\varepsilon) = \Big[F(z,\varepsilon) - D_z F(0,0)z - D_\varepsilon F(0,0)\varepsilon\Big] = \mathcal{O}(2) \tag{20.1.89}$$

in z and ε. Note that the term "$D_\varepsilon F(0,0)\varepsilon$" in (20.1.88) is the new wrinkle—it was zero under our previous assumptions. For notational purposes we let

$$\begin{aligned} D_z F(0,0) &\equiv M \qquad && -(c+s)\times(c+s) \text{ matrix},\\ D_\varepsilon F(0,0) &\equiv \Lambda \qquad && -(c+s)\times p \text{ matrix},\end{aligned}$$

so that (20.1.88) becomes

$$\dot{z} = Mz + \Lambda\varepsilon + G(z,\varepsilon). \tag{20.1.90}$$

Now let T be the $(s+c)\times(s+c)$ matrix that puts M into the following block diagonal form

$$T^{-1}MT = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{20.1.91}$$

where A is a $(c \times c)$ matrix with all eigenvalues having zero real parts and B is an $(s \times s)$ matrix with all eigenvalues having negative real parts. If we let

$$z = Tw, \qquad (x,y) \in \mathbb{R}^c \times \mathbb{R}^s, \tag{20.1.92}$$

where $w = (x,y)$, and apply this linear transformation to (20.1.90), we obtain

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \overline{\Lambda}\varepsilon + \begin{pmatrix} f(x,y,\varepsilon) \\ g(x,y,\varepsilon) \end{pmatrix}, \tag{20.1.93}$$

where

$$\overline{\Lambda} \equiv T^{-1}\Lambda,$$

$$\begin{pmatrix} f(x,y,\varepsilon) \\ g(x,y,\varepsilon) \end{pmatrix} \equiv T^{-1}G(T(x,y),\varepsilon).$$

Note that $f(0,0,0) = 0$, $g(0,0,0) = 0$, $Df(0,0,0) = 0$, and $Dg(0,0,0) = 0$. Next, let

$$\overline{\Lambda} = \begin{pmatrix} \overline{\Lambda}_c \\ \overline{\Lambda}_s \end{pmatrix},$$

where Λ_c corresponds to the first c rows of Λ, and Λ_s corresponds to the last s rows of Λ. Then (20.1.93) can be rewritten as

$$\begin{pmatrix} \dot{x} \\ \dot{\varepsilon} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} A & \overline{\Lambda}_c & 0 \\ 0 & 0 & 0 \\ 0 & \overline{\Lambda}_s & B \end{pmatrix}\begin{pmatrix} x \\ \varepsilon \\ y \end{pmatrix} + \begin{pmatrix} f(x,y,\varepsilon) \\ 0 \\ g(x,y,\varepsilon) \end{pmatrix}. \tag{20.1.94}$$

The reader should recognize that (20.1.94) is "almost" in the standard normal form for application of the center manifold theory. The final step would be to introduce a linear transformation that block diagonalizes the linear part of (20.1.94) into a $(c+p)\times(c+p)$ matrix with eigenvalues all having zero real parts (and p identically zero) and an $(s\times s)$ matrix with all eigenvalues having negative real parts.

a) Carry out this final step and discuss applying the center manifold theorem to the resulting system. In particular, do the relevant theorems from Chapter 18 go through?

Before we work out some specific problems, let us first answer an example.

Consider the vector field

$$\begin{aligned}\dot{x} &= \varepsilon + x^2 + y^2,\\ \dot{y} &= -y + x^2,\end{aligned} \qquad (x,y,\varepsilon) \in \mathbb{R}^3. \tag{20.1.95}$$

It should be clear that $(x,y,\varepsilon) = (0,0,0)$ is a fixed point of (20.1.95). We want to study the orbit structure near this fixed point for ε small. Rewriting (20.1.95) in the form of (20.1.94) gives

$$\begin{pmatrix} \dot{x} \\ \dot{\varepsilon} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ \varepsilon \\ y \end{pmatrix} + \begin{pmatrix} x^2 + y^2 \\ 0 \\ x^2 \end{pmatrix}. \tag{20.1.96}$$

We seek a center manifold of the form

$$h(x,\varepsilon) = ax^2 + bx\varepsilon + c\varepsilon^2 + \mathcal{O}(3).$$

Utilizing the usual procedure for calculating the center manifold, we obtain

$$h(x,\varepsilon) = x^2 - 2x\varepsilon + 2\varepsilon^2 + \mathcal{O}(3).$$

The vector field restricted to the center manifold is then given by

$$\begin{aligned}\dot{x} &= \varepsilon + x^2 + \mathcal{O}(4),\\ \dot{\varepsilon} &= 0.\end{aligned}$$

Hence, a saddle-node bifurcation occurs at $\varepsilon = 0$.

Now consider the following vector fields

b) $\begin{aligned}\dot{x} &= \varepsilon + x^4 + y^2,\\ \dot{y} &= -y + x^3,\end{aligned} \qquad (x,y,\varepsilon) \in \mathbb{R}^3.$

c) $\begin{aligned}\dot{x} &= \varepsilon + x^2 - y^3,\\ \dot{y} &= \varepsilon - y + x^2.\end{aligned}$

d) $\begin{aligned}\dot{x} &= \varepsilon + \varepsilon x + x^2,\\ \dot{y} &= -y + x^2.\end{aligned}$

e) $\dot{x} = \varepsilon + \varepsilon x + x^2,$
$\dot{y} = \varepsilon - y + x^2.$

f) $\dot{x} = \varepsilon + \frac{1}{2}x + y + x^3,$
$\dot{y} = x + 2y - xy.$

g) $\dot{x} = 2\varepsilon + 2x + 2y,$
$\dot{y} = \varepsilon + x + y + y^2.$

h) $\dot{x} = \varepsilon - 2x + 2y - x^4,$
$\dot{y} = 2x - 2y.$

i) $\dot{x} = \varepsilon - 2x + y + z + yz,$
$\dot{y} = x - 2y + z + zx, \qquad (x, y, z, \varepsilon) \in \mathbb{R}^4.$
$\dot{z} = x + y - 2z + xy,$

For each vector field, construct the center manifold and discuss the dynamics near the origin for ε small. What types of bifurcations occur?

8. Consider the vector field

$$\begin{aligned} \dot{x} &= \varepsilon + x^2 + y^2, \\ \dot{y} &= -y + x^2, \end{aligned} \qquad (x, y, \varepsilon) \in \mathbb{R}^3.$$

For this vector field the tangent space approximation is sufficient for approximating the center manifold of the origin. Verify this statement and discuss conditions under which the tangent space approximation might work in general. Consider your ideas in the context of the following examples.

a) $\dot{x} = \varepsilon x + x^2 + y^2,$
$\dot{y} = -y + x^2.$

b) $\dot{x} = \varepsilon + x^2 + xy,$
$\dot{y} = -y + x^2.$

c) $\dot{x} = \varepsilon + y^2,$
$\dot{y} = -y + x^2.$

d) $\dot{x} = \varepsilon + xy + y^2,$
$\dot{y} = -y + x^2.$

9. Consider the block diagonal "normal form" of (20.1.84) to which we first transformed the vector field in order to apply the center manifold theory. Discuss why (or why not) this preliminary transformation was necessary. Is this preliminary transformation necessary for equations of the form of (20.1.94) in order to apply the center manifold theory? Work out several examples to support your views and illustrate the relevant points. (*Hint:* consider the coordinatization of the center manifold and how the invariance condition is manifested in those coordinates.)

10. Consider the following one-parameter family of two-dimensional $\mathbf{C}^r$ (r as large as necessary) vector fields

$$\dot{x} = f(x; \mu), \qquad (x, \mu) \in \mathbb{R}^2 \times \mathbb{R}^1,$$

where $f(0; 0) = 0$ and $D_x f(0, 0)$ has a zero eigenvalue and a negative eigenvalue. Suppose the vector field has the following symmetry

$$f(x, \mu) = -f(-x, \mu).$$

What can you then conclude concerning the symmetry of the vector field restricted to the center manifold for x and μ small? Can the vector field undergo a saddle-node bifurcation at $(x, \mu) = (0, 0)$? Can the vector field undergo a saddle-node bifurcation at other points $(x, \mu) \in \mathbb{R}^2 \times \mathbb{R}^1$?

20.2 A Pure Imaginary Pair of Eigenvalues: The Poincare-Andronov-Hopf Bifurcation

We now turn to the next most simple way that a fixed point can be nonhyperbolic; namely, that the matrix associated with the vector field linearized about the fixed point has a pair of purely imaginary eigenvalues, with the remaining eigenvalues having nonzero real parts. Let us be more precise.

Recall (20.0.1), which we restate here;

$$\dot{y} = g(y, \lambda), \qquad y \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^p, \tag{20.2.1}$$

where g is $\mathbf{C}^r$ ($r \geq 5$) on some sufficiently large open set containing the fixed point of interest. The fixed point is denoted by $(y, \lambda) = (y_0, \lambda_0)$, i.e.,

$$0 = g(y_0, \lambda_0). \tag{20.2.2}$$

We are interested in how the orbit structure near y_0 changes as λ is varied. In this situation the first thing to examine is the linearization of the vector field about the fixed point, which is given by

$$\dot{\xi} = D_y g(y_0, \lambda_0)\xi, \qquad \xi \in \mathbb{R}^n. \tag{20.2.3}$$

Suppose that $D_y g(y_0, \lambda_0)$ has two purely imaginary eigenvalues with the remaining $n - 2$ eigenvalues having nonzero real parts. We know (cf. the remarks at the beginning of this chapter) that since the fixed point is not hyperbolic, the orbit structure of the linearized vector field near $(y, \lambda) = (y_0, \lambda_0)$ may reveal little (and, possibly, even incorrect) information concerning the nature of the orbit structure of the nonlinear vector field (20.2.1) near $(y, \lambda) = (y_0, \lambda_0)$.

Fortunately, we have a systematic procedure for analyzing this problem. By the center manifold theorem, we know that the orbit structure near $(y, \lambda) = (y_0, \lambda_0)$ is determined by the vector field (20.2.1) restricted to the center manifold. This restriction gives us a p-parameter family of vector fields on a two-dimensional center manifold. For now we will assume that we are dealing with a single, scalar parameter, i.e., $p = 1$. If there is more than one parameter in the problem, we will consider all but one of them as fixed.

On the center manifold the vector field (20.2.1) has the following form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \text{Re } \lambda(\mu) & -\text{Im } \lambda(\mu) \\ \text{Im } \lambda(\mu) & \text{Re } \lambda(\mu) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x, y, \mu) \\ f^2(x, y, \mu) \end{pmatrix},$$
$$(x, y, \mu) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1, \tag{20.2.4}$$

where f^1 and f^2 are nonlinear in x and y and $\lambda(\mu)$, $\overline{\lambda(\mu)}$ are the eigenvalues of the vector field linearized about the fixed point at the origin.

Equation (20.2.4) was first discussed in Section 19.2. The reader should recall that in performing the center manifold reduction to obtain (20.2.4), several preliminary steps were first implemented. Namely, first we transformed the fixed point to the origin and, then, if necessary, performed a linear transformation of the coordinates so that the vector field (20.2.1) was in the form of (20.2.4). We further remark that the eigenvalue, denoted $\lambda(\mu)$, should not be confused with the general vector of parameters in (20.2.1), denoted $\lambda \in \mathbb{R}^p$, which we subsequently restricted to a scalar and labeled μ. We will henceforth denote

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu), \tag{20.2.5}$$

and note that by our assumptions we have

$$\begin{aligned} \alpha(0) &= 0, \\ \omega(0) &\neq 0. \end{aligned} \tag{20.2.6}$$

The next step is to transform (20.2.4) into normal form. This was done in Section 19.2. The normal form was found to be

$$\begin{aligned} \dot{x} &= \alpha(\mu)x - \omega(\mu)y + (a(\mu)x - b(\mu)y)(x^2+y^2) + \mathcal{O}(|x|^5, |y|^5), \\ \dot{y} &= \omega(\mu)x + \alpha(\mu)y + (b(\mu)x + a(\mu)y)(x^2+y^2) + \mathcal{O}(|x|^5, |y|^5). \end{aligned} \tag{20.2.7}$$

We will find it more convenient to work with (20.2.7) in polar coordinates. In polar coordinates (20.2.7) is given by

$$\begin{aligned} \dot{r} &= \alpha(\mu)r + a(\mu)r^3 + \mathcal{O}(r^5), \\ \dot{\theta} &= \omega(\mu) + b(\mu)r^2 + \mathcal{O}(r^4). \end{aligned} \tag{20.2.8}$$

Because we are interested in the dynamics near $\mu = 0$, it is natural to Taylor expand the coefficients in (20.2.8) about $\mu = 0$. Equation (20.2.8) thus becomes

$$\begin{aligned} \dot{r} &= \alpha'(0)\mu r + a(0)r^3 + \mathcal{O}(\mu^2 r, \mu r^3, r^5), \\ \dot{\theta} &= \omega(0) + \omega'(0)\mu + b(0)r^2 + \mathcal{O}(\mu^2, \mu r^2, r^4), \end{aligned} \tag{20.2.9}$$

where " $'$ " denotes differentiation with respect to μ and we have used the fact that $\alpha(0) = 0$.

Our goal is to understand the dynamics of (20.2.9) for r small and μ small. This will be accomplished in two steps.

Step 1. Neglect the higher order terms of (20.2.9) and study the resulting "truncated" normal form.

Step 2. Show that the dynamics exhibited by the truncated normal form are qualitatively unchanged when one considers the influence of the previously neglected higher order terms.

Step 1. Neglecting the higher order terms in (20.2.9) gives

$$\begin{aligned} \dot{r} &= d\mu r + ar^3, \\ \dot{\theta} &= \omega + c\mu + br^2, \end{aligned} \tag{20.2.10}$$

where, for ease of notation, we define

$$\begin{aligned} \alpha'(0) &\equiv d, \\ a(0) &\equiv a, \\ \omega(0) &\equiv \omega, \\ \omega'(0) &\equiv c, \\ b(0) &\equiv b. \end{aligned} \tag{20.2.11}$$

In analyzing the dynamics of vector fields we have always started with the simplest situation; namely, we have found the fixed points and studied the nature of their stability. In regard to (20.2.10), however, we proceed slightly differently because of the nature of the coordinate system. To be precise, values of $r > 0$ and μ for which $\dot{r} = 0$, but $\dot{\theta} \neq 0$, correspond to periodic orbits of (20.2.10). We highlight this in the following lemma.

Lemma 20.2.1 *For $-\infty < \frac{\mu d}{a} < 0$ and μ sufficiently small*

$$(r(t), \theta(t)) = \left(\sqrt{\frac{-\mu d}{a}}, \left[\omega + \left(c - \frac{bd}{a} \right) \mu \right] t + \theta_0 \right) \tag{20.2.12}$$

is a periodic orbit for (20.2.10).

Proof: In order to interpret (20.2.12) as a periodic orbit, we need only to insure that $\dot{\theta}$ is not zero. Since ω is a constant independent of μ, this immediately follows by taking μ sufficiently small. □

We address the question of stability in the following lemma.

Lemma 20.2.2 *The periodic orbit is*

i) *asymptotically stable for $a < 0$;*

ii) *unstable for $a > 0$.*

Proof: The way to prove this lemma is to construct a one-dimensional Poincaré map along the lines of Chapter 10 (and in particular, Example 10.1.1), from which the results of this lemma follow. □

We note that since we must have $r > 0$, (20.2.12) is the only periodic orbit possible for (20.2.10). Hence, for $\mu \neq 0$, (20.2.10) possesses a unique periodic orbit having amplitude $\mathcal{O}(\sqrt{\mu})$. Concerning the details of stability of the periodic orbit and whether it exists for $\mu > 0$ or $\mu < 0$, from (20.2.12) it is easy to see that there are four possibilities:

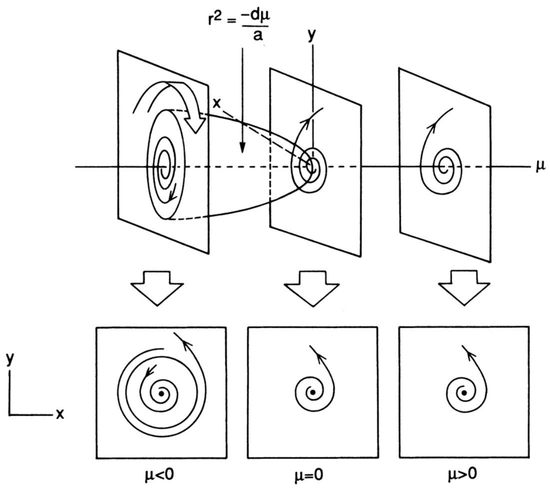

FIGURE 20.2.1. $d > 0$, $a > 0$.

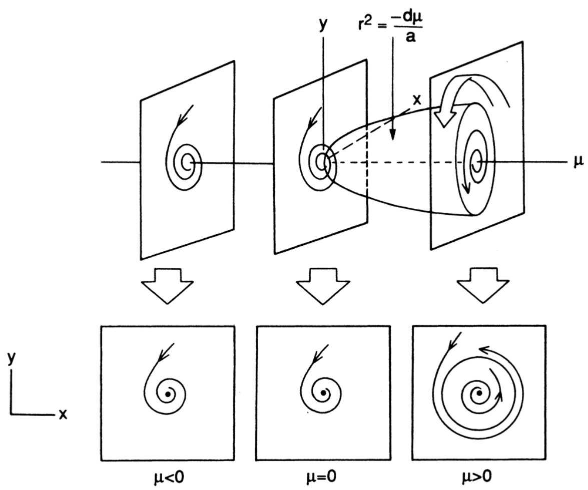

FIGURE 20.2.2. $d > 0$, $a < 0$.

1. $d > 0$, $a > 0$;
2. $d > 0$, $a < 0$;
3. $d < 0$, $a > 0$;

4. $d < 0$, $a < 0$.

We will examine each case individually; however, we note that in all cases the origin is a fixed point which is

$$\text{stable at } \mu = 0 \quad \text{for } a < 0,$$

$$\text{unstable at } \mu = 0 \quad \text{for } a > 0.$$

Case 1: $d > 0$, $a > 0$. In this case the origin is an unstable fixed point for $\mu > 0$ and an asymptotically stable fixed point for $\mu < 0$, with an unstable periodic orbit for $\mu < 0$ (note: the reader should realize that if the origin is stable for $\mu < 0$, then the periodic orbit should be unstable); see Figure 20.2.1.

Case 2: $d > 0$, $a < 0$. In this case the origin is an asymptotically stable fixed point for $\mu < 0$ and an unstable fixed point for $\mu > 0$, with an asymptotically stable periodic orbit for $\mu > 0$; see Figure 20.2.2.

Case 3: $d < 0$, $a > 0$. In this case the origin is an unstable fixed point for $\mu < 0$ and an asymptotically stable fixed point for $\mu > 0$, with an unstable

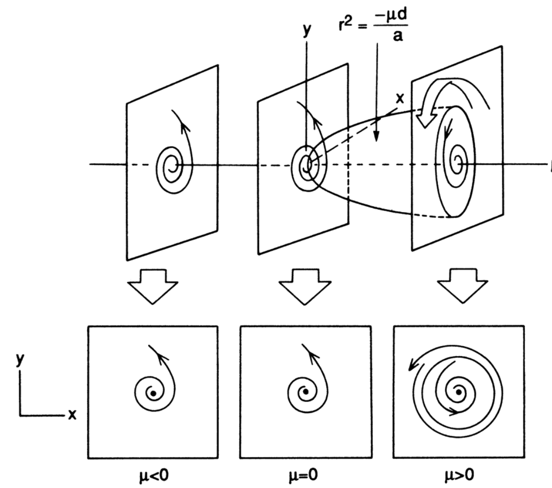

FIGURE 20.2.3. $d < 0$, $a > 0$.

periodic orbit for $\mu > 0$; see Figure 20.2.3.

Case 4: $d < 0$, $a < 0$. In this case the origin is an asymptotically stable fixed point for $\mu < 0$ and an unstable fixed point for $\mu > 0$, with an asymptotically stable periodic orbit for $\mu < 0$; see Figure 20.2.4.

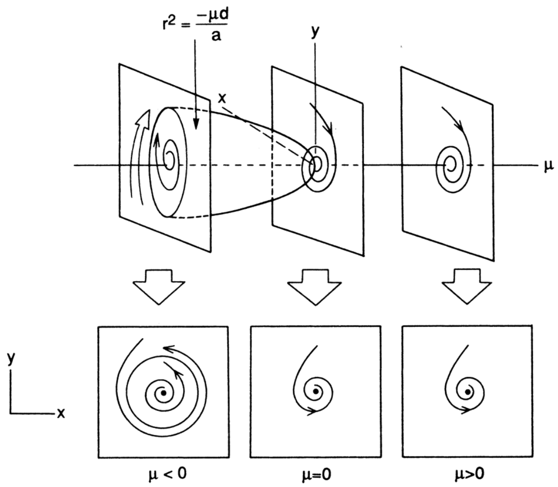

FIGURE 20.2.4. $d < 0$, $a < 0$.

From these four cases we can make the following general remarks.

Remark 1. For $a < 0$ it is possible for the periodic orbit to exist for either $\mu > 0$ (Case 2) or $\mu < 0$ (Case 4); however, in each case the periodic orbit is asymptotically stable. Similarly, for $a > 0$ it is possible for the periodic orbit to exist for either $\mu > 0$ (Case 3) or $\mu < 0$ (Case 1); however, in each case the periodic orbit is unstable. Thus, the number a tells us whether the bifurcating periodic orbit is stable ($a < 0$) or unstable ($a > 0$). The case $a < 0$ is referred to as a *supercritical* bifurcation, and the case $a > 0$ is referred to as a *subcritical* bifurcation.

Remark 2. Recall that

$$d = \frac{d}{d\mu}(\text{Re}\lambda(\mu))\bigg|_{\mu=0}.$$

Hence, for $d > 0$, the eigenvalues cross from the left half-plane to the right half-plane as μ increases and, for $d < 0$, the eigenvalues cross from the right half-plane to the left half-plane as μ increases. For $d > 0$, it follows that the origin is asymptotically stable for $\mu < 0$ and unstable for $\mu > 0$. Similarly, for $d < 0$, the origin is unstable for $\mu < 0$ and asymptotically stable for $\mu > 0$.

Step 2. At this point we have a fairly complete analysis of the orbit structure of the truncated normal form near $(r, \mu) = (0, 0)$. We now must consider Step 2 in our analysis of the normal form (20.2.9); namely, are the dynamics that we have found in the truncated normal form changed when the effects

of the neglected higher order term are considered? Fortunately, the answer to this question is no and is the content of the following theorem.

Theorem 20.2.3 (Poincaré-Andronov-Hopf Bifurcation) *Consider the full normal form (20.2.9). Then, for μ sufficiently small, Case 1, Case 2, Case 3, and Case 4 described above hold.*

Proof: We will outline a proof that uses the Poincaré-Bendixson Theorem. We begin by considering the truncated normal form (20.2.10) and the case $a < 0$, $d > 0$. In this case the periodic orbit is stable and exists for $\mu > 0$, and the r coordinate is given by

$$r = \sqrt{\frac{-d\mu}{a}}.$$

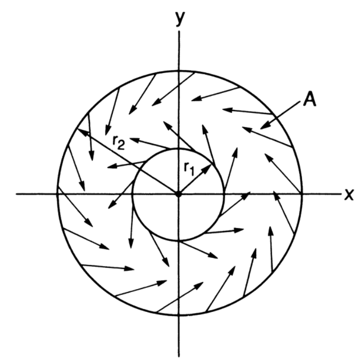

FIGURE 20.2.5.

We next choose $\mu > 0$ sufficiently small and consider the annulus in the plane, A, given by

$$A = \{(r, \theta) |\ r_1 \le r \le r_2\},$$

where r_1 and r_2 are chosen such that

$$0 < r_1 < \sqrt{\frac{-d\mu}{a}} < r_2.$$

By (20.2.10), it is easy to verify that on the boundary of A, the vector field given by the truncated normal form (20.2.10) is pointing *strictly* into the interior of A. Hence, A is a positive invariant region (cf. Definition 3.0.3, Chapter 3); see Figure 20.2.5.

It is also easy to verify that A contains no fixed points so, by the Poincaré-Bendixson theorem, A contains a stable periodic orbit. Of course we already knew this; our goal is to show that this situation still holds when the full normal form (20.2.9) is considered.

Now consider the full normal form (20.2.9). By taking μ and r sufficiently small, the $\mathcal{O}(\mu^2 r, \mu r^3, r^5)$ terms can be made much smaller than the rest of the normal form (i.e., the truncated normal form (20.2.10). Therefore, by taking r_1 and r_2 sufficiently small, A is still a positive invariant region containing no fixed points. Hence, by the Poincaré-Bendixson theorem, A contains a stable periodic orbit. The remaining three cases can be treated similarly; however, in the cases where $a > 0$, the time-reversed flow (i.e., letting $t \to -t$) must be considered. □

To apply this theorem to specific systems, we need to know d (which is easy) and a. In principle, a is relatively straightforward to calculate. We simply carefully keep track of the coefficients in the normal form transformation in terms of our original vector field. However, in practice, the algebraic manipulations are horrendous. The explicit calculation can be found in Hassard, Kazarinoff, and Wan [1980], Marsden and McCracken[1976], and Guckenheimer and Holmes [1983]; here we will just state the result.

At bifurcation (i.e., $\mu = 0$), (20.2.4) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x,y,0) \\ f^2(x,y,0) \end{pmatrix}, \tag{20.2.13}$$

and the coefficient $a(0) \equiv a$ is given by

$$\begin{aligned} a = {} & \frac{1}{16}\left[f^1_{xxx} + f^1_{xyy} + f^2_{xxy} + f^2_{yyy}\right] \\ & + \frac{1}{16\omega}\left[f^1_{xy}\left(f^1_{xx} + f^1_{yy}\right) - f^2_{xy}\left(f^2_{xx} + f^2_{yy}\right)\right. \\ & \left. - f^1_{xx} f^2_{xx} + f^1_{yy} f^2_{yy}\right], \end{aligned} \tag{20.2.14}$$

where all partial derivatives are evaluated at the bifurcation point, i.e., $(x, y, \mu) = (0, 0, 0)$.

We end this section with some historical remarks. Usually Theorem 20.2.3 goes by the name of the "Hopf bifurcation theorem." However, as has been pointed out repeatedly by V. Arnold [1983], this is inaccurate, since examples of this type of bifurcation can be found in the work of Poincaré [1892]. The first specific study and formulation of a theorem was due to Andronov [1929]. However, this is not to say that E. Hopf did not make an important contribution; while the work of Poincaré and Andronov was concerned with two-dimensional vector fields, the theorem due to E. Hopf [1942] is valid in n dimensions (note: this was before the discovery of the center manifold theorem). For these reasons we refer to Theorem 20.2.3 as the Poincaré-Andronov-Hopf bifurcation theorem.

20.2A Exercises

1. This exercise comes from Marsden and McCracken [1976]. Consider the following vector fields

a) $\dot{r} = -r(r-\mu)^2,\quad \dot{\theta} = 1, \qquad (r,\theta) \in \mathbb{R}^+ \times S^1.$

b) $\dot{r} = r(\mu - r^2)(2\mu - r^2)^2,\quad \dot{\theta} = 1.$

c) $\dot{r} = r(r+\mu)(r-\mu),\quad \dot{\theta} = 1.$

d) $\dot{r} = \mu r(r^2 - \mu),\quad \dot{\theta} = 1.$

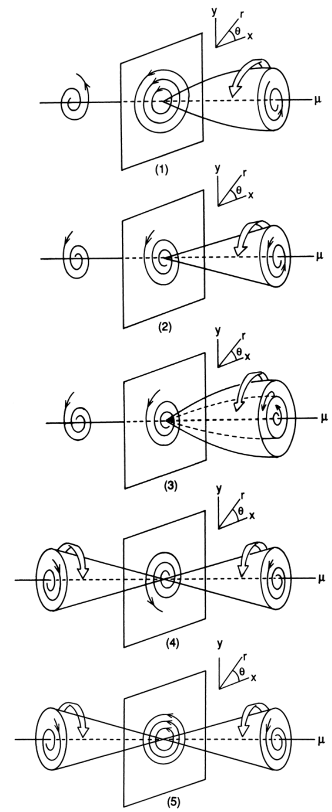

FIGURE 20.2.6.

e) $\dot{r} = -\mu^2 r(r+\mu)^2(r-\mu)^2,\quad \dot{\theta} = 1.$