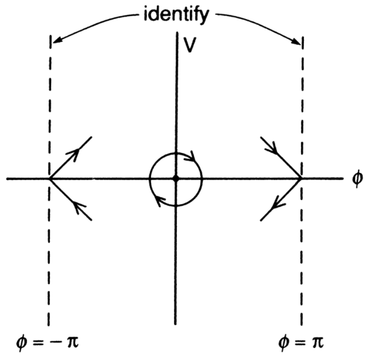

FIGURE 5.1.6. Fixed points of the pendulum.

Denoting the first integral of the unforced, undamped Duffing oscillator by h was meant to be suggestive. The unforced, undamped Duffing oscillator is actually a *Hamiltonian System*, i.e., there exists a function $h = h(x, y)$ such that the vector field is given by

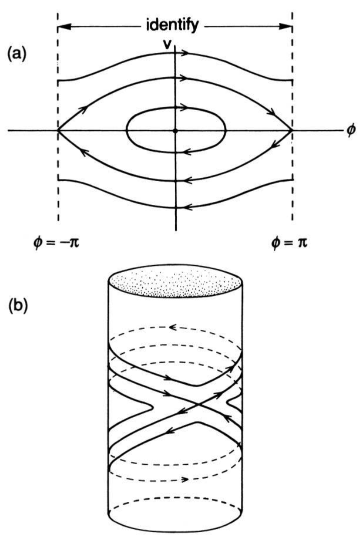

FIGURE 5.1.7. a) Orbits of the pendulum on $\mathbb{R}^2$ with $\phi = \pm\pi$ identified. b) Orbits of the pendulum on the cyliner.

$$\begin{aligned} \dot{x} &= \frac{\partial h}{\partial y}, \\ \dot{y} &= -\frac{\partial h}{\partial x} \end{aligned} \tag{5.1.7}$$

(we will study these in more detail later). Note that all the solutions lie on level curves of h which are topologically the same as S^1 (or T^1). This Hamiltonian system is an *integrable* Hamiltonian system and it has a characteristic of all n-degree-of-freedom integrable Hamiltonian systems in that its bounded motions lie on n-dimensional tori or homoclinic and heteroclinic orbits (see Arnold [1978] or Abraham and Marsden [1978]). (Note that all one-degree-of-freedom Hamiltonian systems are integrable.) More information on Hamiltonian vector fields can be found in Chapters 13 and 14.

Example 5.1.2 (The Pendulum). The equation of motion of a simple pendulum (again, all physical constants are scaled out) is given by

$$\ddot{\phi} + \sin\phi = 0 \tag{5.1.8}$$

or, written as a system,

$$\begin{aligned} \dot{\phi} &= v, \\ \dot{v} &= -\sin\phi, \end{aligned} \qquad (\phi, v) \in S^1 \times \mathbb{R}^1. \tag{5.1.9}$$

This equation has fixed points at $(0, 0)$, $(\pm\pi, 0)$, and simple calculations show that $(0, 0)$ is a *center* (i.e., the eigenvalues are purely imaginary) and $(\pm\pi, 0)$ are saddles, but since the phase space is the cylinder and not the plane, $(\pm\pi, 0)$ are really the same point (see Figure 5.1.6). (Think of the pendulum as a physical object and you will see that this is obvious.)

Now, just as in Example 5.1.1, the pendulum is a Hamiltonian system with a first integral given by

$$h = \frac{v^2}{2} - \cos\phi. \tag{5.1.10}$$

Again, as in Example 5.1.1, this fact allows the global phase portrait for the pendulum to be drawn, as shown in Figure 5.1.7a. Alternatively, by gluing the two lines $\phi = \pm\pi$ together, we obtain the orbits on the cylinder as shown in Figure 5.1.7b.

End of Example 5.1.2

5.2 Two Degree-of-Freedom Hamiltonian Systems and Geometry

We now give an example of a *two degree-of-freedom* Hamiltonian system that very concretely illustrates a number of more advanced concepts that we will discuss later on.