## 3.2 Stable, Unstable, and Center Manifolds for Fixed Points of Nonlinear, Autonomous Vector Fields

Recall that our original motivation for studying the linear system

$$\dot{y} = Ay, \qquad y \in \mathbb{R}^n, \tag{3.2.1}$$

where $A = Df(\bar{x})$, was to obtain information about the nature of solutions near the fixed point $x = \bar{x}$ of the nonlinear equation

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n. \tag{3.2.2}$$

The stable, unstable, and center manifold theorem provides an answer to this question; let us first transform (3.2.2) to a more convenient form.

We first transform the fixed point $x = \bar{x}$ of (3.2.2) to the origin via the translation $y = x - \bar{x}$. In this case (3.2.2) becomes

$$\dot{y} = f(\bar{x} + y), \qquad y \in \mathbb{R}^n. \tag{3.2.3}$$

Taylor expanding $f(\bar{x} + y)$ about $x = \bar{x}$ gives

$$\dot{y} = Df(\bar{x})y + R(y), \qquad y \in \mathbb{R}^n, \tag{3.2.4}$$

where $R(y) = \mathcal{O}(|y|^2)$ and we have used $f(\bar{x}) = 0$. From elementary linear algebra (see Hirsch and Smale [1974]) we can find a linear transformation $T$ which transforms the linear equation (3.2.1) into block diagonal form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A_s & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_c \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{3.2.5}$$

where $T^{-1}y \equiv (u, v, w) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c$, $s + u + c = n$, $A_s$ is an $s \times s$ matrix having eigenvalues with negative real part, $A_u$ is an $u \times u$ matrix having eigenvalues with positive real part, and $A_c$ is an $c \times c$ matrix having eigenvalues with zero real part (note: we point out the (hopefully) obvious fact that the "0" in (3.2.5) are not scalar zero's but rather the appropriately sized block consisting of all zero's. This notation will be used throughout the book). Using this same linear transformation to transform the coordinates of the nonlinear vector field (3.2.4) gives the equation

$$\begin{aligned} \dot{u} &= A_s u + R_s(u, v, w), \\ \dot{v} &= A_u v + R_u(u, v, w), \\ \dot{w} &= A_c w + R_c(u, v, w), \end{aligned} \tag{3.2.6}$$

where $R_s(u, v, w)$, $R_u(u, v, w)$, and $R_c(u, v, w)$ are the first $s$, $u$, and $c$ components, respectively, of the vector $T^{-1}R(T^y)$.



Now consider the linear vector field (3.2.5). From our previous discussion (3.2.5) has an $s$-dimensional invariant stable manifold, a $u$-dimensional invariant unstable manifold, and a $c$-dimensional invariant center manifold all intersecting in the origin. The following theorem shows how this structure changes when the nonlinear vector field (3.2.6) is considered.

**Theorem 3.2.1 (Local Stable, Unstable, and Center Manifolds of Fixed Points)** *Suppose (3.2.6) is* $\mathbf{C}^r$, $r \geq 2$. *Then the fixed point* $(u, v, w) = 0$ *of (3.2.6) possesses a* $\mathbf{C}^r$ *s-dimensional local, invariant stable manifold,* $W^s_{loc}(0)$, *a* $\mathbf{C}^r$ *u-dimensional local, invariant unstable manifold,* $W^u_{loc}(0)$, *and a* $\mathbf{C}^r$ *c-dimensional local, invariant center manifold,* $W^c_{loc}(0)$, *all intersecting at* $(u, v, w) = 0$. *These manifolds are all tangent to the respective invariant subspaces of the linear vector field (3.2.5) at the origin and, hence, are locally representable as graphs. In particular, we have*

$$W^s_{loc}(0) = \big\{(u, v, w) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c \,|\, v = h^s_v(u), w = h^s_w(u);$$
$$Dh^s_v(0) = 0, Dh^s_w(0) = 0; |u| \textit{ sufficiently small}\big\}$$

$$W^u_{loc}(0) = \big\{(u, v, w) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c \,|\, u = h^u_u(v), w = h^u_w(v);$$
$$Dh^u_u(0) = 0, Dh^u_w(0) = 0; |v| \textit{ sufficiently small}\big\}$$

$$W^c_{loc}(0) = \big\{(u, v, w) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c \,|\, u = h^c_u(w), v = h^c_v(w);$$
$$Dh^c_u(0) = 0, Dh^c_v(0) = 0; |w| \textit{ sufficiently small}\big\}$$

*where* $h^s_v(u)$, $h^s_w(u)$, $h^u_u(v)$, $h^u_w(v)$, $h^c_u(w)$, *and* $h^c_v(w)$ *are* $\mathbf{C}^r$ *functions. Moreover, trajectories in* $W^s_{\mathrm{loc}}(0)$ *and* $W^u_{\mathrm{loc}}(0)$ *have the same asymptotic properties as trajectories in* $E^s$ *and* $E^u$, *respectively. Namely, trajectories of (3.2.6) with initial conditions in* $W^s_{\mathrm{loc}}(0)$ *(resp.,* $W^u_{\mathrm{loc}}(0)$*) approach the origin at an exponential rate asymptotically as* $t \to +\infty$ *(resp.,* $t \to -\infty$*).*

*Proof:* See Fenichel [1971], Hirsch, Pugh, and Shub [1977], or Wiggins [1994] for details as well as for some history and further references on invariant manifolds. □

Some remarks on this important theorem are now in order.

*Remark 1.* First some terminology. Very often one hears the terms "stable manifold," "unstable manifold," or "center manifold" used alone; however, alone they are not sufficient to describe the dynamical situation. Notice that Theorem 3.2.1 is entitled stable, unstable, and center manifolds *of fixed points*. The phrase "of fixed points" is the key: one must say the stable, unstable, or center manifold *of something* in order to make sense. The "somethings" studied thus far have been fixed points; however, more general invariant sets also have stable, unstable, and center manifolds. See Wiggins [1994] for a discussion.

*Remark 2.* The conditions $Dh^s_v(0) = 0$, $Dh^s_w(0) = 0$, etc., reflect that the nonlinear manifolds are tangent to the associated linear manifolds at the origin.

*Remark 3.* In the statement of the theorem the term *local*, invariant stable, unstable, or center manifold is used. This deserves further explanation. "Local" refers to the fact that the manifold is only defined in the neighborhood of the fixed point as a graph. Consequently, these manifolds have a boundary. They are therefore only *locally invariant* in the sense that trajectories that start on them may leave the local manifold, but *only* through crossing the boundary. Invariance is still manifested by the vector field being tangent to the manifolds, which we discuss further below.

*Remark 4.* Suppose the fixed point is hyperbolic, i.e., $E^c = \emptyset$. In this case an interpretation of the theorem is that trajectories of the nonlinear vector field in a sufficiently small neighborhood of the origin behave the same as trajectories of the associated linear vector field.

*Remark 5.* In general, the behavior of trajectories in in $W^c_{\text{loc}}(0)$ cannot be inferred from the behavior of trajectories in $E^c$.

*Remark 6. Uniqueness of Stable, Unstable, and Center Manifolds.* Typically the existence of these invariant manifolds are proved through a contraction mapping argument, where the invariant manifold turns out to be the unique fixed point of an appropriately constructed contraction map. From this construction the stable and unstable manifolds are unique. The center manifold is a bit more delicate. In that case, because of the nonhyperbolicity, a "cut-off" function is typically used in the construction of the appropriate contraction map. In this case the center manifold does depend upon the cut-off function. However, it can be shown that the center manifold is unique to all orders of its Taylor expansion. That is, center manifolds only differ by exponentially small functions of the distance from the fixed point. See Wan [1977], Sijbrand [1985] and Wiggins [1994].

### 3.2a Invariance of the Graph of a Function: Tangency of the Vector Field to the Graph

Suppose one has a general surface, or manifold and one wants to check if it is invariant with respect to the dynamics generated by a vector field. How can this be done?

Suppose the vector field is of the form

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y), \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m. \end{aligned}$$

Suppose that the surface in the phase space is represented by the graph of



a function

$$y = h(x),$$

This surface is invariant if the vector field is tangent to the surface. This tangency condition is expressed as follows

$$Dh(x)\dot{x} = \dot{y},$$

or,

$$Dh(x)f(x, h(x)) = g(x, h(x)). \tag{3.2.7}$$

Of course, one must take care that all the functions taking part in these expressions have common domains, and that the appropriate derivatives exist. It is also very important to appreciate the role that specific coordinate representations played in deriving this expression.

## 3.3 Maps

An identical theory can be developed for maps. We summarize the details below. Consider a $\mathbf{C}^r$ diffeomorphism

$$x \mapsto g(x), \qquad x \in \mathbb{R}^n. \tag{3.3.1}$$

Suppose (3.3.1) has a fixed point at $x = \bar{x}$ and we want to know the nature of orbits near this fixed point. Then it is natural to consider the associated linear map

$$y \mapsto Ay, \qquad y \in \mathbb{R}^n, \tag{3.3.2}$$

where $A = Dg(\bar{x})$. The linear map (3.3.2) has invariant manifolds given by

$$\begin{aligned} E^s &= \text{span}\{e_1, \cdots, e_s\}, \\ E^u &= \text{span}\{e_{s+1}, \cdots, e_{s+u}\}, \\ E^c &= \text{span}\{e_{s+u+1}, \cdots, e_{s+u+c}\}, \end{aligned}$$

where $s + u + c = n$ and $e_1, \cdots, e_s$ are the (generalized) eigenvectors of $A$ corresponding to the eigenvalues of $A$ having *modulus less than one*, $e_{s+1}, \cdots, e_{s+u}$ are the (generalized) eigenvectors of $A$ corresponding to the eigenvalues of $A$ having *modulus greater than one*, and $e_{s+u+1}, \cdots, e_{s+u+c}$ are the (generalized) eigenvectors of $A$ corresponding to the eigenvalues of $A$ having *modulus equal to one*. The reader should find it easy to prove this by putting $A$ in Jordan canonical form and noting that the orbit of the linear map (3.3.2) through the point $y_0 \in \mathbb{R}^n$ is given by

$$\{\cdots, A^{-n}y_0, \cdots, A^{-1}y_0, y_0, Ay_0, \cdots, A^n y_0, \cdots\}. \tag{3.3.3}$$

Now we address the question of how this structure goes over to the nonlinear map (3.3.1). In the case of maps Theorem 3.2.1 holds identically.