to put it more cynically, one needs to know the answer before asking the question. It might therefore seem that these ideas are of little use to the applied scientist; however, this is not exactly true, since the theorems describing structural stability and generic properties do give one a good idea of what to *expect*, although they cannot tell what is precisely happening in a specific system. Also, the reader should always ask him or herself whether or not the dynamics are stable and/or typical in some sense. Probably the best way of mathematically quantifying these two notions for the applied scientist has yet to be determined.

12.2 Transversality

Before leaving this section let us introduce the idea of *transversality*, which will play a central role in many of our geometrical arguments.

Transversality is a geometric notion which deals with the intersection of surfaces or manifolds. Let M and N be differentiable (at least $\mathbf{C}^1$) manifolds in $\mathbb{R}^n$.

Definition 12.2.1 (Transversality) *Let p be a point in $\mathbb{R}^n$; then M and N are said to be* transversal at p *if $p \notin M \cap N$; or, if $p \in M \cap N$, then $T_pM + T_pN = \mathbb{R}^n$, where T_pM and T_pN denote the tangent spaces of M and N, respectively, at the point p. M and N are said to be* transversal *if they are transversal at every point $p \in \mathbb{R}^n$; see Figure 12.2.1.*

Whether or not the intersection is transversal can be determined by knowing the dimension of the intersection of M and N. This can be seen as follows. Using the formula for the dimension of the intersection of two

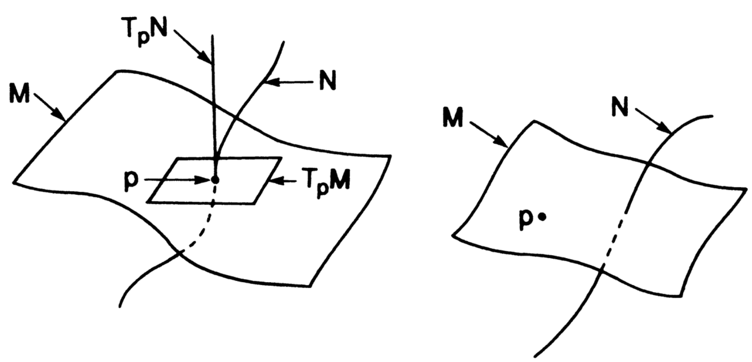

FIGURE 12.2.1. M and N transversal at p.

vector subspaces we have

$$\dim(T_pM + T_pN) = \dim T_pM + \dim T_pN - \dim(T_pM \cap T_pN). \quad (12.2.1)$$

From Definition 12.2.1, if M and N intersect transversely at p, then we have

$$n = \dim T_pM + \dim T_pN - \dim(T_pM \cap T_pN). \quad (12.2.2)$$

Since the dimensions of M and N are known, then knowing the dimension of their intersection allows us to determine whether or not the intersection is transversal.

Note that transversality of two manifolds at a point requires more than just the two manifolds geometrically piercing each other at the point. Consider the following example.

Example 12.2.1. Let M be the x axis in $\mathbb{R}^2$, and let N be the graph of the function $f(x) = x^3$; see Figure 12.2.2. Then M and N intersect at the origin in $\mathbb{R}^2$, but they are not transversal at the origin, since the tangent space of M is just the x axis and the tangent space of N is the span of the vector $(1, 0)$; thus, $T_{(0,0)}N = T_{(0,0)}M$ and, therefore, $T_{(0,0)}N + T_{(0,0)}M \neq \mathbb{R}^2$.

End of Example 12.2.1

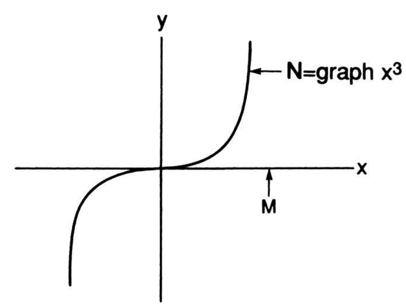

FIGURE 12.2.2. Nontransversal manifolds.

The most important characteristic of transversality is that it persists under sufficiently small perturbations. This fact will play a useful role in many of our geometric arguments; we remark that a term often used synonymously for transversal is *general position*, i.e., two or more manifolds which are transversal are said to be in general position.

Let us end this section by giving a few "dynamical" examples of transversality.

Example 12.2.2. Consider a hyperbolic fixed point of a $\mathbf{C}^r$, $r \geq 1$, vector field on $\mathbb{R}^n$. Suppose the matrix associated with the linearization of the vector