(b) $C^1[a,b]$, the space of real-valued functions that are continuous and that have continuous derivatives on the closed interval $[a,b]$;

(c) $C^2[a,b]$, the space of real-valued functions that are continuous and that have continuous first and second derivatives on the closed interval $[a,b]$;

(d) $D[a,b]$, the space of real-valued functions that are piecewise continuous on the closed interval $[a,b]$; and

(e) $D^1[a,b]$, the space of real-valued functions that are continuous and that have piecewise continuous derivatives on the closed interval $[a,b]$.

A piecewise continuous function can have a finite number of jump discontinuities in the interval $[a,b]$. The right-hand and left-hand limits of the function exist at the jump discontinuities. A function that is piecewise continuously differentiable is continuous but may have a finite number of corners.

We wish to find the *extremum* of a functional. Extremum is a word that was first introduced by Paul du Bois-Reymond (1879b). Du Bois-Reymond got tired of always having to say "maximum or minimum" and so he introduced a single term, extremum, to talk about both maxima and minima. The term stuck.

We will take our lead from (ordinary) calculus. We will look for a condition analogous to setting the first derivative equal to zero in calculus. The resulting *Euler–Lagrange equation* is quite important, so much so that we will derive this equation in three ways. We will begin with Euler's heuristic derivation (Euler, 1744) and then move on to Lagrange's 1755 derivation (the traditional approach). We will then consider Paul du Bois-Reymond's modification of Lagrange's derivation (du Bois-Reymond, 1879a).

2.2. Euler's approach

Leonhard Euler was the first person to systematize the study of variational problems. His 1744 opus, *A Method for Finding Curved Lines Enjoying Properties of Maximum or Minimum, or Solution of Isoperimetric Problems in the Broadest Accepted Sense*, is a compendium of

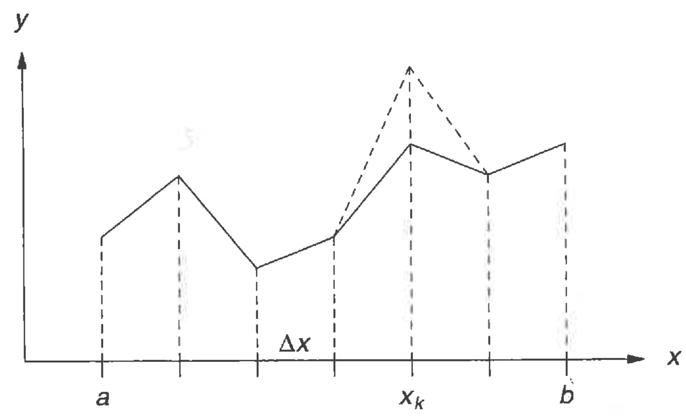

Figure 2.1. Polygonal curves

handling these problems. Euler dropped his method for Lagrange's more elegant "method of variations" after receiving Lagrange's (August 12, 1755) letter. Euler also named this subject the *calculus of variations* in Lagrange's honor.

Euler's essential idea was to first go from a variational problem to an n-dimensional problem and to then pass to the limit as $n \to \infty$. We will borrow from the modernized treatment of Euler's method found in Elsgolc (1961) and Gelfand and Fomin (1963). See Goldstine (1980) and Fraser (2003) for more on the original approach.

Let us divide the closed interval $[a,b]$ into $n+1$ equal subintervals (see Figure 2.1). We will assume that the subintervals are bounded by the points

$$x_0 \;=\; a,\, x_1, \ldots, x_n,\, x_{n+1} \;=\; b\,. \tag{2.3}$$

Each subinterval is of width

$$\Delta x \;=\; x_{i+1} - x_i \;=\; \frac{(b-a)}{n+1}\,. \tag{2.4}$$

We will also replace the smooth function $y(x)$ by the polygonal curve with vertices

Here, $y_i = y(x_i)$. We can now approximate the functional $J[y]$ by the sum

$$J(y_1, \ldots, y_n) \equiv \sum_{i=0}^{n} f\left(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\right) \Delta x, \qquad (2.6)$$

a function of n variables. (Remember that $y_0 = y_a$ and $y_{n+1} = y_b$ are fixed.)

What is the effect of raising or lowering one of the free y_i? To answer this question, let us choose one of the free y_i, y_k, and take the partial derivative with respect to y_k. Since y_k appears in only two terms in our sum, the partial derivative is just

$$\begin{aligned}\frac{\partial J}{\partial y_k} &= f_y\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right) \Delta x \qquad (2.7)\\ &+ f_{y'}\left(x_{k-1}, y_{k-1}, \frac{y_k - y_{k-1}}{\Delta x}\right)\\ &- f_{y'}\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right).\end{aligned}$$

To find an extremum, we would ordinarily set this partial derivative equal to zero for each k. We also, however, want to take the limit as $n \to \infty$. In this limit, $\Delta x \to 0$ and the right-hand side of equation (2.7) goes to zero. The equation $0 = 0$, while true, is, sadly, not very helpful. To obtain a nontrivial result, we must first divide by Δx,

$$\begin{aligned}\frac{1}{\Delta x}\frac{\partial J}{\partial y_k} &= f_y\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right) \qquad (2.8)\\ &- \frac{1}{\Delta x}\left[f_{y'}\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right) - f_{y'}\left(x_{k-1}, y_{k-1}, \frac{y_k - y_{k-1}}{\Delta x}\right)\right].\end{aligned}$$

As we now let $n \to \infty$ and $\Delta x \to 0$, equation (2.8) yields the *variational derivative*

$$\frac{\delta J}{\delta y} = f_y(x, y, y') - \frac{d}{dx} f_{y'}(x, y, y'). \qquad (2.9)$$

This variational derivative plays the same role for functionals that the partial derivative plays for multivariate functions. For a relative

leaving us with the *Euler–Lagrange equation*

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \qquad (2.10)$$

This condition must be modified if the minimizing curve lies on the boundary rather than in the interior of the region of interest. Moreover, the Euler–Lagrange equation is only a *necessary* condition, in the same sense that $f'(x) = 0$ is a necessary, but not a sufficient, condition in calculus.

I should, perhaps, add that the above discussion is misleading to the extent that the formal notion of a variational or functional derivative was not introduced until much later, by Vito Volterra (1887), in the early stages of the development of functional analysis. See the recommended reading at the end of this chapter for more information about variational derivatives.

Example 2.1 (Shortest curve in the plane).

Let's see what the Euler–Lagrange equation has to say about the shape of the shortest curve between two points, (a, y_a) and (b, y_b), in the plane. We clearly wish to minimize the arc-length functional

$$J[y] = \int_a^b ds = \int_a^b \sqrt{1 + y'^2}\, dx. \qquad (2.11)$$

The integrand,

$$f(x, y, y') = \sqrt{1 + y'^2}, \qquad (2.12)$$

does not depend on y and so the Euler–Lagrange equation reduces to

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0. \qquad (2.13)$$

Integrating once produces

$$\frac{y'}{\sqrt{1 + y'^2}} = \text{constant} \qquad (2.14)$$

and we quickly conclude that