Physics in Spacetime

Lecture notes

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Chapter 1

Space and Time

This course deals with the special theory of relativity laid out by Einstein more than 100 years ago. The traditional way of introducing special relativity is to derive it in much the same way that Einstein did from the two postulates of

1. The principle of relativity
2. The constancy of the speed of light

From these assumptions the notion of a spacetime with (inertial) observers being connected by Lorentz transformations follows. This is a natural way to proceed if one starts from a knowledge of classical mechanics and of Maxwell's equations of electrodynamics. However it is not the best way to understand the geometrical aspects of spacetime which is part of the reason it took Einstein another ten years to come up with the general theory of relativity, where the geometry of spacetime is the key player.

In this course we will follow a different route[1] which leads more directly to a geometric picture. Rather than follow the traditional route starting with the principles described above we will derive the same physics from something called

- The principle of maximum proper time

Before we explain what this principle says we must first decide what we mean by the concepts of space and time in physics.

1.1 What is space and time?

The notions of Space and Time are central to physics. We typically want to answer questions of the sort

Given some configuration, say of particles with given momenta in space, at some initial time t_i what will the configuration look like at some later time t_f.

[1]The approach to relativity taken here is inspired by lecture notes and a book by B. Laurent (*Introduction To Spacetime*, World Scientific, 1994).

For such questions to make sense we must have a precise way of defining what we mean by time and also what we mean by a particle's position in space. So what are space and time? Rather than get into a philosophical discussion of the nature of space and time, a more useful approach when faced with such deep questions questions in physics is to try to replace it by a different, more down to earth, question. After all, in physics we can only talk about things that can be measured and therefore a better question to ask is

> *How do we* **measure** *space and time?*

In fact answering this question lets us define *operationally*, i.e. by measurements, what space and time are. How do we measure distances in space? The most basic way is to take a reference object, say a 1 meter ruler, and use it to measure the distance between two points. Of course a good ruler should not bend or change its length with temperature etc. so we will assume it is always possible to find a sufficiently good ruler (or equivalent) that we can always measure lengths to the precision we need. How do we measure time? To measure time we need a clock. It does not have to be what we normally think of as a clock, it can be any physical system which has a known time dependence, e.g. a pendulum or atoms in an excited state with known half-life. Again we will assume that there exist such clocks with good enough precision for the time measurements we need to perform.

We have established that each person, or more generally, each *observer* measures time with his clock and spatial distances with his reference ruler. We will assume that these are small enough that the observer can carry them with him, i.e. they will be in the same state of motion as the observer and experience the same forces he experiences. How are the measurements of two *different* observers related? Newton assumed that there was an absolute notion of time so that all observers clocks tick at the same rate. We now know that this assumption was wrong. Taking two synchronized atomic clocks, putting one on a plane circling the earth and leaving one on the ground, one finds when comparing them at the end that they differ (by a few hundred nanoseconds). This observation is clearly inconsistent with the Newtonian idea of an absolute time.

1.2 The principle of maximum proper time

Experiments show that time runs differently for different observers. We must assign each observer her own time, her *proper time*, which is the time measured by her clock. We now come to the principle that will allow us to compare the measurements of different observers

> *If two observers are separated and then meet the one that does not experience any acceleration (inertial observer) always measures the* **longest** *time.*

This is the *principle of maximum proper time*, that proper time is maximized for inertial, or unaccelerated, observers. There is plenty of experimental evidence to support this principle, such as the experiments with atomic clocks on planes, or the operation of GPS satellites. We will take this principle as the starting point from which we will derive the theory of special relativity.

We are familiar with the fact that to specify the position of an object in our three dimensions we need to give three numbers – the coordinates with respect to some coordinate system. For positions on the earth we might for example give the longitude, the latitude and the height above sea level. To specify an *event* – something happening at a certain place at a certain time – we must give one more number namely the time on a clock associated to our coordinate system. In our example this could be the time GMT.

1.3 Spacetime

We have seen that we must allow each observer to measure distances and times according to her own coordinate system defined by her standard ruler and clock. Each observer will therefore associate to a given event four numbers (t, x, y, z) – the *spacetime* coordinates relative to her coordinate system. Note that we are defining an event here in an idealized way as a single point in spacetime, i.e. something that happens at a point in space at a single instant of time. The set of all events make up the four-dimensional spacetime. Note that each observer will in general assign different coordinates to the same event because they are using different coordinate systems, there is no special preferred coordinate system in spacetime. One of our first tasks will be to understand how to relate the observations of different observers.

The trajectory of an object traces out a path in spacetime – a *worldline*, or, if the object is not point-like, a tube-like worldvolume. In ordinary Euclidean space we are familiar with that fact the there is a shortest path between any two points. This path is called a straight line. It is the path a particle follows if it is not acted upon by any external forces. Similarly we will assume that there is precisely one straight worldline connecting any two events in spacetime and that any object not acted on by external forces, i.e. not experiencing any acceleration, follows such a straight line. To a worldline connecting two events in spacetime we can associate a number – the proper time along that worldline. Recall that this is the time an observer traveling along the worldline measures on her clock between the two events. The principle of maximum proper time says that a straight line in spacetime corresponds to the *longest* proper time. Therefore the analog of shortest length in Euclidean space is longest proper time in spacetime and a clock can be thought of as measuring distance in spacetime. When we draw spacetime diagrams we will draw the worldlines of unaccelerated objects as straight lines. Curved lines will correspond to worldlines of accelerated objects (Figure 1.1).

1.4 Parallel worldlines

An important notion in Euclidean geometry is the notion of two lines being parallel. In spacetime we can similarly have the notion of two observers being on the same course. How can two observers, e.g. two spaceships traveling in outer space, determine whether they are on the same course? One way to do so uses a construction from Euclidean space

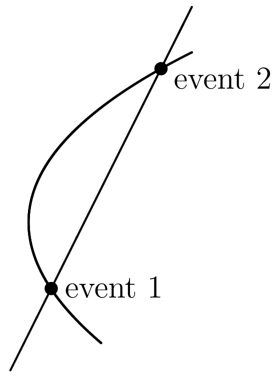


Figure 1.1: Spacetime diagram showing and accelerated (curved line) and an unaccelerated (straight line) observer meeting at the spacetime points marked event 1 and event 2.

adapted to spacetime. Imagine that the two observers each send out a probe fitted with a clock, which travels freely until it is picked up by the other observer at some later time (Figure 1.2). If the two probes happen to meet halfway, i.e. after half of the proper time (from being emitted to being picked up) has elapsed on each clock, then we will say that the observers are on the same course, or that their worldlines are *parallel*. From the figure we see that this also implies that the lines AB and CD (not drawn) are parallel.

Note that to carry out the experiment we really need to send clocks that also have a recording device that records the time they were sent and the time they met. We would also need to do the experiment several times to get the clocks to meet halfway. It is also important to note that we never refer to space or time separately, only to the full spacetime or the proper time measured by a particular clock. This is in sharp distinction to how we would describe such an experiment in Newtonian physics.

Just like in Euclidean space the line AC in Figure 1.2 defines a vector which we can draw as an arrow starting at A and ending at C. The length of the vector is given by the proper time elapsed from A to C. The construction in the figure gives us a way to *parallel transport* vectors. The vector AC can be parallel transported to the vector BD. Taking the two worldlines to approach each other we obtain the special case of parallel transport of the vector along the worldline. A general parallel transport is obtained by a sequence of such "elementary" parallel transports.

We will now make a very important assumption. We will assume that the vector one obtains by such a sequence of elementary parallel transports from a point A to a point A' in spacetime does not depend on how one chooses the sequence of parallel transports, i.e. it does not depend on the path taken. This assumption is not true close to gravitating bodies and in that case one must use *general relativity*. The assumption is true if gravity

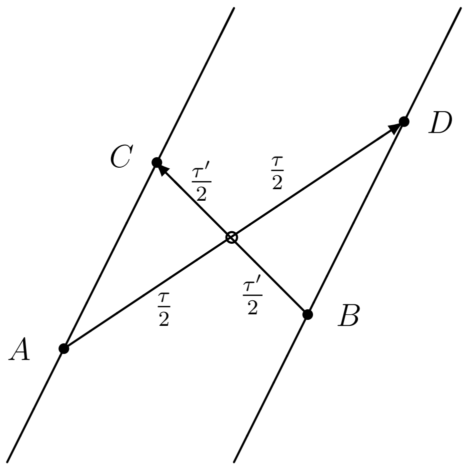


Figure 1.2: The worldlines of two observers are parallel if they can send out clocks, at A and B, that meet halfway before encountering the other observer at D and C.

is very weak, which is the case we consider in this course, and we say that we work with *special relativity.* In fact the change of a vector under parallel transport is related to the curvature of a space so in special relativity we are working in a *flat* spacetime.

Chapter 2

Spacetime vectors

In the last chapter we defined a vector on the straight worldline of an observer as an arrow from one event to another on the worldline with length given by the proper time elapsed between the two events. The notion of vectors is familiar from Euclidean space and we will use the same notation $\bar{v}$ for a vector in spacetime. Such vectors are often called *four-vectors* since spacetime is four-dimensional. Just as any point in Euclidean space $\mathbb{R}^3$ can be associated with a vector going from the origin to that point, any event in spacetime can be associated to a spacetime vector from an origin (which we can choose as we please) to the point in question. We have also seen that we can move vectors around using parallel transport. Two vectors related by parallel transport will be considered the same vector.

Spacetime vectors obey the usual axioms familiar from Euclidean space

- Associativity of addition: $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$
- Commutativity of addition: $\bar{u} + \bar{v} = \bar{v} + \bar{u}$
- Identity element of addition: $\bar{v} + \bar{0} = \bar{v}$
- Inverse elements of addition: Given $\bar{v}$ the exists a vector $-\bar{v}$ such that $\bar{v} + (-\bar{v}) = 0$
- Compatibility of scalar multiplication: $a(b\bar{v}) = (ab)\bar{v}$
- Identity element of scalar multiplication: $1\bar{v} = \bar{v}$
- Distributivity of scalar multiplication with respect to vector addition: $a(\bar{u} + \bar{v}) = a\bar{u} + a\bar{v}$
- Distributivity of scalar multiplication with respect to addition: $(a + b)\bar{v} = a\bar{v} + b\bar{v}$

Addition of spacetime vectors can be done by the geometric construction familiar from Euclidean space, Figure 2.1. Recall that a *basis* for a vector space is a set of linearly independent vectors $\bar{v}_i$ with $i = 1, \ldots, n$ which span the space, so that any vector is expressed uniquely as a linear combination

$$a_1\bar{v}_1 + a_2\bar{v}_2 + \ldots + a_n\bar{v}_n \,, \tag{2.1}$$

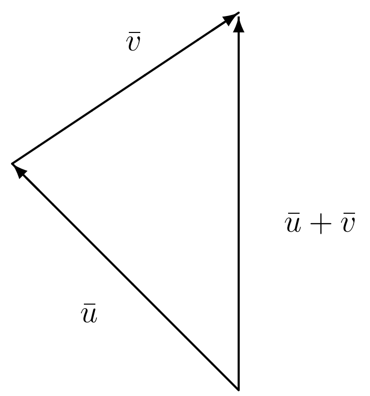


Figure 2.1: Addition of vectors $\bar{u}$ and $\bar{v}$ to produce a third vector $\bar{u}+\bar{v}$.

for some numbers a_i with $i = 1, \ldots, n$. The vector can be denoted in this basis as $(a_1, a_2, \ldots, a_n)$ and n is called the dimension of the vector space. A basis of spacetime vectors therefore consists of four spacetime vectors.

2.1 Inner product

An important operation in linear algebra is the *inner product* (also often called scalar product) between two vectors. Given two vectors their inner product is a number. We will denote the inner product with a dot, e.g. $\bar{u} \cdot \bar{v}$ denotes the inner product between vectors $\bar{u}$ and $\bar{v}$. The inner product satisfies the following axioms

- Symmetry: $\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$
- Linearity: $(a\bar{u}) \cdot \bar{v} = a(\bar{u} \cdot \bar{v})$ and $(\bar{u}+\bar{v}) \cdot \bar{w} = \bar{u} \cdot \bar{w} + \bar{v} \cdot \bar{w}$
- Non-degenerate: If $\bar{u} \cdot \bar{v} = 0$ for all vectors $\bar{v}$ then $\bar{u} = \bar{0}$

Often the inner product is required to be positive definite, so that $\bar{u}^2 = \bar{u} \cdot \bar{u} \geq 0$. In Euclidean space we are used to identifying $\bar{u}^2$ with the length-squared of a vector, which is obviously positive. We will see below that this is not true for spacetime vectors, in fact, for a vector that is the straight worldline of an object from point A to point B we will take

$$\bar{u}^2 = -\tau^2 \,, \tag{2.2}$$

where τ is the proper time along the worldline from A to B. The minus sign seems strange at this point but we will see shortly that it is needed if we want lengths in space to have positive square. All the differences between Euclidean space and spacetime are due to the fact that the inner product of spacetime vectors are not positive definite. This is what makes it possible to separate a time-direction from the spatial directions.

The assumption the $\bar{u}^2 = -\tau^2$ for vectors corresponding to a piece of a straight worldline determines also $\bar{u} \cdot \bar{v}$ for some other straight worldline vector $\bar{v}$. To see this consider three such worldline vectors satisfying

$$a\bar{u} = \bar{v} + \bar{w}\,, \tag{2.3}$$

for some number a. Writing this as $\bar{w} = a\bar{u} - \bar{v}$ and squaring both sides we get

$$\bar{w}^2 = (a\bar{u})^2 - 2(a\bar{u}) \cdot \bar{v} + \bar{v}^2 = a^2\bar{u}^2 - 2a\bar{u} \cdot \bar{v} + \bar{v}^2\,. \tag{2.4}$$

Rearranging this we have

$$\bar{u} \cdot \bar{v} = \frac{1}{2a}\left(a^2\bar{u}^2 + \bar{v}^2 - \bar{w}^2\right) \tag{2.5}$$

and the right-hand-side is clearly expressed only in terms of the proper times corresponding to the length-squared of $\bar{u}, \bar{v}$ and $\bar{w}$.

It is important to understand that the assumption that there exists an inner product for spacetime vectors satisfying the above axioms is not a trivial statement. The mere existence of this inner product has physical consequences. To see this consider the identity

$$(\bar{u} + \bar{v})^2 + (\bar{u} - \bar{v})^2 = 2\bar{u}^2 + 2\bar{v}^2\,. \tag{2.6}$$

Consider the case that all these vectors are part of straight worldlines of observers. Since the expression contains only squares it only involves proper times measured by these observers. With four spaceships traveling along these straight worldlines it is then possible to arrange an experiment (Figure 2.2) test whether the proper times they measure satisfy the above identity which amounts to $\tau_1^2 + \tau_2^2 = 2\tau_3^2 + 2\tau_4^2$.

2.2 Timelike, Spacelike and Null

Let $\bar{u}$, $\bar{v}$ be two straight worldline vectors. Then

$$\bar{u}^2 = -\tau_u^2\,, \qquad \bar{v}^2 = -\tau_v^2\,. \tag{2.7}$$

We now define a new vector which is a linear combination of $\bar{u}$ and $\bar{v}$

$$\bar{y} = a\bar{u} + b\bar{v}\,. \tag{2.8}$$

Its inner product with $\bar{u}$ is

$$\bar{u} \cdot \bar{y} = a\bar{u}^2 + b\bar{u} \cdot \bar{v} \tag{2.9}$$

and picking $a = -b\frac{\bar{u}\cdot\bar{v}}{\bar{u}^2}$ we find

$$\bar{u} \cdot \bar{y} = 0\,. \tag{2.10}$$

We say that $\bar{y}$ is *orthogonal* to $\bar{u}$.

Now consider the following situation. Two spaceships part and then meet again:

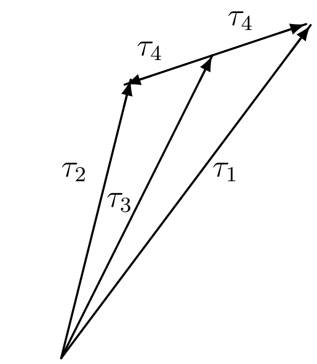
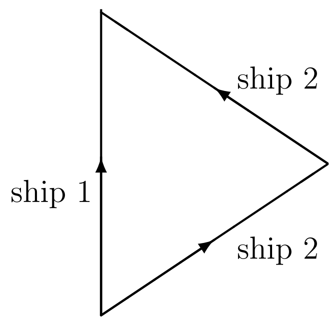


Figure 2.2: Experiment involving four spaceships to test equation (2.6).



Spaceship 1 is unaccelerated throughout the whole duration of its journey, while spaceship two flies unaccelerated for awhile then accelerates hard for a short time to reverse its direction of motion and then again floats freely until it meets spaceship 1 again.

We take the spacetimes vectors corresponding to this situation as in Figure 2.3 with

$$\bar{v}_1 = \frac{1}{2}\bar{u} + \epsilon\bar{y}\,, \qquad \bar{v}_2 = \frac{1}{2}\bar{u} - \epsilon\bar{y}\,, \tag{2.11}$$

where $\bar{y}$ is the vector introduced above which is orthogonal to $\bar{u}$ and ϵ is a small number. Note that $\bar{v}_1 + \bar{v}_2 = \bar{u}$ so that the spaceships indeed meet at the end. The proper time for

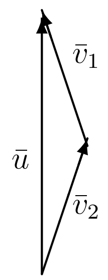


Figure 2.3: Spacetime vectors corresponding to two spaceships parting and meeting again.

the journey of ship 1 is

$$\tau_1 = \sqrt{-\bar{u}^2}\,. \tag{2.12}$$

The proper time for the journey of ship 2 is the sum of the proper time for the two segments of the journey

$$\tau_2 = \sqrt{-\bar{v}_1^2} + \sqrt{-\bar{v}_2^2}\,. \tag{2.13}$$

Since $\bar{u} \cdot \bar{y} = 0$ we have $\bar{v}_1^2 = \frac{1}{4}\bar{u}^2 + \epsilon^2\bar{y}^2 = \bar{v}_2^2$ so that finally

$$\tau_2 = \sqrt{-\bar{u}^2 - 4\epsilon^2\bar{y}^2} = \sqrt{-\bar{u}^2}\sqrt{1 + \frac{4\epsilon^2\bar{y}^2}{\bar{u}^2}}\,. \tag{2.14}$$

The principle of maximum proper time says that the unaccelerated observer measures the longest proper time, i.e. $\tau_1 > \tau_2$ (provided that $\epsilon\bar{y} \neq 0$). This in turn implies that

$$\bar{y}^2 > 0\,. \tag{2.15}$$

This result was obtained assuming $\bar{u}^2 < 0$. With the opposite convention that $\bar{u}^2 > 0$ we would instead find $\bar{y}^2 < 0$. We see that, in contrast to what we are used to from Euclidean geometry, it is not possible for all spacetime vectors to have positive square.

Clearly no observer can travel along $\bar{y}$ because then his clock would need to show an imaginary time, which is absurd.

Consider the vector

$$\bar{w} = a\bar{u} + b\bar{y}\,. \tag{2.16}$$

Squaring we find

$$\bar{w}^2 = a^2\bar{u}^2 + b^2\bar{y}^2\,. \tag{2.17}$$

If we take $a^2 = -\frac{b^2\bar{y}^2}{\bar{u}^2}$ (note that the RHS is positive which is consistent with a, b being real numbers) we get $\bar{w}^2 = 0$! We therefore conclude that there exist spacetime vectors $\bar{w} \neq \bar{0}$ such that $\bar{w}^2 = 0$.

To summarize we have learned that there are 3 classes of spacetime vectors $\bar{v}$ which we give names as follows

- $\bar{v}^2$ negative: *Timelike*
- $\bar{v}^2$ positive: *Spacelike*
- $\bar{v}^2 = 0$: *Null* (light-like)

Vectors that are part of a straight worldline of an observer are timelike. We have also seen above that if $\bar{u}$ is timelike and $\bar{u} \cdot \bar{v} = 0$ then either $\bar{v} = \bar{0}$ or $\bar{v}$ is spacelike. This is a very useful result to remember when working with spacetime vectors.

Chapter 3

Simultaneity and spatial distance

An observer traveling along in a spaceship has only direct access to the interior of the spaceship. Nevertheless she must be able to make statements and inferences about what happens in the outside world. To be able to do this she needs in particular to be able to say when an event, which is not on her worldline, occurred. To do this she needs to have a way to determine whether an event far away is *simultaneous* with an event on her worldline, e.g. a supernova explosion far away happens when her clock shows 10:00.

The natural way for here to define simultaneity is via the construction in figure 3.1. She sends out a probe which travels on a straight worldline to the event and on a straight worldline back. She arranges it so that the probe reaches the event precisely when half the proper time of its journey has elapsed. Then she will say that the event on her worldline halfway between sending out and receiving the probe is simultaneous with the distant event.

From the figure we have

$$\tau^2 = -(\bar{v}+\bar{r})^2 = -(\bar{v}-\bar{r})^2 \qquad \Rightarrow \qquad \bar{v}\cdot\bar{r} = 0\,, \tag{3.1}$$

so that $\bar{r}$ is spacelike.

Let us now consider a family of straight parallel worldlines L_0, L_1, ... defined by the equation

$$\bar{R}_n = \lambda_n \bar{u} + n\bar{\rho}\,, \qquad n = 0, 1, 2, \ldots \tag{3.2}$$

where $\bar{u}$, $\bar{\rho}$ are timelike vectors and λ_n parameterizes a point on L_n. This is illustrated in figure 3.2. This could be a fleet of identical spaceships traveling unaccelerated arranged head to tail. Consider an observer traveling from the front of the fleet to the back counting how many ships he passes. This number is a measure of how far it is from the head of the fleet to the tail. The distance is expressed in units of 'standard spaceship'.

There is an alternative way to measure this distance. There is only one vector going

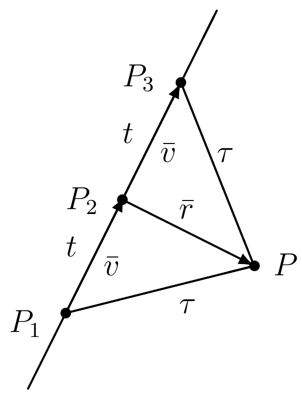


Figure 3.1: Via this construction the observer decides that P_2, halfway between P_1 and P_3, is simultaneous with P.

from L_0 to L_n with the property that it is orthogonal to $\bar{u}$.[1] It is given by

$$\bar{r}_n = n\bar{\rho} - \left(\frac{n\bar{\rho} \cdot \bar{u}}{\bar{u}^2}\right) \bar{u} \, . \tag{3.3}$$

Note that $\bar{r}_n$, and therefore also its length, is proportional to n, the number of spaceships. We can use $\sqrt{\bar{r}_n^2}$ as a measure of the distance. We just need to work out the conversion factor to go between $\sqrt{\bar{r}_n^2}$ and the number of spaceships.

Looking at figure 3.1 (replacing $\bar{v}$ with $\bar{u}$ and dropping the subscript on $\bar{r}_n$ in the following) we read off

$$\tau^2 = -(\bar{u} + \bar{r})^2 = t^2 - \bar{r}^2 \, , \qquad \text{or} \qquad l^2 = \bar{r}^2 = t^2 - \tau^2 \, . \tag{3.4}$$

The advantage of this method is that we don't need the fleet of spaceships (other than to fix the unit of distance). Later we will find an even more practical way to measure distance.

How we pick the unit of distance is up to us. Nothing prevents us from choosing units such that $\sqrt{\bar{r}^2}$ itself is the distance. This is in fact the most natural choice to make. From (3.4) we see that now space and time acquire the same dimensions. In the theory of relativity this is as natural as height and width having the same dimensions and being measured in the same units.

[1] Proof: Assume there are two such vectors $\bar{r}_1$, $\bar{r}_2$. We may assume their foot-point is the same point on L_0. The fact the $\bar{u} \cdot \bar{r}_1 = \bar{u} \cdot \bar{r}_2 = 0$ implies $\bar{u} \cdot (\bar{r}_1 - \bar{r}_2) = 0$. But $\bar{r}_1 - \bar{r}_2 = \lambda \bar{u}$ and the previous equation implies $\lambda = 0$ so that $\bar{r}_1 = \bar{r}_2$. □

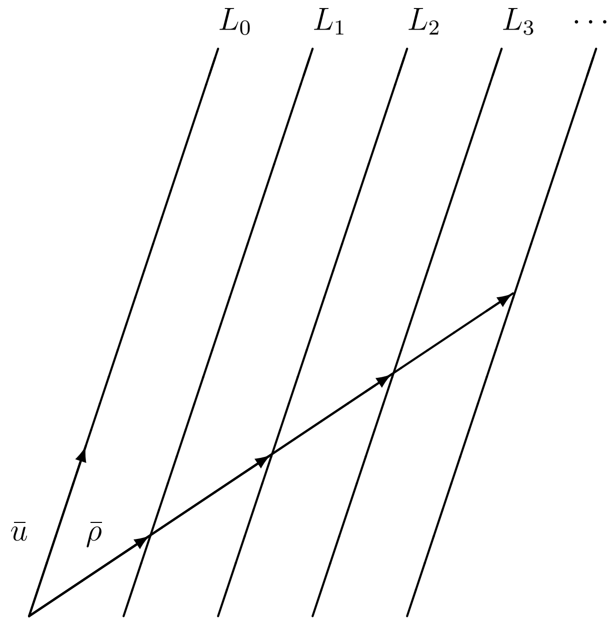


Figure 3.2: A family of parallel worldlines.

3.1 Orthogonal space

To every unaccelerated observer there corresponds a straight worldline. Such a worldline is characterized by a timelike vector which we can normalize to a *unit vector* $\hat{u}$. We will always us a 'hat' to denote a unit vectors. A timelike unit vector satisfies $\hat{u}^2 = -1$ and a spacelike unit vector $\hat{r}^2 = 1$. Given an observer with unit vector $\hat{u}$ there there exist vectors $\bar{r}$ such that

$$\hat{u} \cdot \bar{r} = 0\,. \tag{3.5}$$

They form the *orthogonal space* to the observers worldline. This is a vector space since linear combinations of such vectors clearly belong to the space. In fact, since all such $\bar{r}$ are spacelike (or zero), it is a Euclidean vector space. Since we are imposing one condition on the four components of $\bar{r}$ this orthogonal space is three-dimensional. Recall that $\sqrt{\bar{r}^2}$ is the (spatial) distance from the observer. The orthogonal space is the space used in Newtonian physics. The difference is that in the theory of relativity each observer constructs her own orthogonal space.

Given the worldline of an observer with direction $\hat{u}$ we can split any spacetime vector $\bar{R}$ into a component along $\hat{u}$ and a component orthogonal to it as

$$\bar{R} = t\hat{u} + \bar{r} \qquad \text{with} \qquad \hat{u} \cdot \bar{r} = 0\,. \tag{3.6}$$

This is illustrated in figure 3.3. According to the figure an observer following the worldline

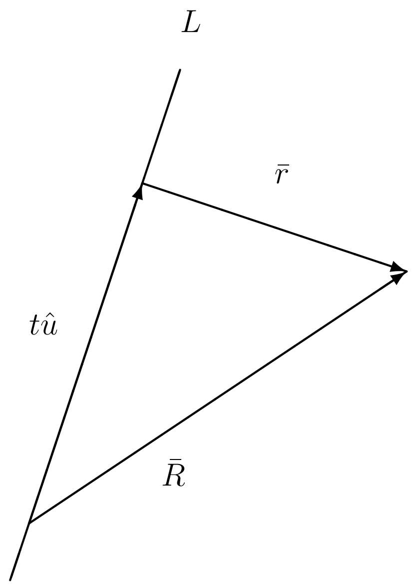


Figure 3.3: Split of a spacetime $\bar{R}$ vector with respect to a timelike direction $\hat{u}$.

L measures the event corresponding to $\bar{R}$ to happen at a time t and spatial position $\bar{r}$. The distance to the event is $\sqrt{\bar{r}^2}$. From the equation we find $t = -\hat{u} \cdot R$ and $\bar{r} = \bar{R} + (\hat{u} \cdot R)\hat{u}$ from which we see that t and $\bar{r}$ are uniquely fixed in terms of $\hat{u}$ and $\bar{R}$.

3.2 Linearly independent vectors

Consider four spacetime vectors

$$\bar{A},\, \bar{B},\, \bar{C},\, \bar{D}\,. \tag{3.7}$$

They are linearly independent of none of them can be expressed as a linear combination of the others, or equivalently if the equation

$$a\bar{A} + b\bar{B} + c\bar{C} + d\bar{D} = 0 \tag{3.8}$$

has only the trivial solution $a = b = c = d = 0$. Note that since spacetime is four-dimensional we cannot have more than four linearly independent vectors.

In Euclidean space we are used to two orthogonal vectors being linearly independent. This is **not** true in spacetime, e.g. a null vector is orthogonal to itself but clearly not linearly independent of itself. What is true is that if $\dot{\bar{v}}u = 0$ for *all* $\bar{u}$ then $\bar{v} = 0$. To see this take $\bar{u}$ timelike. We conclude that $\bar{v}$ must be spacelike but if it is orthogonal to all other spacelike vectors it must vanish since they form a Euclidean vector space.

To test if $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ are lineraly independent we consider the determinant of the matrix of inner products

$$\begin{vmatrix} \bar{A}\cdot\bar{A} & \bar{A}\cdot\bar{B} & \bar{A}\cdot\bar{C} & \bar{A}\cdot\bar{D} \\ \bar{B}\cdot\bar{A} & \bar{B}\cdot\bar{B} & \bar{B}\cdot\bar{C} & \bar{B}\cdot\bar{D} \\ \bar{C}\cdot\bar{A} & \bar{C}\cdot\bar{B} & \bar{C}\cdot\bar{C} & \bar{C}\cdot\bar{D} \\ \bar{D}\cdot\bar{A} & \bar{D}\cdot\bar{B} & \bar{D}\cdot\bar{C} & \bar{D}\cdot\bar{D} \end{vmatrix} \tag{3.9}$$

The vectors are linearly dependent if and only if this determinant vanishes. To see this assume they are linearly dependent. Then (3.8) holds for some non-zero coefficients. Taking this linear combination of rows in the matrix we obtain a row of zeros so that the determinant vanishes. Conversely, if the determinant vanishes there exists a linear combination of the rows that gives zero. This means that there exists a vector $a\bar{A} + b\bar{B} + c\bar{C} + d\bar{D}$, with a, b, c, d not all zero, which is orthogonal to $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$. Assuming they are linearly independent gives a contradiction, therefore they are linearly dependent.

Note that this test works only for four vectors. It does not work for lower-dimensional subspaces of spacetime.

Chapter 4

Velocity and light signals

Consider an observer and an object following straight worldlines L and L' respectively, figure 4.1. The observer describes the objects position at a time t to be $\bar{r}$, with $\bar{r} = 0$ at

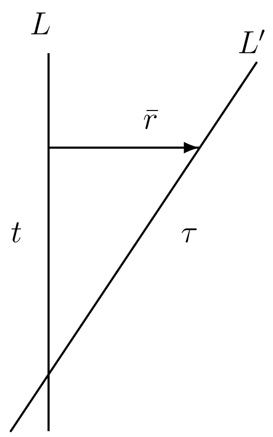


Figure 4.1: Observer L describes the object L' as having position $\bar{r}$ at time t.

$t = 0$. He assigns the object the velocity

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{\bar{r}}{t}\,, \tag{4.1}$$

where the last equality follows from the fact that they are following straight worldlines so $\bar{r}$ is proportional to t. This velocity clearly depends on the observer, since $\bar{r}$ and t refer to the observer. For this reason it is called the *relative velocity* of the observer and the object.

The unit vector along the objects worldline $\hat{v}$ tells us how the relative velocity $\bar{v}$ is directed relative to $\hat{u}$. We call $\hat{v}$ the *four-velocity* of the object. Any timelike unit vector (pointing forwards in time) can be a four-velocity since it could be the direction of an objects worldline.

4.1 Standard velocity split

Let $\hat{v}$ be the four-velocity of the object and $\hat{u}$ that of the observer. The objects position is given by $\bar{R} = \tau\hat{v}$ and from figure 4.1 we see that

$$\tau\hat{v} = t\hat{u} + \bar{r} = t(\hat{u} + \bar{v})\,, \tag{4.2}$$

where we used the definition of the relative velocity in (4.1). We can write this as

$$\hat{v} = \gamma(\hat{u} + \bar{v}) \qquad \text{where} \qquad \gamma = \frac{t}{\tau} \qquad \text{and} \qquad \hat{u}\cdot\bar{v} = 0\,. \tag{4.3}$$

This formula for the split of the four velocity of the object into the four-velocity of the observer and the relative velocity is very useful and has many applications. For example, squaring this equation gives

$$1 = \gamma^2(1 - v^2) \qquad \text{or} \qquad \frac{t}{\tau} = \gamma = \frac{1}{\sqrt{1 - v^2}}\,, \tag{4.4}$$

where $v^2 = \bar{v}^2$, the relative velocity squared. Note that $v^2 = 1 - \frac{\tau^2}{t^2} \leq 1$. This is the famous formula for *time dilation*. Since $t = \gamma\tau$ and $\gamma \geq 1$ the observer sees the objects clocks running slower that his by a factor of γ.

Taking the inner product of (4.3) with $\hat{u}$ we find

$$-\hat{u}\cdot\hat{v} = \gamma = \frac{1}{\sqrt{1 - v^2}}\,. \tag{4.5}$$

Notice that this implies that the definition of the relative velocity v is symmetric between the observer and object. The object judges the observer to have the same velocity as the observer assigns to the object.

4.2 Light signals

So far we have not discussed null lines, which are on the border between timelike and spacelike lines. Now let us assume that L' in figure 4.1 is such a null line. Then

$$\tau^2 = -\bar{R}^2 = 0\,, \tag{4.6}$$

but it is still true that

$$\tau\hat{v} = t\hat{u} + \bar{r}\,. \tag{4.7}$$

Squaring this we get

$$-t^2 + \bar{r}^2 = 0 \qquad \text{or} \qquad v^2 = 1\,. \tag{4.8}$$

Note that this result is independent of the observer. All observers would measure the relative velocity of such a signal to have magnitude $v = 1$.

It does not follow from the theory of relativity itself that particles following such null lines exist. It would be meaningless to assign them a clock since it would not tick ($\tau = 0$ along such a line).

Nevertheless experience tells us that there exist signals in nature that can travel along null lines. Light being the most important example and $v = 1$ is called the "velocity of light". More generally, as we will see later, a particle can follow a null worldline if and only if it is massless. The quantum of light is the massless particle called the photon.

The existence of such signals is of great practical importance. Recall that to determine simultaneity we used the setup in figure 3.1. The observer has to arrange the situation so that the proper time of the probe to and from the event are the same. To achieve this in practice presents great difficulties. The existence of light, or more generally electromagnetic, signals solves this problem since using such a signal instead of the probe $\tau = 0$ always. Equivalently the observer knows that $v = 1$ and can therefore calculate the distance directly from the time it takes there and back.

Imagine an observer who sends out a flash of light at $t = 0$. The light pulse travels out in all directions with unit velocity forming a sphere of light. To draw the corresponding spacetime diagram we must go down to two dimensions of space where the light forms a circle traveling outwards from the observer, figure 4.2. In this three-dimensional spacetime picture the light forms a cone. For this reason it is referred to as the *light-cone*. It is the surface given by the equation

$$r = t\,, \qquad \text{where} \qquad r = \sqrt{\bar{r}^2}\,. \tag{4.9}$$

What we have drawn is really only half of the light-cone, called the *future light-cone*. There is also the *past light-cone* given by

$$r = -t\,. \tag{4.10}$$

It describes a sphere of light contracting towards the origin. The full light cone is given by the equation

$$r^2 = t^2 \tag{4.11}$$

and is illustrated in figure 4.3. Note that this equation is true for any observer. This means that the light-cone looks the same to any observer. Don't be mislead by the picture which might seem to suggest otherwise!

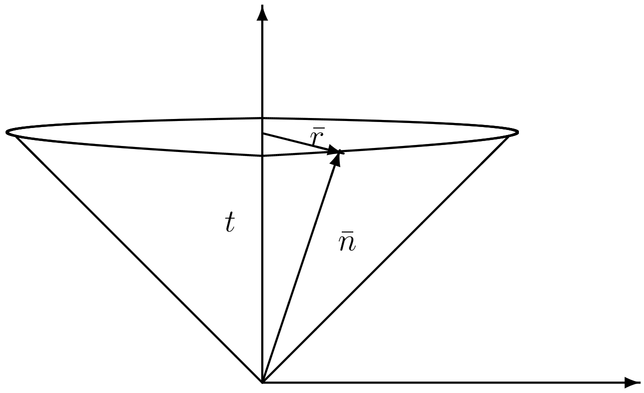


Figure 4.2: An observer's future light-cone. The circle of light is at distance r at time t and traveling out in all null directions, e.g. $\bar{n}$.

4.3 Split of null vectors

As we have seen any spacetime vector can be split with respect to some four-velocity $\hat{u}$ into a component along $\hat{u}$ and a component orthogonal to it. For a null vector $\bar{n}$ we get

$$\bar{n} = a(\hat{u} + \bar{m}) \tag{4.12}$$

but $\bar{n}^2 = 0$ implies $\bar{m}^2 = 1$ so that

$$\bar{n} = a(\hat{u} + \hat{m}) \,, \tag{4.13}$$

proportional to the sum of a timelike and a spacelike unit vector. If a vector $\bar{k}$ is orthogonal to $\bar{n}$ it is either spacelike or proportional to $\bar{n}$.[1] In particular we have that two orthogonal null vectors must be parallel.

[1]Proof: Writing $\bar{k} = b(\hat{u} + \bar{r})$ we find the condition (we may assume $a, b \neq 0$)

$$\hat{m} \cdot \bar{r} = 1 \,.$$

since $\hat{m}$ and $\bar{r}$ belong to the orthogonal subspace they are Euclidean. Therefore we must have either $\sqrt{\bar{r}^2} > 1$, in which case $\bar{k}$ is spacelike, or $\sqrt{\bar{r}^2} = 1$ and $\bar{r}$ is parallel to $\hat{m}$, in which case $\bar{k} = \frac{b}{a}\bar{n}$. $\Box$

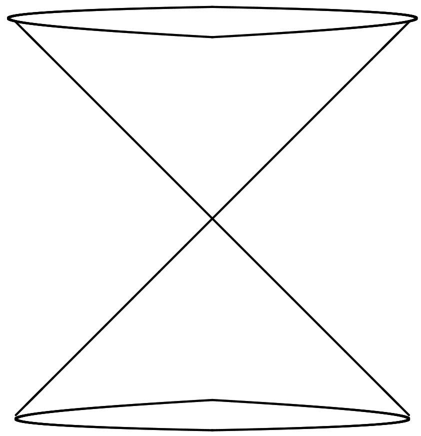


Figure 4.3: The full light-cone.

4.4 Future and past

Consider a timelike vector $\bar{v}$ at the origin. We can split it with respect to the four-velocity of an observer as

$$\bar{v} = t\hat{u} + \bar{r}\,, \qquad \hat{u} \cdot \bar{r} = 0\,. \tag{4.14}$$

Squaring we find

$$\bar{v}^2 = r^2 - t^2 < 0\,, \tag{4.15}$$

which implies that every timelike vector is pointing insider the light-cone (see figure 4.2). Replacing $\bar{v}$ with a spacelike vector we similarly find that every spaclike vector points outside the light-cone. Null vectors point along the light-cone.

Let $\bar{u}$, $\bar{v}$ be timelike vectors with negative inner product

$$\bar{u} \cdot \bar{v} < 0\,. \tag{4.16}$$

Construct the linear combination

$$\bar{w} = a\bar{u} + b\bar{v}\,, \qquad \text{with} \qquad a, b \geq 0\,, \quad a + b \neq 0\,. \tag{4.17}$$

Squaring we find

$$\bar{w}^2 = a\bar{u}^2 + 2ab\bar{u} \cdot \bar{v} + \bar{v}^2 < 0\,, \tag{4.18}$$

since all terms are negative. By varying a, b we can continuously go from the vector $\bar{u}$ to the vector $\bar{v}$ via only timelike vectors. This would be impossible if one was pointing into the future light-cone and the other into the past light-cone. We therefore conclude that two timelike vectors with negative inner product must be pointing in the same direction, i.e. either both into the future light-cone or both into the past light-cone. It is easy to see that the same argument goes through if we take one timelike and one null vector.

If instead $\bar{u} \cdot \bar{v} > 0$ then $\bar{u}$ and $-\bar{v}$ are point in the same direction so that $\bar{u}$ and $\bar{v}$ are pointing in opposite directions. Conversely two timelike vectors pointing inside the same part (future/past) of the light-cone have $\bar{u} \cdot \bar{v} < 0$ (note that $\bar{u} \cdot \bar{v}$ cannot vanish). Again this goes through if one of them is null.

Let $\hat{u}$, $\hat{v}$ be timelike and future directed. Then

$$(\hat{u} + \hat{v})^2 = -2 + 2\hat{u} \cdot \hat{v} < 0 \qquad \text{and} \qquad (\hat{u} + \hat{v}) \cdot \hat{v} = \hat{u} \cdot \hat{v} - 1 < 0\,, \tag{4.19}$$

from which we conclude that $\hat{u} + \hat{v}$ is also timelike and future directed. This must hold for any sum of timelike future directed vectors. An important consequence of this is that no spaceship (or other object) can reverse its four-velocity and travel to the past to arrive before it departed. Time travel is therefore impossible within the theory of relativity. Related to this, if two spaceships part and then meet again they will always agree they part before they meet.

We have seen that timelike (and null) vectors can be divided into two classes: future directed and past directed. Note that no such division is possible for spacelike vectors.

Chapter 5

Lorentz transformation

In this chapter we will restrict to situations where all the interesting spacetime vectors lie in a two-dimensional plane. This means that they can all be expressed in terms of two linearly independent vectors. Many problems one encounters are in fact of this type.

In the cases of interest here this plane contains a timelike future directed unit vector which could be the four-velocity of an observer. Every vector can be split with respect to this four-velocity, figure 3.3. Note that this corresponds to a one-dimensional problem in Newtonian physics.

Consider the situation in Euclidean space with two sets of orthogonal vectors rotated with respect to each other. Thinking of this as two coordinate systems we are familiar with how two relate them via the rotation. Consider not the corresponding situation in spacetime, illustrated in figure 5.1, which occurs often. We have two sets of orthogonal unit vectors $\hat{u}$, $\hat{r}$ with $\hat{u}\cdot\hat{r} = 0$ and $\hat{v}$, $\hat{s}$ with $\hat{v}\cdot\hat{s} = 0$ and $\hat{u}^2 = \hat{v}^2 = -1$, $\hat{r}^2 = \hat{s}^2 = 1$. Obviously $\hat{u}$ and $\hat{r}$ are lineraly independent since one is timelike and one is spacelike. Since we are restricting to a two-dimensional plane can express $\hat{v}$, $\hat{s}$ in terms of $\hat{u}$, $\hat{r}$. Up to a proportionality factor we have

$$\hat{v} \propto \hat{u} + \alpha\hat{r}\,, \qquad \hat{s} \propto \hat{r} + \alpha\hat{u}\,, \tag{5.1}$$

where we have used the fact that $\hat{v}\cdot\hat{s} = 0$. Let us compute the length of the vectors on the RHS to fix the normalization. We have

$$(\hat{r} + \alpha\hat{u})^2 = -(\hat{u} + \alpha\hat{r})^2 = 1 - \alpha^2 \tag{5.2}$$

and therefore

$$\hat{v} = \pm\frac{1}{\sqrt{1-\alpha^2}}(\hat{u} + \alpha\hat{r})\,, \qquad \hat{s} = \pm\frac{1}{\sqrt{1-\alpha^2}}(\hat{r} + \alpha\hat{u})\,, \tag{5.3}$$

so that $\hat{s}^2 = -\hat{v}^2 = 1$. Demanding that $\hat{v} \to \hat{u}$ and $\hat{s} \to \hat{r}$ as $\alpha \to 0$ we see that we need to pick the plus signs.

Recalling the standard velocity split, eq. (4.3)

$$\hat{v} = \frac{1}{\sqrt{1-v^2}}(\hat{u} + \bar{v})\,, \qquad \hat{u}\cdot\bar{v} = 0\,, \tag{5.4}$$

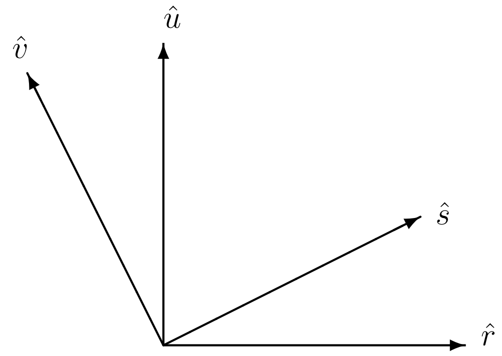


Figure 5.1: Two sets of orthogonal unit vectors in spacetime.

we read off $\alpha = \pm v$ with $v = \sqrt{\bar{v}^2}$, the magnitude of the relative velocity. We therefore have

$$\hat{v} = \gamma(\hat{u} + v\hat{r})\,, \qquad \hat{s} = \gamma(\hat{r} + v\hat{u})\,, \qquad \gamma = \frac{1}{\sqrt{1-v^2}}\,, \tag{5.5}$$

where we have absorbed the sign into v so that v is positive if the relative velocity is along $\hat{r}$ and negative if it is along $-\hat{r}$. This is the famous *Lorentz transformation*. It tells us how the measurements of two inertial observers are related. To see this consider an event specified by the spacetime vector $\bar{R}$. We can write it in two ways as

$$\bar{R} = t\hat{u} + x\hat{r} \qquad \text{or} \qquad \bar{R} = t'\bar{v} + x'\bar{s}\,. \tag{5.6}$$

Observer u assigns coordinates (t, x) to the event while observer v assigns it coordinates (t', x'). Equating the two expressions for $\bar{R}$ and using the formula for the Lorentz transformation we find

$$t\hat{u} + x\hat{r} = t'\gamma(\hat{u} + v\hat{r}) + x'\gamma(\hat{r} + v\hat{u}) \tag{5.7}$$

or

$$t = \gamma(t' + vx')\,, \qquad x = \gamma(x' + vt')\,. \tag{5.8}$$

This is the Lorentz transformation that relates the time and space measurements of the two observers.

5.1 Addition of velocities

In Euclidean space we can perform first one rotation and then another. The result is a third rotation. Similarly we can perform first one Lorentz transformation with parameter (relative velocity) v_1 and then another with parameter v_2. This clearly gives a third Lorentz transformation. The only question is how the parameter of the third Lorentz transformation is related to v_1, v_2.

To answer this question we note that the final timelike unit vector is proportional to

$$(\hat{u} + v_1\hat{r}) + v_2(\hat{r} + v_1\hat{u}) = (1 + v_1v_2)\hat{u} + (v_1 + v_2)\hat{r}\,. \tag{5.9}$$

From the equation $\hat{v} = \gamma(\hat{u} + v\hat{r})$ we see that we can read off v as $\hat{r}$-component/$\hat{u}$-component. In this way we find

$$v = \frac{v_1 + v_2}{1 + v_1v_2}\,. \tag{5.10}$$

This is the relativistic formula for addition of velocities. It has been tested to high accuracy in experiments with light propagating through flowing liquids.

In the special case $v_2 = -v_1$ we get $v = 0$. This means that the inverse Lorentz transformation is obtained by changing the sign of v. This is easily verified directly by performing first a Lorentz transformation with parameter v and then one with parameter $-v$.

5.2 Lorentz contraction

Consider a spaceship of length ℓ. The observer on the spaceship has four-velocity $\hat{u}$. Another observer has four-velocity $\hat{v}$. Both measure the length of the spaceship from their perspective getting the answer ℓ and ℓ' respectively. The setup is illustrated in figure 5.2. From the figure we see that

$$\ell'\hat{s} - \ell\hat{r} \propto \hat{u} \tag{5.11}$$

and taking the inner product with $\hat{r}$ gives

$$\ell'\hat{r}\cdot\hat{s} - \ell = 0\,. \tag{5.12}$$

Using the Lorentz transformation $\hat{s} = \gamma(\hat{r} + v\hat{u})$ we find $\hat{r}\cdot\hat{s} = \gamma$ so that

$$\ell'/\ell = 1/\gamma = \sqrt{1 - v^2}\,, \tag{5.13}$$

where we is the relative velocity of the two observers.

This is the equation for the *Lorentz contraction*. A moving object appears shortened, or contracted, in its direction of motion relative to the observer. This effect is similar to slicing a sausage in that the length of the slice depends on the cutting angle. In that case the perpendicular slice has the *shortest* length, whereas in the spaceship case the observer at rest measures the *longest* length. Note that in this respect the Euclidean figure 5.2 is very misleading.

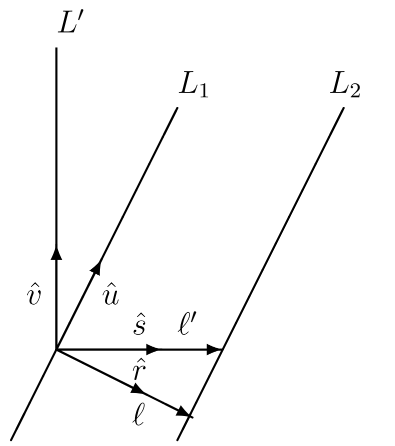


Figure 5.2: A spaceship stretching between L_1 and L_2, with length ℓ, is observed by the observer following the worldline L' who measures its length to be ℓ'.

Chapter 6

Waves

Sound waves are familiar from Newtonian physics. The pressure $P(\bar{r}, t)$ varies in space and with time around the mean pressure P_0. For a plane sound wave $p = P - P_0$ has the form

$$p = A\sin(-\omega t + \bar{k} \cdot \bar{r} + \phi_0)\,. \tag{6.1}$$

The constants A and ϕ_0 are the *amplitude* and *phase shift* of the wave, while ω is the *angular frequency* and the three-vector $\bar{k}$ is the *wave vector*. We may assume that $\omega > 0$. The surfaces where p takes a constant value, e.g. its maximum, are 2-dimensional planes in our 3-dimensional space, see figure 6.1, which is the reason for the name *plane waves*. Sound waves are just one example. Many other types of waves exist in nature and they can all be described in the same way.

Consider now a plane wave in spacetime given by

$$\psi = A\sin(\bar{K} \cdot \bar{R} + \phi_0)\,, \tag{6.2}$$

where now $\bar{K}$ is the *wave four-vector* and $\bar{R}$ is the four-vector of a spacetime point (event). Introduce an observer with four-velocity $\hat{u}$. We can then write

$$\bar{R} = t\hat{u} + \bar{r}\,, \qquad \bar{r} \cdot \hat{u} = 0\,, \tag{6.3}$$

$$\bar{K} = \omega\hat{u} + \bar{k}\,, \qquad \bar{k} \cdot \hat{u} = 0\,. \tag{6.4}$$

Using these expressions the wave takes the form

$$\psi = A\sin(-\omega t + \bar{k} \cdot \bar{r} + \phi_0)\,, \tag{6.5}$$

which is the same as before. Notice that ω and $\bar{k}$ depend on the observer since they are defined using his four-velocity. We get

$$\omega_u = -\hat{u} \cdot \bar{K}\,, \tag{6.6}$$

the angular frequency of the wave as measured by the observer u.

The *superposition principle* says that we can add together waves of the form (6.2) to form new waves. For physical waves the angular frequency is determined by the wave vector

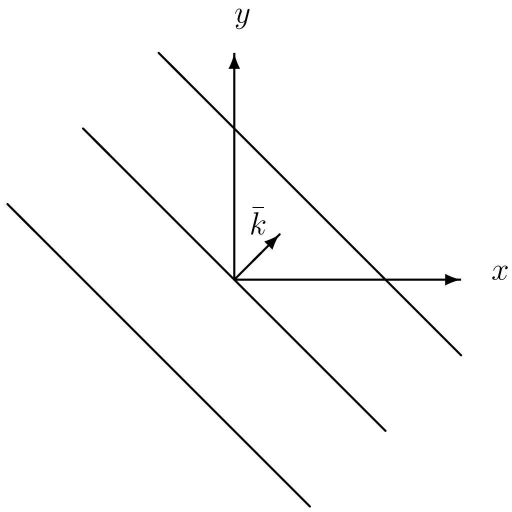


Figure 6.1: Plane wave in two space dimensions. The wavefronts, surfaces of constant phase at say $t = 0$, given by $\bar{k} \cdot \bar{r} =$ const, are straight parallel lines orthogonal to $\bar{k}$. In 3 dimensions they are planes.

$\omega = \omega(\bar{k})$. The relation between ω and $\bar{k}$ is called the *dispersion relation* and depends on the type of wave. Let us consider the sum of two one-dimensional plane waves with similar wave number $k_1 = (1 - \epsilon)k$ and $k_2 = (1 + \epsilon)k$ and $\phi = 0$,

$$\psi = \sin(-\omega_1 t + k_1 x) + \sin(-\omega_2 t + k_2 x)\,. \tag{6.7}$$

We have

$$-\omega_1 t + k_1 x = -\omega(k_1)t + k_1 x = -\omega(k)t + kx + \epsilon k(\omega' t - x)\,, \tag{6.8}$$

where $\omega' = d\omega/dk$, and for k_2 we find the same expression with the sign of the last term changed. Therefore, using the formula for sin of a sum of angles, we have

$$\begin{aligned}\psi =& \sin(-\omega t + kx + \epsilon k(\omega' t - x)) + \sin(-\omega t + kx - \epsilon k(\omega' t - x))\\ =& \sin(-\omega t + kx)\cos(\epsilon k(\omega' t - x)) + \cos(-\omega t + kx)\sin(\epsilon k(\omega' t - x))\\ &+ \sin(-\omega t + kx)\cos(-\epsilon k(\omega' t - x)) + \cos(-\omega t + kx)\sin(-\epsilon k(\omega' t - x))\\ =& 2\cos(\epsilon k(\omega' t - x))\sin(-\omega t + kx)\,.\end{aligned} \tag{6.9}$$

This is a plane wave with wave number k and frequency $\omega = \omega(k)$ but with an amplitude which is *modulated* by the cos factor. The cosine factor describes the envelope of the wave packet. The waves packet depends on the position through $x - \omega' t$ and therefore travels with the *group velocity*

$$v = \omega' = \frac{d\omega}{dk}\,. \tag{6.10}$$

This is the velocity of signals sent with such waves. For light we should clearly have $v = 1$ and indeed light (electromagnetic radiation) has the dispersion relation

$$\omega = k = \sqrt{\bar{k}^2} \tag{6.11}$$

or

$$\bar{K}^2 = (\omega\hat{u} + \bar{k})^2 = -\omega^2 + \bar{k}^2 = 0\,, \tag{6.12}$$

i.e. light rays have a wave four-vector which is null.

6.1 Doppler shift and aberration

Consider two observers with four-velocities $\hat{u}$ and $\hat{v}$ observing a light wave ($\bar{K}^2 = 0$). Splitting vectors with respect to the first observer we have

$$\hat{v} = \gamma(\hat{u} + \bar{v}) \qquad \bar{v}\cdot\hat{u} = 0 \tag{6.13}$$

and

$$\bar{K} = \omega_u(\hat{u} + \hat{k}_u) \qquad \hat{k}_u \cdot \hat{u} = 0\,. \tag{6.14}$$

We have out a subscript u to emphasize that ω_u and $\hat{k}_u$ are the quantities measure by observer u. Note also that $\hat{k}_u$ is a unit vector which guarantees that the wave four-vector $\bar{K}$ is null as it should be for light. Since $\bar{v}$ and $\hat{k}_u$ are in the orthogonal space to $\hat{u}$ they can be treated as ordinary Euclidean vectors and we can write

$$\bar{v}\cdot\hat{k}_u = v\cos\theta_u\,, \tag{6.15}$$

where θ_u is the angle between the relative velocity $\bar{v}$ of observers u and v and the direction the light is traveling, given by $\hat{k}_u$, as measured by observer u.

The angular frequency measured by observer v is instead

$$\omega_v = -\hat{v}\cdot\bar{K} = -\omega_u\gamma(-1 + \bar{v}\cdot\hat{k}_u) = \omega_u\gamma(1 - v\cos\theta_u)\,, \tag{6.16}$$

or

$$\frac{\omega_v}{\omega_u} = \frac{1 - v\cos\theta_u}{\sqrt{1-v^2}}\,. \tag{6.17}$$

This is the formula for the *Doppler shift*. It is very important in astronomy where it is used for example to measure the velocity of stars and galaxies relative to us by looking at the shifts of spectral lines.

We could have used the observer v instead of u and we would have found

$$\frac{\omega_u}{\omega_v} = \frac{1 - v\cos\theta_v}{\sqrt{1-v^2}}\,. \tag{6.18}$$

Multiplying this with the previous formula for the Doppler shift we find

$$(1 - v\cos\theta_v)(1 - v\cos\theta_u) = 1 - v^2\,. \tag{6.19}$$

This is the formula for *aberration* derived by Einstein in his 1905 paper. It can also be written as

$$\cos\theta_v = \frac{v - \cos\theta_u}{1 - v\cos\theta_u}\,. \tag{6.20}$$

For $\theta_u << 1$ we have $\theta_v \approx \pi$ since $\bar{v}$ is directed from u to v. Changing the sign of $\cos\theta_v$ so that θ_u and θ_v are measured in the same we we find, using $\cos\theta_u \approx 1 - \theta_u^2/2$, that

$$\theta_v \approx \sqrt{\frac{1+v}{1-v}}\,\theta_u\,, \tag{6.21}$$

which shows that an observer traveling away from the object (since $\bar{v}$ and $\hat{k}$ almost point in the same direction in this case) measures a larger angle. Conversely, switching the sign of v the observer is traveling toward where the light is coming from and measures a smaller angle by a factor $1 - v/(1+v) < 1$. Therefore light is concentrated in the direction of motion and an observer traveling towards a star sees it as smaller and brighter while one traveling away sees it as larger and fainter. This effect again has important consequences in astronomy where it leads to seasonal variations in the angular separation of objects in the sky.

Chapter 7

Particle kinematics

An important application of special relativity is to processes involving elementary particles, e.g. in particle accelerators, where they can often reach velocities close to the speed of light. In order to discuss what happens in particle collisions we first need to introduce the notion of four-momentum.

7.1 Four-momentum

Recall the split of a four-velocity $\hat{v}$ with respect to another one $\hat{u}$

$$\hat{v} = \gamma(\hat{u} + \bar{v}) \qquad \hat{u} \cdot \bar{v} = 0\,, \quad \gamma = \frac{1}{\sqrt{1 - v^2}}\,. \tag{7.1}$$

If v is an object and u an observer $\bar{v}$ is the velocity of the object relative to the observer. In the *Newtonian limit* $v << 1$, i.e. for velocities much smaller than the speed of light, we have, dropping terms of order v^3 and higher,

$$\hat{v} \approx (1 + \frac{v^2}{2})\hat{u} + \bar{v}\,. \tag{7.2}$$

To relate to something familiar let us multiply this by the mass of the object m giving

$$m\hat{v} \approx (m + \frac{mv^2}{2})\hat{u} + m\bar{v}\,. \tag{7.3}$$

We recognize the kinetic energy of the particle $\frac{1}{2}mv^2$ and its momentum $m\bar{v}$. Note that every object has a well-defined mass (just consider an observer traveling along with the object, i.e. whose $\hat{u}$ is (nearly) parallel to $\hat{v}$, who can define the mass in the usual Newtonian way). We define the *four-momentum* of an object to be its mass times its four-velocity

$$\bar{P} = m\hat{v}\,. \tag{7.4}$$

It satisfies

$$\bar{P}^2 = -m^2\,. \tag{7.5}$$

The split of $\hat{v}$ gives a corresponding split of the four-momentum

$$\bar{P} = E\hat{u} + \bar{p} \qquad \hat{u} \cdot \bar{p} = 0\,, \tag{7.6}$$

with

$$E = m\gamma = \frac{m}{\sqrt{1-v^2}} \approx m + \frac{mv^2}{2}\,, \qquad \bar{p} = m\bar{v}\gamma = \frac{m\bar{v}}{\sqrt{1-v^2}} \approx m\bar{v}\,. \tag{7.7}$$

In analogy with the Newtonian case we call E the energy and $\bar{p}$ the momentum (or three-momentum) of the object. Note that they are not intrinsic properties of the object but depend on the observer.

The four-momentum $\bar{P} = m\hat{v}$ is time-like since $\hat{v}$ is time-like. Furthermore, since $m > 0$, it is also future directed. The energy measure by observer u can be written

$$E = -\hat{u} \cdot \bar{P} \tag{7.8}$$

and we see that it is always positive.

The usefulness of the notion of momentum in Newtonian mechanics comes from the fact that it is *conserved* (in the absence of external forces): The sum of all momenta before a collision is equal to the sum of momenta after the collision. This is actually a special case of the relativistic conservation of four-momentum

$$\sum_{\text{before}} \bar{P}_i = \sum_{\text{after}} \bar{P}_i\,. \tag{7.9}$$

It says that there are four conserved quantities: The total energy and the three components of the momentum.

The fact that four-momentum is conserved is intimately tied to symmetries in nature. In fact you will learn later in your studies that the conservation of momentum is equivalent to the statement that physical laws look the same regardless of position in space. The laws of physics are the same here as on the moon or in the Andromeda galaxy. Similarly the conservation of energy is equivalent to the fact that the laws of physics look the same at all times. Just like the theory of relativity combines space and time into a spacetime it combines the notions of momentum and energy into the single concept of four-momentum.

Note that in the Newtonian limit the total energy is

$$E \approx \sum_i \left(m_i + \frac{m_i \bar{v}_i^2}{2} \right)\,. \tag{7.10}$$

But we know that the kinetic energy is not conserved in general, it is only conserved in elastic collisions. This means that the first term must change to keep the total energy conserved, i.e. the sum of the masses before and after the collision will in general differ contrary to what is assumed in Newtonian mechanics. In inelastic collisions the kinetic energy decreases so the total mass must increase. Of course in everyday situations $v << 1$ so the kinetic energy is much smaller than the total mass, $\frac{1}{2}m\bar{v}^2 << m$.

This fact, that mass is also a form of energy, is the content of one of the most famous equations in physics

$$E = mc^2 . \tag{7.11}$$

Remember that we are using units where the speed of light is unity, $c = 1$. Note also that $E = m$ is true only for an observer at rest with respect to the object in question. It is often referred to as the *rest energy*.

7.2 Massless particles

Knowing all but one four-momentum of the particles in a given reaction one can use the conservation of four-momentum to determine the remaining one. In this way one can experimentally establish the existence of particles with null four-momentum

$$\bar{P}^2 = 0 . \tag{7.12}$$

Since we may define the mass through $\bar{P}^2 = -m^2$ we call such particles *massless*. The most important example of such a particle is the photon, the quantum of light (more generally electromagnetic radiation). The four-momentum and energy of such a particle does not vanish, if it did we would not be able to detect them, and therefore we conclude from the expression

$$E = \frac{m}{\sqrt{1-v^2}} \tag{7.13}$$

that such a particle must have $v = 1$, i.e. travel at the speed of light. It therefore follows a null line with four-velocity proportional to the null vector $\bar{P}$. Note that for a massless particle we have

$$\bar{P} = E\hat{u} + \bar{p} = E(\hat{u} + \hat{p}) , \tag{7.14}$$

since it is null.

7.3 Tachyons

Sometimes on hears about particles with space-like four-momentum

$$\bar{P}^2 > 0 . \tag{7.15}$$

Such particles are referred to as *tachyons*. Such a particle would have an imaginary mass from $\bar{P}^2 = -m^2$ and always travel along space-like lines. This turns out not to be consistent with the laws of quantum mechanics and such particles therefore cannot exist in nature.

Instead in our current best theories particles emerge as "ripples" of a field with a quantum of energy. In such *quantum field theories* it is not unusual to have $m^2 < 0$. However, the resulting particles do not travel faster than light. Instead such an imaginary mass signals an instability of the vacuum. In fact this is part of the mechanism by which particles can acquire mass through the so-called Higgs mechanism, related to the famous Higgs particle discovered at CERN.

7.4 Particle reactions and kinematics

The two most important types of particle reactions are

- Decay: One particle goes into two or more
- Collision: Two particles go into one or more

The allowed configurations of four-momenta are constrained by

1. $\bar{P}_i^2 = -m_i^2$ for all particles involved
2. Conservation of four-momentum

$$\sum_i \bar{P}_{i,\text{in}} = \sum_j \bar{P}_{j,\text{out}}$$

These equations form the basis of *particle kinematics*. It is often convenient to refer all quantities to some particular observer, say with four-velocity $\hat{u}$. Then we have

$$\bar{P}_i = E_i \hat{u} + \bar{p}_i \qquad \hat{u} \cdot \bar{p}_i = 0\,. \tag{7.16}$$

The inner product the four-momenta of two particles becomes

$$\bar{P}_i \cdot \bar{P}_j = -E_i E_j + \bar{p}_i \cdot \bar{p}_j\,, \tag{7.17}$$

and in the special case $i = j$ we find

$$m_i^2 = -\bar{P}_i^2 = E_i^2 - \bar{p}_i^2\,. \tag{7.18}$$

A typical kinematical calculation involves two steps

1. Take the inner product of the conservation of for momentum

$$\sum_i \bar{P}_{i,\text{in}} = \sum_j \bar{P}_{j,\text{out}}$$

   with some $\bar{P}_i$, or rearrange it and take the square.

2. Replace $\bar{P}_i^2$ by $-m_i^2$ everywhere and inner products by (7.17).

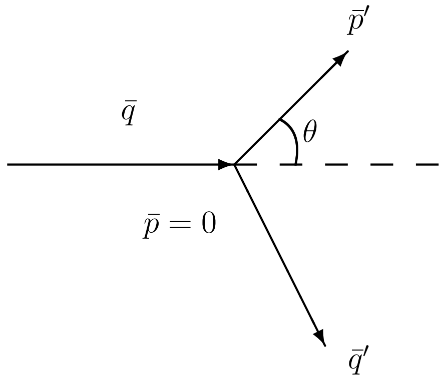


Figure 7.1: Scattering of electron off a proton at rest ($\bar{p} = 0$) in the observers orthogonal space. The recoil angle of the proton is θ.

Example

Consider an electron (e^-) scattering off a proton (p^+). We set $m_{e^-} = m$ and $m_{p^+} = M$ and let $\bar{P}$, $\bar{Q}$ be the initial four-momenta and $\bar{P}'$, $\bar{Q}'$ the final ones. We have

$$\bar{P}^2 = \bar{P}'^2 = -M^2\,, \qquad \bar{Q}^2 = \bar{Q}'^2 = -m^2 \tag{7.19}$$

while the conservation of four-momentum says that

$$\bar{P} + \bar{Q} = \bar{P}' + \bar{Q}'\,. \tag{7.20}$$

Let's say we are interested in the situation where the proton is at rest before the collision and we want to find the recoil angle of the proton. The situation is illustrated in the observers orthogonal space in figure 7.1. One there is one particle which we do not know anything about, in this case the outgoing electron with four-momentum $\bar{Q}'$, it is often a useful strategy to rearrange the conservation of four momentum so that its four momentum appears alone on one side and then square. We therefore write

$$\bar{P} + \bar{Q} - \bar{P}' = \bar{Q}'\,. \tag{7.21}$$

Squaring this equation we get

$$\bar{P}^2 + \bar{Q}^2 + \bar{P}'^2 + 2\bar{P} \cdot \bar{Q} - 2\bar{P} \cdot \bar{P}' - 2\bar{Q} \cdot \bar{P}' = \bar{Q}'^2\,. \tag{7.22}$$

Using fact that $\bar{P}^2 = \bar{P}'^2 = -M^2$ and $\bar{Q}^2 = \bar{Q}'^2 = -m^2$ this can be written

$$\begin{aligned} 0 =& - M^2 + \bar{P}\cdot\bar{Q} - (\bar{P}+\bar{Q})\cdot\bar{P}' \\ =&(\bar{P}+\bar{Q})\cdot(\bar{P}-\bar{P}')\,. \end{aligned} \tag{7.23}$$

This is a useful equation for two-particle elastic collisions. Since we are assuming the proton is initially at rest with respect to the observer we have

$$\bar{P} = M\hat{u}\,, \qquad \bar{Q} = E\hat{u} + \bar{q}\,, \qquad \bar{P}' = E'\hat{u} + \bar{p}'\,. \tag{7.24}$$

Using this we find

$$0 = (\bar{P}+\bar{Q})\cdot(\bar{P}-\bar{P}') = ((M+E)\hat{u}+\bar{q})\cdot((M-E')\hat{u}-\bar{p}') = -(M+E)(M-E')-\bar{q}\cdot\bar{p}'\,. \tag{7.25}$$

Since $\bar{q}$ and $\bar{p}'$ are in the orthogonal space to $\hat{u}$ they can be treated as ordinary Euclidean vector and we can write

$$\bar{q}\cdot\bar{p}' = qp'\cos\theta\,, \tag{7.26}$$

where θ is the recoil angle, see figure 7.1. We have

$$-m^2 = \bar{Q}^2 = -E^2 + q^2 \qquad \Rightarrow \qquad q = \sqrt{E^2 - m^2} \tag{7.27}$$

and similarly we have $p' = \sqrt{E'^2 - M^2}$ giving

$$\cos\theta = \frac{\bar{q}\cdot\bar{p}'}{qp'} = \frac{(E+M)(E'-M)}{\sqrt{E^2-m^2}\sqrt{E'^2-M^2}} = \frac{E+M}{\sqrt{E^2-m^2}}\sqrt{\frac{E'-M}{E'+M}}\,. \tag{7.28}$$

This expresses the recoil angle of the proton in terms of the initial energy of the electron E and the final energy of the proton E'. □

7.5 Center-of-mass observer

For more complicated processes it is often useful to regard all (or part) of the out-going particles together. Their total four-momentum is

$$\bar{P} = \sum_i \bar{P}_i\,. \tag{7.29}$$

Then the "mass" defined by $\bar{P}^2 = -M^2$ does not have a fixed value. It depends on the relative motion of the particles. It does have a lower bound though. Since a sum of time-like future directed vectors is again time-like and future directed we can consider an observer with four-velocity $\hat{u} = \bar{P}/M$. Then we find

$$\bar{P}^2 = -E^2 = -\left(\sum_i E_i\right)^2 = -\left(\sum_i \sqrt{m_i^2 + p_i^2}\right)^2\,. \tag{7.30}$$

Therefore

$$\bar{P}^2 = \left(\sum_i \bar{P}_i\right)^2 \geq \left(\sum_i m_i\right)^2 \tag{7.31}$$

with equality occurring only if $\bar{p}_i = 0$ for all i, i.e. if all particles are at rest with respect to each other. This inequality leads to so-called *threshold conditions* for when certain particles can be produced, for example a Higgs boson at the LHC. Note that there is no reference to any observer in the last equation – the result is observer independent. Nevertheless it was convenient to introduce an observer in the derivation of the result and this turns out to often be the case.

An observer, such as the one considered above, whose four-momentum is parallel to the total four-momentum $\sum_i \bar{P}_i$ is called a *center-of-mass observer* (often center-of-mass frame or center-of-mass system), since such an observer sees the particles with zero total three-momentum, $\sum_i \bar{p}_i = 0$. The introduction of such an observer can often simplify the calculations. Note that using Lorentz transformations one can of course go freely from one observer to another.

Chapter 8

Curved worldlines

So far we have dealt almost exclusively with straight worldlines. Unaccelerated spaceships and particles not influenced by external forces travel along such worldlines. Conversely spaceships that run their engines or particles influenced by external forces follow *curved worldlines*. While it is possible to treat accelerated observers in special relativity it is usually avoided for practical reasons. We will therefore continue to assume that all observers are unaccelerated, unless otherwise stated.

The natural parameter along a (massive) particle's worldline is the proper time τ. The spacetime position of the particle is given as a function of τ

$$\bar{R} = \bar{R}(\tau)\,. \tag{8.1}$$

We can also calculate the rate of change of the position vector with τ as

$$\frac{d\bar{R}}{d\tau} = \lim_{\Delta\tau\to 0} \frac{\bar{R}(\tau+\Delta\tau) - \bar{R}(\tau)}{\Delta\tau}\,. \tag{8.2}$$

Since τ is the proper time along the worldline, which measures its length, we have

$$d\tau^2 = -(d\bar{R})^2\,. \tag{8.3}$$

Therefore

$$\hat{v} = \frac{d\bar{R}}{d\tau} \tag{8.4}$$

is a time-like unit vector, $\hat{v}^2 = -1$. It is tangent to the worldline at the event $\bar{R}(\tau)$. For a straight worldline $d\bar{r}/d\tau$ =constant and $\hat{v}$ is the four-velocity. We will continue to define it as the four-velocity also for curved worldlines.

The *four-acceleration* is defined as

$$\bar{A} = \frac{d\hat{v}}{d\tau} = \frac{d^2\bar{R}}{d\tau^2}\,. \tag{8.5}$$

Differentiating the condition $\hat{v}^2 = -1$ we find

$$\hat{v}\cdot\bar{A} = 0\,. \tag{8.6}$$

This implies that $\bar{A}$ is space-like. Note that it is not, in general, a unit vector.

Let us now analyze how an unaccelerated observer views an accelerated worldline. Consider an observer whose worldline is tangent to the curved worldline at a certain point $\bar{R}(\tau_0)$, as depicted in figure 8.1. Splitting with respect to the four-velocity $\hat{u}$ of the observer

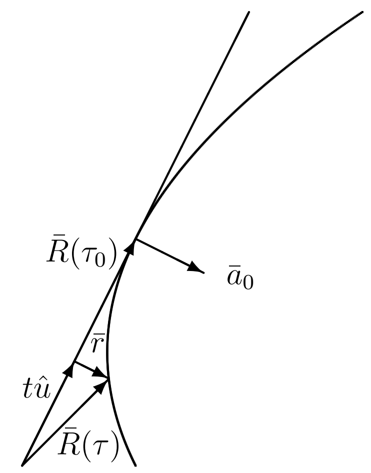


Figure 8.1: An observers worldline is tangent to the curved worldline of an accelerated object at the point $\bar{R}(\tau_0)$.

we have

$$\bar{R}(\tau) = t\hat{u} + \bar{r}\,, \qquad \hat{u}\cdot\bar{r} = 0\,. \tag{8.7}$$

Considering a small interval along the curve this gives

$$d\bar{R} = dt\hat{u} + d\bar{r} \tag{8.8}$$

and squaring this we find

$$-d\tau^2 = -dt^2 + d\bar{r}^2 \tag{8.9}$$

or

$$d\tau = dt\sqrt{1-v^2}\,, \qquad \bar{v} = \frac{d\bar{r}}{dt}\,. \tag{8.10}$$

Note that $\bar{v}$ is the relative velocity and we can also write $d\tau = dt/\gamma$. This gives the four-velocity of the object

$$\hat{v} = \frac{d\bar{R}}{d\tau} = \frac{d\bar{R}}{d\tau} = \gamma \frac{d\bar{R}}{dt} = \gamma(\hat{u} + \bar{v})\,. \tag{8.11}$$

In fact this is the same equation we have before when expressing a constant four-velocity $\hat{v}$ in terms of another one $\hat{u}$, here however $\hat{v}$ and $\bar{v}$ are not constant. At the point $\bar{R}(\tau_0)$ we have $\hat{v} = \hat{u}$ so the relative velocity $\bar{v} = 0$ at this point. For the four-acceleration we find

$$\bar{A}_0 = \left(\frac{d\hat{v}}{d\tau}\right)_0 = \left(\frac{d\hat{v}}{dt}\right)_0 = (\bar{a})_0\,, \tag{8.12}$$

where $\bar{a}$ is the ordinary Newtonian acceleration measured by the observer momentarily at rest with respect to the accelerated object. If this object happens to be a spaceship the vector $\bar{A}$ on a certain point of its worldline is the *acceleration experience by the crew at that instant.*

8.1 Constant acceleration

Consider a two-dimensional spacetime, which could be the history of a spaceship traveling in a straight line as seen by an observer. In two-dimensions we need only one condition to determine a curve and we take

$$\bar{R}^2 = R^2\,, \tag{8.13}$$

with $\bar{R}$ the spacetime position and R a constant. Taking the derivative we find

$$\bar{R} \cdot \hat{v} = 0 \tag{8.14}$$

and another derivative gives

$$\bar{R} \cdot \bar{A} = 1\,. \tag{8.15}$$

The vectors $\bar{R}$, $\hat{v}$ and $\bar{A}$ lie in a 2-plane and since both $\bar{R}$ and $\bar{A}$ are orthogonal to $\hat{v}$ they must be parallel, so that using the equation above we have

$$\bar{A} = \frac{\bar{R}}{R^2} \qquad \Rightarrow \qquad \bar{A}^2 = \frac{1}{R^2}\,, \tag{8.16}$$

i.e. the four-acceleration is constant. This could be the worldline of a spaceship running its engines so that the crew experiences a constant acceleration.

Let $\bar{R}^2 = R_1^2$ and $\bar{R}^2 = R_2^2$ be two such curves in the same 2-plane and with the same origin. From (8.14) we see that any line through the origin cuts the curves orthogonally. Therefore $R_1 - R_2$ is the orthogonal distance between the curves.

In fact one of the lines could be the worldline of the front of a spaceship and the other the worldline of its tail. The ship's length is constant but the acceleration of the front and the tail are unequal, being $1/R_1^2$ and $1/R_2^2$ respectively.

Introducing an observer with four-velocity $\hat{u}$ we can split

$$\bar{R} = t\hat{u} + \bar{r}\,, \qquad \hat{u}\cdot\bar{r} = 0 \tag{8.17}$$

and

$$\bar{R}^2 = R^2 \qquad \Rightarrow \qquad -t^2 + \bar{r}^2 = R^2\,. \tag{8.18}$$

This means that in the $(\bar{r}, t)$-plane of the observer the worldlines of constant acceleration are hyperbolas, as illustrated in figure 8.2. Important properties are hidden in this figure,

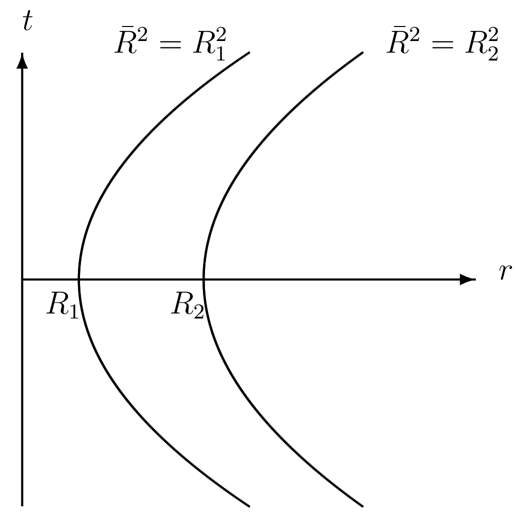


Figure 8.2: Worldlines of constant acceleration as viewed by an observer.

e.g. that all points on the curves are equivalent and that all observers see the same picture. It is also hard to see that lines through the origin are orthogonal to the curves and that they have constant separation. These problems arise because we are representing a spacetime situation in Euclidean space. The corresponding situation in Euclidean space would be two concentric circles of radius R_1 and R_2 respectively, and here they properties mentioned above become obvious. Note however that this Euclidean picture is also not a reliable representation of the spacetime situation, e.g. the curves are closed which is impossible in spacetime.

8.2 Fitting a car into a garage

A famous 'paradox' in special relativity is this:

> Imagine you just bought a very fast car. Unfortunately the car turns out to be slightly too long to fit in your garage. Is it possible to exploit the Lorentz contraction and drive the car very fast into the garage and slam the door?

From the point of view of the garage the car appears shorter due to the Lorentz contraction which suggests that it should work. On the other hand, from the point of view of the car, the garage appears shorter making the situation worse, not better. This is the 'paradox'. Of course, when you analyze things carefully there is no contradiction.

Let $\hat{u}$ be the four-velocity of the garage and $\hat{v}$ that of the car. Lets also take units where the length of the garage is 1 and let $\hat{g}$ be orthogonal to $\hat{u}$ and stretching from the doors to the back of the garage. Similarly we let $\bar{c}$ be orthogonal to $\hat{v}$ and stretch from the front of the car to the tail. We assume the front (tail) of the car continues unaccelerated until it reaches the back wall (doors) of the garage at which point it suddenly stops. (This is of course an idealized situation involving infinite acceleration.)

The spacetime diagram is given in figure 8.3. The following important events are

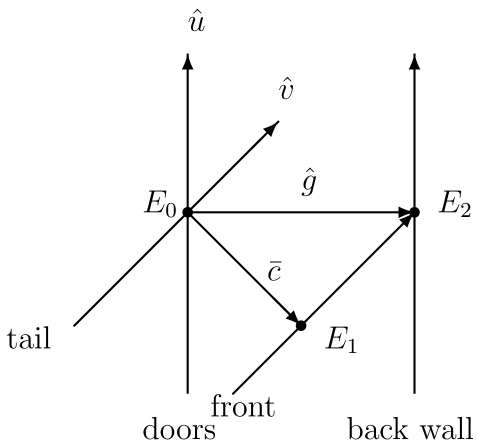


Figure 8.3: Spacetime diagram for the problem of fitting the car into the garage.

indicated in the figure

- E_0 : The garage doors close
- E_1 : The simultaneous event at the front of the car from the cars point of view
- E_2 : The front of the car collides with the back wall of the garage. Simultaneous with E_0 from the point of view of the garage.

If $\bar{v}$ is the relative velocity we can apply the Lorentz transformation relating the observations from the garage to those from the car

$$\hat{v} = \gamma(\hat{u} + v\hat{g}), \qquad \hat{c} = \gamma(\hat{g} + v\hat{u}), \tag{8.19}$$

or the inverse transformations

$$\hat{u} = \gamma(\hat{v} - v\hat{c}), \qquad \hat{g} = \gamma(\hat{c} - v\hat{v}). \tag{8.20}$$

The last of these equations will interest us here. Let $\bar{V}$ be the vector from E_1 to E_2. From figure 8.3 we have

$$\bar{c} + \bar{V} = \hat{g} = \gamma(\hat{c} - v\hat{v}) \tag{8.21}$$

and we find

$$\bar{c} = \gamma\hat{c}, \qquad \bar{V} = -\gamma v\hat{v}. \tag{8.22}$$

The length of the car is $\ell = \sqrt{\bar{c}^2} = \gamma > 1$, so the car is longer than the garage, consistent with our assumptions.

Since $\bar{V}$ is negative relative to $\hat{v}$, from the point of view of the car the front collides with the back wall *before* the doors close, although these events are simultaneous from the point of view of the garage. However, since both $\hat{g}$ and $\bar{c}$ are space-like no signal from the collision can reach the tail of the car before it passes the doors. Therefore no material strength can stop the car from being compressed (from its point of view) and you can shut the door behind it. You have to be quick though if the car is elastic and tends to regain its original length. This example illustrates the fact that no perfectly rigid body exists.

We have seen that it is indeed possible (though not recommended) to fit the car into the garage by exploiting the Lorentz contraction. We have also seen that what appears at first sight to be a paradox is just due to not analyzing the situation carefully enough, in particular neglecting the issues due to the relativity of simultaneity. When the problem is carefully analyzed there are of course no contradictions.

8.3 Rotating wheel

In a wheel at rest all constituent particles follows straight worldlines. When the wheel starts spinning the worldlines become tilted and form helices in spacetime. Only the center continues on a straigth worldline.

Using the center as an observer and thinking of a ring of matter a distance ρ around it we realize that such a ring must have circumference $2\pi\rho$ whether the wheel is rotation or not. In a sense the Lorentz contraction is prevented from taking place. Instead the tilting of the worldlines has the effect that the orthogonal distance between them increases, i.e. a *deformation* takes place.

This is an inevitable deformation which is always connected with rotation. How much energy is needed to produce the deformation depends on the stresses in the wheel. Therefore the moment of inertia of a wheel depends on the stresses within it. This is an important consequence of the theory of relativity though it plays little role in everyday circumstances.