Linear Algebra - lecture 4

Real Quadratics Forms

Theorem (Sylvester law of inertia)

Let U be a real vector space of finite dimension. For every quadratic form $g: U \to \mathbb{R}$ there is a basis $\beta = (u_1, u_2, \dots u_n)$ such that in the coordinates of β

$$g(u) = 1\cdot x_1^2 + 1\cdot x_2^2 + \dots + 1\cdot x_p^2 - 1\cdot y_{p+1}^2 - \dots - 1\cdot x_s^2 + 0\cdot x_{s+1}^2 + \dots + 0\cdot x_n^2$$

where $(u)_\alpha = (x_1, x_2, \dots, x_n)^T$.

The numbers of $1, -1, 0$ do not depend of the choice of the basis β.

Proof - is.muni.cz/auth/el/1431/jaro 2016/M2110/um/54759737/la2-04_2015.pdf pages 3 ~~...~~ - 6. Not necessary to know.

Question What are coordinates of a vector in a basis $\beta = (u_1, u_2, \dots, u_n)$?

Definition Signature of the real quadratic form is the triple (s_1, s_{-1}, s_0) where s_1 is the number of 1, s_{-1} is the number of -1 and s_0 is the number of 0. $s_0 + s_1 + s_{-1} = \dim U$

<u>Signature of a symmetric matrix</u> $A=(a_{ij})$ is the signature of the corresponding quadratic form $g=\sum_{j,i=1}^{n} a_{ij}\cdot x_i\cdot x_j$.

Two symmetric matrices A and B are <u>congruent</u> if there is a regular matrix P such that $B = P^T A P$.

This is an equivalence. (A is congruent to A, if A is congruent to B, then B is congruent to A, if A is congruent to B, B is congruent to C, then A is congruent to C.)

<u>Consequence of Sylvester law</u>

Symmetric matrices A and B are congruent if and only if they have the same signature.

Classification of quadratic forms over $\mathbb{R}$

Name	Definition	Signature
positive definite	$\forall u \in U \setminus \{\vec{o}\}$ $g(u) > 0$	$s_{-1} = s_0 = 0$
negative definite	$\forall u \in U \setminus \{\vec{o}\}$ $g(u) < 0$	$s_{+1} = s_0 = 0$

Indefinite	$\exists u \; g(u) > 0$ $\exists v \; g(v) < 0$	$s_{+1} > 0, s_{-1} > 0$
positive semidefinite	$\forall u \in \mathcal{U} \; g(u) \geq 0$	$s_{-1} = 0$
negative semidefinite	$\forall u \in \mathcal{U} \; g(u) \leq 0$	$s_{+1} = 0$

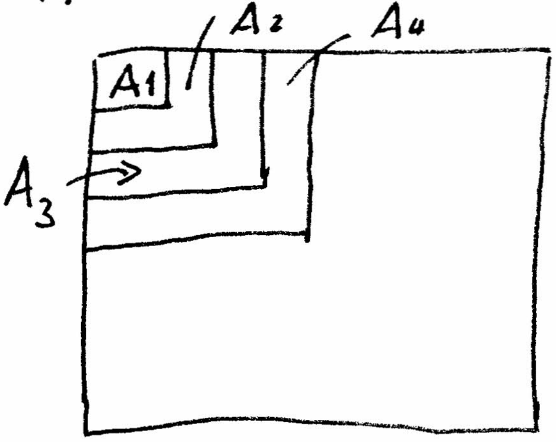
<u>Sylvester criterion</u> Real quadratic form $g : \mathcal{U} \to \mathbb{R}$ is positive definite iff the main minors of its matrix are positive, i.e

$$s_1 > 0, \; s_2 > 0, \; \dots, \; s_n > 0$$

$g : \mathcal{U} \to \mathbb{R}$ is negative definite iff the main minors satisfy

$$(-1)^i s_i > 0 \quad \text{for } i = 1, 2, \dots, n.$$

Main minors of a matrix A



$A = A_n$

$s_i = \det A_i$

<u>Example</u>: $g : \mathbb{R}^3 \to \mathbb{R}$

$$g(x) = 3x_1^2 + 2x_1x_2 + x_2^2 + 4x_1x_3 + 7x_3^2$$

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 7 \end{pmatrix} \qquad \begin{aligned} s_1 &= \det(3) = 3 > 0 \\ s_2 &= \det\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = 3\cdot 1 - 1\cdot 1 = 2 > 0 \\ s_3 &= \det A = 10 > 0 \end{aligned}$$

g is positive definite.

<u>Example</u>: $g : \mathbb{R}^n \to \mathbb{R}$

$g(x) = -x_1^2 - x_2^2 - \dots - x_n^2$ is negative definite according to the definition.

Main minors are
$$s_1 = \det(-1) = -1 < 0$$
$$s_2 = \det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1 > 0$$
$$s_3 = \det\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1 < 0$$
$$\dots\dots\dots$$

<u>Homework</u>: For a given quadratic fom on $\mathbb{R}^3$ find a basis with ± 1 and 0 on the diagonale.

<u>Repeat spaces with scalar product over $\mathbb{R}$ and $\mathbb{C}$.</u>

- norm of a vector $\|u\| = \sqrt{\langle u, u \rangle}$

  $\langle -, - \rangle$ is the notation for the scalar product in my lectures
- Cauchy inequality: $|\langle u, v \rangle| \le \|u\| \, \|v\|$