LINEAR ALGEBRA – WEEK 5

Repeat the notion of the vector space with a scalar product. I will use notation $\langle u, v\rangle$ for the scalar product of two vector.

<u>Ortogonal vectors</u> Vectors $u_1, u_2, \dots, u_k$ are orthogonal if $\langle u_i, u_j\rangle = 0$ for all $i \neq j$.

Gramm - Schmidt Ortogonalization

Let $u_1, u_2, \dots u_k$ be linearly independent in U. Then there are uniquely determined <u>orthogonal</u> vectors $v_1, v_2, \dots, v_k$ such that

$$
\begin{aligned}
v_1 &= u_1 \\
v_2 &= u_2 - a_1 v_1 \\
v_3 &= u_3 - b_1 v_1 - b_2 v_2 \\
&\dots\dots\dots\dots \\
v_k &= u_k - c_1 v_1 - c_2 v_2 - \dots - c_{k-1} v_{k-1}.
\end{aligned}
$$

It means that for linear hulls we have

$$[v_1, v_2, \dots, v_i] = [u_1, u_2, \dots, u_i]$$

for all $i \in \{1, 2, \dots, k\}$.

<u>Proof</u>: By induction. Suppose we have orthogonal $v_1, v_2, \dots v_i$. Then

$$v_{i+1} = u_{i+1} - d_1 v_1 - d_2 v_2 - \dots - d_i v_i$$

Scalar product with v_1 gives

$$0 = \langle v_{i+1}, v_1\rangle = \langle u_{i+1}, v_1\rangle - d_1 \langle v_1, v_1\rangle \underbrace{- d_2 \langle v_2, v_1\rangle - \dots}_{0}$$

$$d_1 = \frac{\langle u_{i+1}, v_1\rangle}{\langle v_1, v_1\rangle}$$

Similarly we compute $d_2, \dots, d_i$.

<u>Example</u>: In $\mathbb{R}^3$ apply Gramm-Schmidt ortogonalization on vectors
$u_1 = (1,0,0)$, $u_2 = (1,2,0)$, $u_3 = (1,1,2)$.

<u>Orthonormal basis in U</u> – a basis $(u_1, u_2, \dots, u_n)$ such that $\langle u_i, u_j \rangle = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$

<u>Theorem</u> Let $\alpha = (u_1, u_2, .. u_n)$ be an orthonormal basis in U. Then

(1) the coordinates of a vector u in the basis α are

$$(u)_\alpha = (\langle u, u_1 \rangle, \langle u, u_2 \rangle, \dots, \langle u, u_n \rangle)^T$$

(2) the scalar product of vectors u and v in the coordinates of α is

$$\langle u, v \rangle = x_1 \overline{y_1} + x_2 \overline{y_2} + \dots + x_n \overline{y_n}$$
$$= (x_1, \dots x_n) \cdot \begin{pmatrix} \overline{y_1} \\ \vdots \\ \overline{y_n} \end{pmatrix} = x^T \cdot \overline{y}$$

where

$$(u)_\alpha = (x_1, \dots x_n)^T, \ (v)_\alpha = (y_1, \dots, y_n)^T.$$

<u>Notation</u> For complex number $z = a + ib$, $a, b \in \mathbb{R}$, the conjugate number is $\overline{z} = a - ib$.

If z is real, then $\bar{z} = z$.

<u>Orthogonal complement</u> $V^\perp$ to a vector subspace $V \subseteq U$ is the vector subspace

$$V^\perp = \{u \in U : (\forall v \in V)\ \langle u, v\rangle = 0\}$$

<u>Theorem</u> Let U be a vector space with a scalar product, $V \subseteq U$ vector subspace. Then

$$U = V \oplus V^\perp$$

which according to definition means

(1) $\forall u \in U\ \exists v \in V\ \exists w \in V^\perp \quad u = v + w$

(2) $V \cap V^\perp = \{\vec{0}\}$.

The condition (1) and (2) are equivalent to the condition

(*) $\forall u \in U\ \exists! v \in V\ \exists! w \in V^\perp \quad u = v + w$.

(The symbol $\exists!$ means: „there is just one.")

<u>Example</u> In R^4 we have a subspace $V = \{x \in \mathbb{R}^4,\ a x_1 + b x_2 + c x_3 + d x_4 = 0\}$ where $(a, b, c, d) \neq (0,0,0,0)$. Compute an orthogonal complement of V.

<u>Orthogonal projection</u> of the space U into a subspace V is the mapping $P: U \to V$ such that for every $u \in U$

$$u = Pu + (u - Pu)$$

where $Pu \in V$ and $u - Pu \in V^\perp$.

So having decomposition

$$u = v + w, \quad v \in V, \quad w \in V^\perp$$

then $Pu = v$.

It is a linear map.

<u>Computation of orthogonal projection</u>

Let $V = [u_1, u_2, .., u_k] \subseteq U$. Then we look for Pu in the form

$$Pu = a_1 u_1 + a_2 u_2 + .. + a_k u_k$$

and we want

$$u - Pu \in V^\perp \quad \text{i.e.} \quad u - Pu \perp u_1, u_2, \dots, u_k$$

So we get a system of k equations for unknowns $a_1, a_2, .., a_k$

$$\left\langle u - \sum_{i=1}^{k} a_i u_i, \, u_j \right\rangle = 0$$

which is

$$\sum_{i=1}^{k} a_i \langle u_i, u_j \rangle = \langle u, u_j \rangle \quad \text{for } j = 1, 2, .., k$$

Example Compute orthogonal projection of the vector $u = (1,2,3,4,5)$ in $\mathbb{R}^5$ into the subspace

$$V = [(3,3,2,1,3), (5,1,4,-1,1)].$$

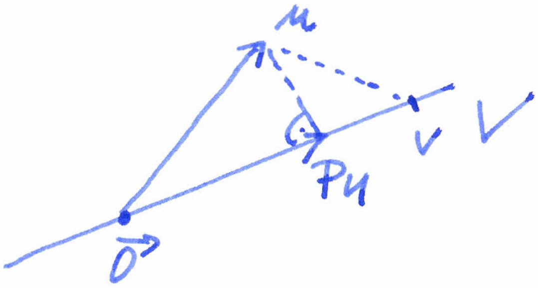
Theorem Properties of orthogonal projection

Let $P : U \to V$ be an ortogonal projection into a subspace V. Then Pu is the only vector with the property

$$\|u - Pu\| = \min\{\|u-v\|, \ v \in V\}.$$

Proof:

$u \in U, \ v \in V$



$$\|u-v\|^2 = \|\underbrace{u-Pu}_{\in V^\perp} + \underbrace{Pu-v}_{\in V}\|^2 = \langle (u-Pu)+(Pu-v), (u-Pu)+(Pu-v)\rangle$$

$$= \langle u-Pu, u-Pu\rangle + \underbrace{\langle u-Pu, Pu-v\rangle}_{=0} + \underbrace{\langle Pu-v, u-Pu\rangle}_{=0} + \langle Pu-v, Pu-v\rangle$$

$$= \|u-Pu\|^2 + \|Pu-v\|^2 \geq \|u-Pu\|^2$$

The equality occurs just for $v = Pu$.