LA - WEEK 9 ORTHOGONAL OPERATORS III

<u>Exercise 3</u> Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation around the axis $x_1 = x_2,\ x_3 = 0$ by the angle $\frac{\pi}{2}$ in such a way, that $\varphi(1,0,0)$ has all three components positive. Find a matrix A such that in standard coordinates

$$\varphi(x) = Ax.$$

<u>Solution:</u> ① We will find ~~the~~matrix $(\varphi)_{\alpha,\alpha}$ in a suitable orthonormal basis and then we can compute

$$A = (\varphi)_{\varepsilon,\varepsilon} = (id)_{\varepsilon,\alpha}(\varphi)_{\alpha,\alpha}(id)_{\alpha,\varepsilon} = P(\varphi)_{\alpha,\alpha}P^{-1}$$

where ε is the standard basis $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

<u>Basis α</u> The first vector is the generator of the axis, i.e.

$$v_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

Other two vector are perpendicul to it and also mutually orthogonal.

$$v_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \qquad v_3 = (0,0,1)$$

$\alpha = (v_1, v_2, v_3)$ is an orthonormal basis and

$\varphi(v_1) = v_1, \quad \varphi(v_3) = -v_2$ (since $\varphi(1,0,0) = (x,y,z)$, $x>0, y>0, z>0$ gives the direction)

$\varphi(v_2) = v_3$. Hence

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \pi/2 & -\sin \pi/2 \\ 0 & \sin \pi/2 & \cos \pi/2 \end{pmatrix}$$

$$P = (id)_{\varepsilon,\alpha} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an orthogonal matrix, since both bases ε and α are orthonormal. Hence

$$(id)_{\alpha,\varepsilon} = (id)^{-1}_{\varepsilon,\alpha} = P^{-1} = P^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now

$$A = P(\varphi)_{\alpha,\alpha}P^{-1} = P(\varphi)_{\alpha,\alpha}P^T = \begin{pmatrix} 1/2 & 1/2 & -\frac{\sqrt{2}}{2} \\ 1/2 & 1/2 & -\frac{\sqrt{2}}{2} \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \end{pmatrix}.$$

2 Another solution uses the fact that the columns of the matrix A are Ae_1, Ae_2, Ae_3. So we compute $\varphi(e_1), \varphi(e_2), \varphi(e_3)$ using geometric considerations.

SELFADJOINT OPERATORS

Let U and V be spaces with scalar product over $\mathbb{R}$ or $\mathbb{C}$. Let $\varphi : U \to V$ be a linear mapping. A linear mapping

$$\varphi^* : V \to U$$

is called <u>adjoint to φ</u>, if for all $u \in U$ and $v \in V$

$$\langle \varphi(u), v \rangle_V = \langle u, \varphi^*(v) \rangle_U .$$

<u>Example 1</u> $U = \mathbb{R}^n$, $V = \mathbb{R}^k$, $\varphi(x) = Ax$, where A is a matrix $k \times n$. Then $\varphi^* : \mathbb{R}^k \to \mathbb{R}^n$ is the mapping $\varphi^*(y) = A^T y$, since for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$ we have

$$\langle \varphi(x), y \rangle = \langle Ax, y \rangle = (Ax)^T \cdot y = x^T A^T y$$

$$\langle x, \varphi^*(y) \rangle = \langle x, A^T y \rangle = x^T (A^T y) = x^T A^T y .$$

<u>Example 2</u> $U = \mathbb{C}^n$, $V = \mathbb{C}^k$, $\varphi(x) = Ax$, $\varphi^*(y) = \bar{A}^T y$, where $\bar{\ }$ stands for complex conjugation. The proof is very similar to the one above.

<u>Theorem</u> Let α be an orthonormal basis in U, β an orthonormal basis in V, $\varphi : U \to V$ and $\varphi^* : V \to U$. Then

$$(\varphi^*)_{\alpha,\beta} = \overline{(\varphi)_{\beta,\alpha}}^{\,T}.$$

<u>Proof</u>: Let $(\varphi)_{\beta,\alpha} = A$, $(\varphi^*)_{\alpha,\beta} = B$. Then for $u \in U$ and $v \in V$ we get

$$\langle \varphi(u), v \rangle = (\varphi(u))_\beta^T \cdot \overline{(v)_\beta} = (A\,(u)_\alpha)^T\, \overline{(v)_\beta} = (u)_\alpha^T A^T \overline{(v)_\beta}$$

$$\langle u, \varphi^*(v) \rangle = (u)_\alpha^T\, \overline{(\varphi^*(v))_\alpha} = (u)_\alpha^T \cdot \overline{B(v)_\beta} = (u)_\alpha^T\, \overline{B}\, \overline{(v)_\beta}$$

Hence $A^T = \overline{B}$, and consequently

$$B = \overline{A}^T.$$

<u>Corollary</u> To every linear mapping $\varphi : U \to V$ there is an adjoint mapping φ^*.

<u>Definition of selfadjoint mapping</u>

Let U be a vector space with a scalar product. An operator $\varphi : U \to U$ is called <u>selfadjoint</u> if $\varphi = \varphi^*$, i.e. for all $u, v \in U$

$$\langle \varphi(u), v \rangle = \langle u, \varphi(v) \rangle.$$

<u>Example 1</u> $U = \mathbb{R}^n$, $A = A^T$ is a symmetric matrix. Then $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(x) = Ax$ is a selfadjoint operator.

<u>Example 2</u> $U = \mathbb{C}^n$, $A = \bar{A}^T$ is a hermitian matrix. Then $\varphi : \mathbb{C}^n \to \mathbb{C}^n$, $\varphi(x) = Ax$ is a selfadjoint operator.

Hermitian matrix $A = \begin{pmatrix} 3 & i & 2-i \\ -i & 4 & 3+4i \\ 2+i & 3-4i & -8 \end{pmatrix}$

<u>Example 3</u> $\varphi : U \to U$ is an orthogonal projection onto the subspace $V \subseteq U$.
The φ is a selfadjoint operator.
Let $u = u_1 + u_2$, $v = v_1 + v_2$, $u_1, v_1 \in V$ and $u_2, v_2 \in V^\perp$. Then

$$\langle \varphi(u), v \rangle = \langle u_1, v \rangle = \langle u_1, v_1 + v_2 \rangle = \langle u_1, v_1 \rangle$$

$$\langle u, \varphi(v) \rangle = \langle u, v_1 \rangle = \langle u_1 + u_2, v_1 \rangle = \langle u_1, v_1 \rangle .$$

Eigenvalues and eigenvectors of selfadjoint operators

1. Every eigenvalue of a selfadjoint operator is real (even if we work over $\mathbb{C}$).
2. Eigenvectors to different eigenvalues are penpedicular each to another.

Proof 1. Let $\varphi(u) = \lambda u$, $u \neq \vec{0}$. Then

$$\lambda \langle u, u\rangle = \langle \lambda u, u\rangle = \langle \varphi(u), u\rangle = \langle u, \varphi(u)\rangle = \langle u, \lambda u\rangle = \bar{\lambda}\langle u, u\rangle$$

which gives $\lambda = \bar{\lambda}$, and so λ is real.

2. Let $\varphi(u_1) = \lambda_1 u_1$, $\varphi(u_2) = \lambda_2 u_2$, $\lambda_1 \neq \lambda_2$, $u_1 \neq \vec{0}$, $u_2 \neq \vec{0}$. Then

$$\lambda_1 \langle u_1, u_2\rangle = \langle \lambda_1 u_1, u_2\rangle = \langle \varphi(u_1), u_2\rangle = \langle u_1, \varphi(u_2)\rangle = $$
$$= \langle u_1, \lambda_2 u_2\rangle = \bar{\lambda}_2 \langle u_1, u_2\rangle =$$
$$= \lambda_2 \langle u_1, u_2\rangle.$$

Hence

$$(\lambda_1 - \lambda_2)\langle u_1, u_2\rangle = 0.$$

Since $\lambda_1 - \lambda_2 \neq 0$, $\langle u_1, u_2\rangle = 0$.

<u>Theorem</u> Let $\varphi : U \to U$ be a selfadjoint operator. Then in U there is an orthonormal basis $\alpha = (u_1, u_2, \ldots, u_n)$ formed by eigenvectors. In this basis

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigenvalues.

(This is an analogue to similar theorem for unitary operators!)

<u>Proof</u> by induction on $\dim U$. I will not give it.

<u>Corollary</u> - spectral decomposition of self-adjoint operators.

Every selfadjoint operator is a linear combination of orthogonal projection on (mutually orthogonal) eigenspaces:

$$\varphi = \lambda_1 P_1 + \cdots + \lambda_k P_k$$

where λ_i are different eigenvalues and P_i is a projection on $\ker(\varphi - \lambda_i \mathrm{id})$.

EXERCISE 1 Find eigenvalues and an orthonormal basis in $\mathbb{R}^3$ formed by eigenvectors of the linear operator $\varphi(x) = Ax$, where

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

APPLICATION TO QUADRATIC FORMS

Every quadratic form on $\mathbb{R}^n$

$$f(x) = \sum_{i,j=1}^{n} a_{ij} \cdot x_i x_j = x^T A x$$

is given by a symmetric matrix A. This matrix determines a selfadjoint operator

$$\varphi : \mathbb{R}^n \to \mathbb{R}^n \text{ given by } \varphi(x) = Ax.$$

Now we can take the (orthonormal) basis $\alpha = (u_1, u_2, \ldots, u_n)$ formed by eigenvectors of the operator φ (of the matrix A). In this basis α the matrix of the quadratic form is given by the corresponding symmetric bilinear form $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$g(u_i, u_j) = u_i^T \cdot A u_j = u_i^T \lambda_j u_j = \lambda_j \langle u_i, u_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \lambda_j & \text{if } i = j \end{cases}$$

So in the coordinates of the orthonormal basis α we have

$$f(y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A.

EXERCISE 2 Find an orthonormal basis in which the quadratic form
$f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x) = 4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$$

has a digonal form.

HOMEWORK 10 Find an orthogonal basis in which the quadratic form $g : \mathbb{R}^3 \to \mathbb{R}^3$

$$g(x) = 17x_1^2 + 4x_1x_2 - 4x_1x_3 + 14x_2^2 + 8x_2x_3 + 14x_3^2.$$

has a diagonal form. (Hint: One of the eigenvalues is 18.)