LA - WEEK 8 EIGENVALUES AND EIGENVECTORS, II

<u>Algebraic multiplicity</u> of an eigenvalue λ_0 is the multiplicity of λ_0 as a root of characteristic polynomial

$$\det(A-\lambda E) = (\lambda-\lambda_0)^k p(\lambda), \quad p(\lambda_0) \neq 0.$$

k is the multiplicity of the root λ_0.

<u>Geometric multiplicity</u> of an eigenvalue λ_0 is

$$\dim \ker(\varphi - \lambda_0 \mathrm{id}).$$

(the dimension of the corresponding eigenspace).

<u>Theorem</u> algebraic multiplicity $\geq$ geom. multiplicity

<u>Example 1</u> $\varphi : \mathbb{R}^3 \to \mathbb{R}^3 \quad \varphi(x) = Ax$

$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ characteristic polynomial is $\det(A-\lambda E) = (2-\lambda)^3$

Eigenvalue 2 has alg. multiplicity 3.

$\ker(A-2E) = \mathbb{R}^3$

Eigenvalue 2 has geom. multiplicity $= \dim \mathbb{R}^3 = 3$.

<u>Example 2</u> $\varphi : \mathbb{R}^3 \to \mathbb{R}^3 \quad \varphi(x) = Bx$

$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ characteristic polynomial is $\det(B - \lambda E) = (2-\lambda)^3$

Eigenvalue 2 has alg. multiplicity 3, but

$\ker(B - 2E) = \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$

has $\dim = 1$. The geom. multiplicity is 1.

<u>Spectrum</u> of a linear operator is the set of all its eigenvalues. The sum of algebraic multiplicities is $\leq n = \dim$ of the space, if we are over $\mathbb{R}$, and is $= n = \dim$ of the space, if we are over $\mathbb{C}$, since every polynomial of the degree n has in $\mathbb{C}$ n roots.

<u>Theorem</u> The eigenvectors corresponding to different eigenvalues are linearly independent.

<u>Theorem</u> Let $\dim U = n$ and $\varphi : U \to U$ is a linear operator, which has n different eigenvalues. Then the corresponding eigenvectors form a basis α in U and

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, .. \lambda_n$ are the eigenvalues.

<u>Proof</u>: Let u_i be a eigenvector to λ_i, $i = 1, 2, .., n$. According to the previous theore[m] $u_1, u_2, .., u_n$ are linearly independent. Since $n = \dim U$, the vectors $u_1, u_2, .., u_n$ form a basis of U. In this basis α

$$\varphi(u_i) = \lambda_i u_i = 0 \cdot u_1 + 0 \cdot u_2 + .. + \lambda_i u_i + 0 u_{i+1} + .. 0 u_n$$

Hence the i-th column of $(\varphi)_{\alpha,\alpha}$ is

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th element}$$

$\square$

<u>Example 3</u> $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ $\varphi(x) = Cx$

$$C = \begin{pmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{pmatrix} \qquad \det(C - \lambda E) = (1-\lambda)(2-\lambda)(3-\lambda)$$

Eigenvalue	Eigen vector
$\lambda_1 = 1$	$v_1 = (1,1,2)^T$
$\lambda_2 = 2$	$v_2 = (1,0,1)^T$
$\lambda_3 = 3$	$v_3 = (1,2,2)^T$

In the basis $\alpha = (v_1, v_2, v_3)$ the matrix of φ is

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Example 4 $\quad U = \mathbb{R}_2[x] \quad \varphi : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$

$\varphi(p) = p'$ (the derivation of p)

We look for eigenvalues

$$p = a_2 x^2 + a_1 x + a_0 \qquad \varphi(p) = 2a_2 x + a_1$$

$$\varphi(p) = \lambda p$$

$$2a_2 x + a_1 = \lambda a_2 x^2 + \lambda a_1 x + \lambda a_0$$

Comparing the coefficients at the same powers of x, we get

$$0 = \lambda a_2$$
$$2a_2 = \lambda a_1$$
$$a_1 = \lambda a_0$$

Which gives either $a_2 = a_1 = a_0 = 0$ and λ arbitrary, or $\lambda = 0$ and $a_2 = a_1 = 0$, a_0 arbitrary.

So the only eigenvalue is $\lambda = 0$.

Algebraic multiplicity

$$(\varphi)_{\mathcal{E},\mathcal{E}} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{for the basis } \mathcal{E} = (x^2, x, 1)$$

$$\det \begin{pmatrix} -\lambda & 0 & 0 \\ 2 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} = \lambda^3$$

Algebraic multiplicity is 3, geom. multiplicity is 1.

Similar matrices: Matrices A and B are similar, if

$$B = P^{-1}AP \qquad (\Leftrightarrow PB = AP)$$

(all matrices are of the form $n \times n$).

Last week we showed that similar matrices have the same characteristic polynomial, hence they have the same eigenvalues with the same algebraic multiplicity. Moreover, their eigenvalues have the same geometric multiplicity.

If $u_1, u_2, \dots u_k$ are eigenvectors for B corresponding to the eigenvalue λ, then $Pu_1, Pu_2, \dots Pu_k$ are eigenvectors for A.

$$A(Pu_i) = (PB)u_i = P(Bu_i) = P(\lambda u_i) = \lambda(Pu_i).$$

ORTHOGONAL AND UNITARY OPERATORS

Definition $\mathcal{U}$ is a vector space with a scalar product. $\varphi : \mathcal{U} \to \mathcal{U}$ is a linear operator with the property

$$\langle \varphi(u), \varphi(v) \rangle = \langle u, v \rangle$$

for all $u, v \in \mathcal{U}$.

If $\mathcal{U}$ is a real vector space, then φ is called orthogonal operator.

If $\mathcal{U}$ is a complex vector space, then φ is called unitary operator.

Properties

(1) Such operators preserves the norms of vectors and the angles between vectors.

$$\|\varphi(u)\| = \sqrt{\langle \varphi(u), \varphi(u) \rangle} = \sqrt{\langle u, u \rangle} = \|u\|$$

$$\cos(\sphericalangle\, \varphi(u), \varphi(v)) = \frac{\langle \varphi(u), \varphi(v) \rangle}{\|\varphi(u)\| \, \|\varphi(v)\|} = \frac{\langle u, v \rangle}{\|u\| \, \|v\|} = \cos(\sphericalangle\, u, v)$$

(2) Such operators $\varphi : \mathcal{U} \to \mathcal{U}$ are isomorphisms.

$$\varphi(u) = 0 \iff \|\varphi(u)\| = 0 \iff \|u\| = 0 \iff u = 0.$$

(3) Such operators map orthonormal bases into orthonormal bases.

<u>Theorem</u> Let $\varphi : U \to U$ be unitary operator (orthogonal operator). Then its matrix A in any orthonormal basis α satisfies

$$A^{-1} = \bar{A}^{T} \iff A\bar{A}^{T} = E$$

where $\bar{A}$ is the matrix formed from A by complex conjugation.

($A^{-1} = A^{T} \iff A A^{T} = E$ over $\mathbb{R}$).

Such matrices are called <u>unitary</u> or <u>orthogonal</u>.

<u>Example</u>:

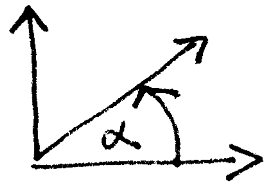
$$A = \begin{pmatrix} 2+i & 3-i \\ i & 4 \end{pmatrix} \qquad \bar{A} = \begin{pmatrix} 2-i & 3+i \\ -i & 4 \end{pmatrix}$$

$$\bar{A}^{T} = \begin{pmatrix} 2-i & -i \\ 3+i & 4 \end{pmatrix}$$

<u>Examples of orthogonal operators</u>

1. $\varphi : \mathbb{R}^2 \to \mathbb{R}^2 \quad \varphi(x) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Geometrically it is rotation around the origin in the plane by the angle α.



2. $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$ geometrically rotation around an axis going through the origin

3. $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$ reflexion with respect to a line or a plane going through the origin.

4. $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad \varphi(x) = Ax$

$$A = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

Determinant of an orthogonal matrix is ± 1.

$$A A^T = E$$
$$\det(A A^T) = \det E$$
$$\det A \cdot \det A^T = 1$$
$$(\det A)^2 = 1 \iff \det A = \pm 1$$

Determinant of a unitary matrix is a complex number with absolute value 1.

$$A \bar{A}^T = E$$
$$\det A \cdot \det \bar{A}^T = 1$$
$$(\det A) \cdot \overline{(\det A)} = 1$$
$$|\det A|^2 = 1$$