LR rozklad

$A = L \cdot R \qquad Ax = b \;\dots\; L \cdot \underbrace{Rx}_{y} = b$

$Ly = b$ – systém s dolní △ maticí

$y_1 = \dots$
$y_2 = \dots$

Pak $Rx = y$ – systém s horní △ maticí

$x_n = \dots$
$x_{n-1} = \dots$

kvě 3-7:49

Př.

$2x_1 + 4x_2 - x_3 = -5$
$x_1 + x_2 - 3x_3 = -9$
$4x_1 + x_2 + 2x_3 = 9$

$A = \begin{bmatrix} 2 & 4 & -1 \\ 1 & 1 & -3 \\ 4 & 1 & 2 \end{bmatrix} \overset{(-1/2)\,(-2)}{\sim} \begin{bmatrix} 2 & 4 & -1 \\ 0 & -1 & -5/2 \\ 0 & -7 & 4 \end{bmatrix} \overset{(-7)}{\sim} \begin{bmatrix} 2 & 4 & -1 \\ 0 & -1 & -5/2 \\ 0 & 0 & 43/2 \end{bmatrix} = R$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & 7 & 1 \end{bmatrix} \qquad Ly = b \qquad L \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -9 \\ 9 \end{bmatrix}$

$y_1 = -5$
$y_1/2 + y_2 = -9 \Rightarrow y_2 = -9 + \frac{5}{2} = -13/2$
$2y_1 + 7y_2 + y_3 = 9 \Rightarrow y_3 = 9 + 10 + 91/2 = 129/2$

$Rx = y \qquad R \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -13/2 \\ 129/2 \end{bmatrix}$

$43/2\, x_3 = 129/2 \Rightarrow \underline{x_3 = 3}$
$-x_2 - 5/2\, x_3 = -13/2 \Rightarrow x_2 = 13/2 - 15/2 = -1$
$2x_1 + 4x_2 - x_3 = -5 \Rightarrow 2x_1 = -5 + 4 + 3 = 2 \Rightarrow \underline{x_1 = 1}$

kvě 3-8:15

LR rozklad s výměnou řádků a výběrem ved. prvku (pivotal)

$A = \begin{bmatrix} 2 & 4 & -1 \\ 1 & 1 & -3 \\ 4 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 4 & 1 & 2 \\ 1 & 1 & -3 \\ 2 & 4 & -1 \end{bmatrix} \overset{(-1/4)\,(-1/2)}{\sim} \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3/4 & -7/2 \\ 0 & 7/2 & -2 \end{bmatrix} \sim \begin{bmatrix} 4 & 1 & 2 \\ 0 & 7/2 & -2 \\ 0 & 3/4 & -7/2 \end{bmatrix} \overset{(-3/14)}{\sim}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 3/14 & 1 \end{bmatrix} = L \qquad \sim \begin{bmatrix} 4 & 1 & 2 \\ 0 & 7/2 & -2 \\ 0 & 0 & -43/14 \end{bmatrix} = R$

$P = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \sim \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \qquad L \cdot R = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 4 & -1 \\ 1 & 1 & -3 \end{bmatrix} = P \cdot A$

$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

kvě 3-8:33

Věta o rozkladu na součin dolní a horní △ matice

Dk: - nenulové hl. minory
- indukcí

1) $n = 1 \Rightarrow A = [a_{11}] \quad L = [1],\ R = [a_{11}] \quad \left(L = [c],\ R = \left[\frac{a_{11}}{c}\right] \right)$

2) .. tvrzení platí pro $n-1$

$A = \begin{bmatrix} A_{n-1} & b \\ d & a_{nn} \end{bmatrix}$ c – sloupc. vektor $(n-1)\times 1$, d – řádk. –"– $1 \times (n-1)$; $A_{n-1} = L_{n-1} \cdot R_{n-1}$, L_{n-1}, R_{n-1} – regulární

$L \cdot R = A$

$R = \begin{bmatrix} R_{n-1} & x \\ O_{1\times(n-1)} & z \end{bmatrix},\ L = \begin{bmatrix} L_{n-1} & 0 \\ y & w \end{bmatrix}, \quad \begin{bmatrix} L_{n-1} & 0 \\ y & w \end{bmatrix} \begin{bmatrix} R_{n-1} & x \\ 0 & z \end{bmatrix} = \begin{bmatrix} L_{n-1} R_{n-1} & L_{n-1} x \\ y \cdot R_{n-1} & y \cdot x + wz \end{bmatrix} = \begin{bmatrix} A_{n-1} & c \\ d & a_{nn} \end{bmatrix}$

$L_{n-1} x = c \Rightarrow x = L_{n-1}^{-1} c$
$y \cdot R_{n-1} = d \Rightarrow y = d \cdot R_{n-1}^{-1}$

$y \cdot x + wz = a_{nn} \Rightarrow wz = a_{nn} - yx \qquad w = 1,\ z = a_{nn} - yz \qquad \left(w = c,\ z = \frac{1}{c}(a_{nn} - yz) \right)$

kvě 3-8:48

LR rozklad sym. matice A – Chol. rozklad

$A = T^T \cdot T$, T – horní △, $T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & \dots \\ 0 & t_{22} & t_{23} & t_{24} & \dots \\ 0 & 0 & t_{33} & t_{34} & \dots \\ \vdots & & & & \\ 0 & \dots & 0 & t_{ii} & t_{i,i+1} \dots \\ \vdots & & & \ddots & t_{nn} \end{bmatrix}$

$A = \begin{bmatrix} t_{11} & 0 & 0 & \dots \\ t_{12} & t_{22} & 0 & \\ t_{13} & t_{23} & t_{33} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} & t_{13} & \dots \\ 0 & t_{22} & t_{23} & \dots \\ 0 & 0 & t_{33} & \dots \\ \vdots & & & \ddots \end{bmatrix}$

$t_{11} \cdot t_{11} = a_{11} \Rightarrow t_{11} = \sqrt{a_{11}}$

1. ř.: $a_{1k} = t_{11} \cdot t_{1k} \Rightarrow t_{1k} = \frac{a_{1k}}{t_{11}}$

2. ř.: $a_{22} = t_{12}^2 + t_{22}^2$

$t_{22} = \sqrt{a_{22} - t_{12}^2}$

$t_{12} t_{13} + t_{22} \cdot t_{23} = a_{23} \Rightarrow t_{23} = \frac{1}{t_{22}} (a_{23} - t_{12} t_{13})$

$t_{12} \cdot t_{1k} + t_{22} \cdot t_{2k} = a_{2k} \Rightarrow t_{2k} = \frac{1}{t_{22}} (a_{2k} - t_{12} \cdot t_{1k})$

i-tý řádek:

$t_{ii} = \sqrt{a_{ii} - \sum_{j=1}^{i-1} t_{ji}^2}$

$t_{ik} = \frac{1}{t_{ii}} \left(a_{ik} - \sum_{j=1}^{i-1} t_{ji} t_{jk} \right)$

kvě 3-9:07

Př: $x_1 + 2x_2 - x_3 = 4$
$2x_1 + 2x_2 + 4x_3 = 1$
$-x_1 + 4x_2 + 8x_3 = -8$

$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ -1 & 4 & 8 \end{bmatrix}$

$t_{11} = \sqrt{1} = 1$
$t_{1k} = \frac{1}{t_{11}} a_{1k}$
$t_{22} = \sqrt{a_{22} - t_{12}^2} = \sqrt{2 - 2^2} = \sqrt{-2} = \sqrt{2}\, i$
$t_{23} = \frac{1}{t_{22}} (a_{23} - t_{12} t_{13}) = \frac{1}{\sqrt{2}\, i} (4 - 2(-1)) = \frac{1}{\sqrt{2}\, i} \cdot 6 = -3\sqrt{2}\, i$
$t_{33} = \sqrt{a_{33} - t_{13}^2 - t_{23}^2} = \sqrt{8 - 1 - (-3\sqrt{2}\, i)^2} = \sqrt{7 + 18} = 5$

$A = T^T \cdot T$

$Ax = T^T \underbrace{T \cdot x}_{y} = b \Rightarrow T^T y = b$, $Tx = y$

$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & \sqrt{2}\, i & -3\sqrt{2}\, i \\ 0 & 0 & 5 \end{bmatrix}$

$T^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & \sqrt{2}\, i & 0 \\ -1 & -3\sqrt{2}\, i & 5 \end{bmatrix}$

$\underline{y_1 = 4}$
$2y_1 + \sqrt{2}\, i\, y_2 = 1$
$\sqrt{2}\, i\, y_2 = -7$
$\underline{y_2 = \frac{7}{2} \sqrt{2}\, i}$

$-y_1 - 3\sqrt{2}\, i\, y_2 + 5y_3 = -8$
$-4 + 21 + 5y_3 = -8$
$5y_3 = -25$
$\underline{y_3 = -5}$

$Tx = y$
$5 \cdot x_3 = -5 \Rightarrow \underline{x_3 = -1}$
$\sqrt{2}\, i\, x_2 - 3\sqrt{2}\, i (-1) = \frac{7}{2} \sqrt{2}\, i$
$x_2 + 3 = \frac{7}{2} \Rightarrow \underline{x_2 = \frac{1}{2}}$
$x_1 + 2x_2 - x_3 = 4$
$x_1 + 1 + 1 = 4$
$\underline{x_1 =}$

kvě 3-9:21