Radon transforms

The mathematics behind Computertomography

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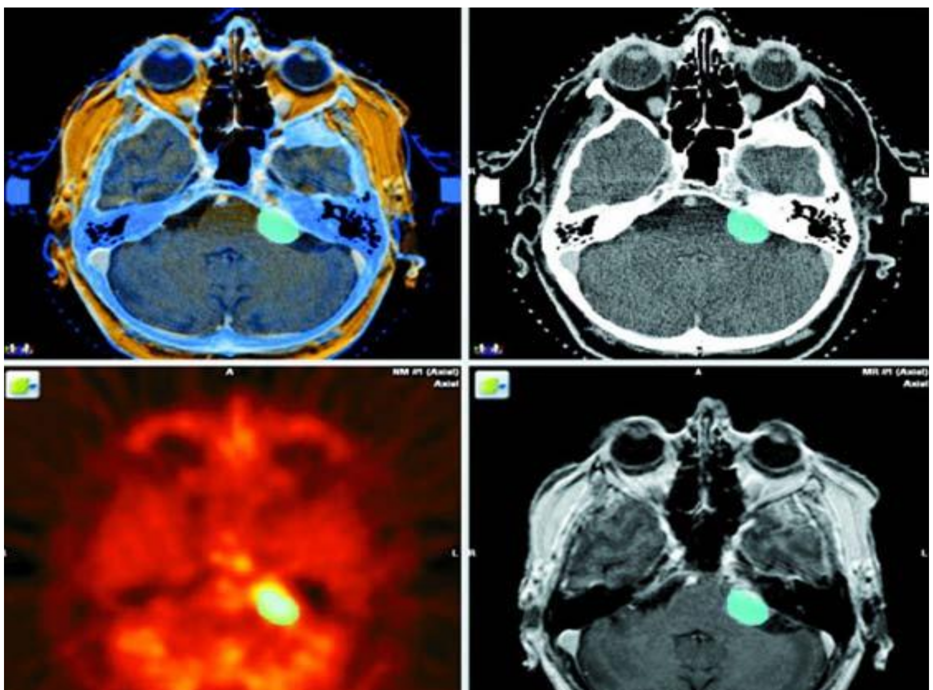
International Summer academic course 2008,

„Modelling and Simulation of Technical Processes“



first row left: image fusion of CT and MRI

first row right: Computertomography (CT)

second row right: Magnetic Resonance Imaging (MRI)

second row left: tumor search

Historical remarks

- C. Röntgen (1895) – X-rays
- J. Radon (1917)– Mathematical Model
- G. Grossmann (1935) – Tomography
- G. Hounsfield, McCormack (1972) – Computerized assited tomography (CAT scan)

Abb. 1.4: moderner CT Scanner

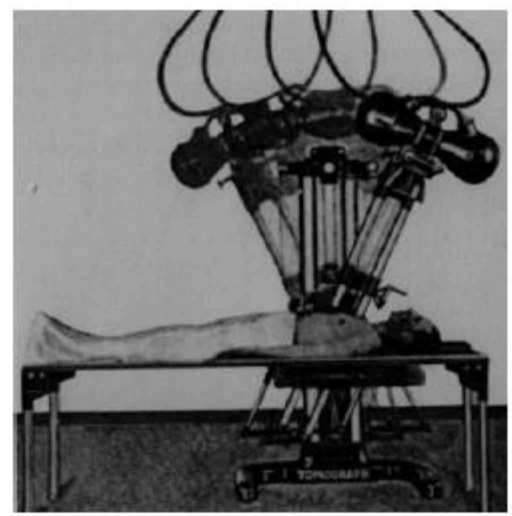


Abb. 1.1: historischer Grossmann-Tomograph

Why does it work?
The physical priniples.

- Tomography means *slice imaging,*
- Quantification of the tendency of objects to absorb or scatter x-rays by the *attunation coefficient,* involving *Beer‘s law.*

Model

- *No refraction or diffraction:* X-ray beams travel along straight lines that are not „bent“ by the objects they pass through.

  This is a good approximation because x-rays have very high energies, and therefore very short wavelength.

- *The X-rays used are monochromatic:* The waves making up the x-ray beams are all of the same frequency.

  This is not a realistic assumption, but it is needed to construct a *linear model* for the measurements.

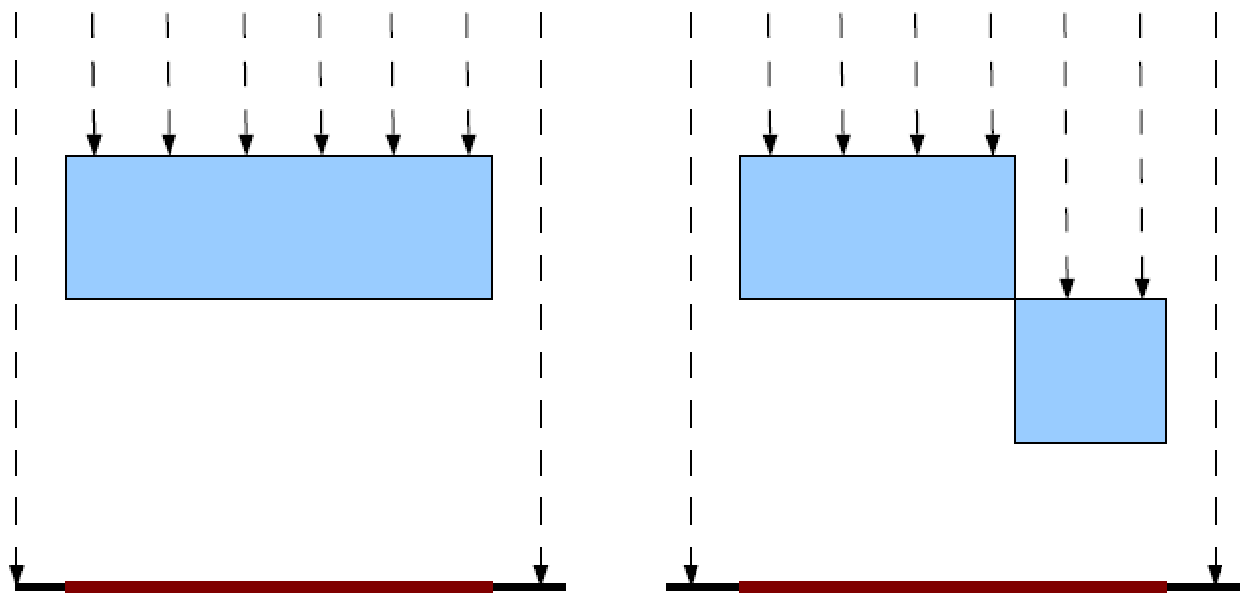
Model

- *Beer's law:* Each material encountered has a characteristic linear attenuation coefficient μ for x-rays of a given energy.
- The *intensity*, *I* of the x-ray beam satisfies Beer's law:

$$\frac{dI}{ds} = -\mu(x)I$$

Here, s is the arc-length along the straight line trajectory of the x-ray beam.

The failure to distinguish objects

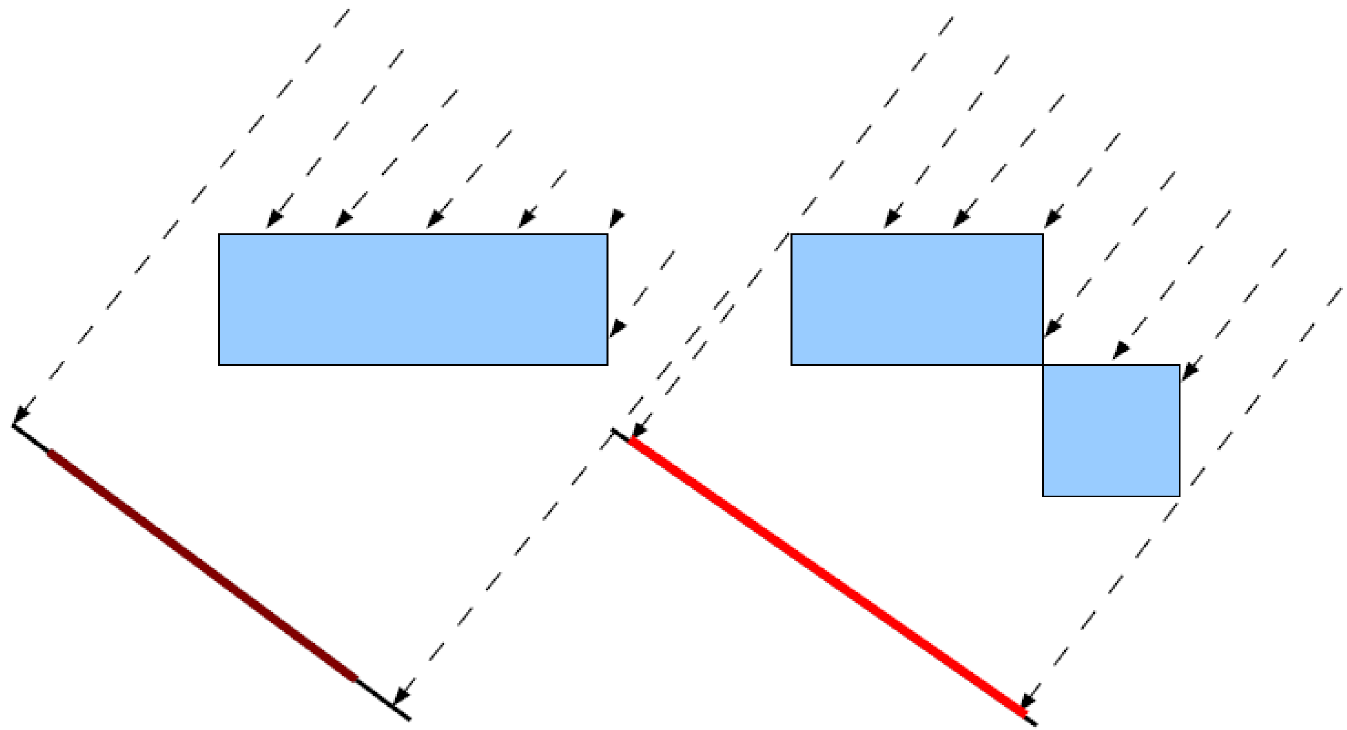


one object

two objects

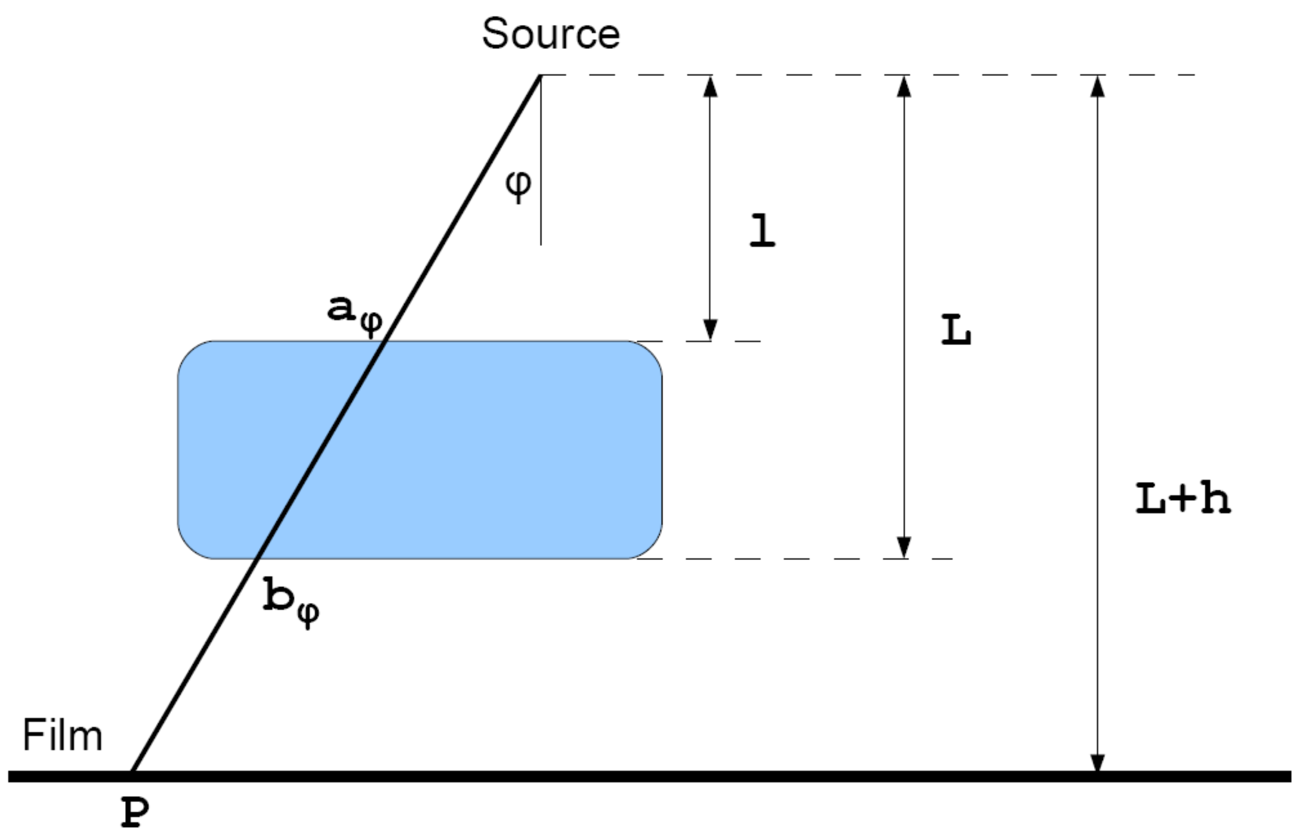
same projection

Solution: more directions



Different angles lead to different projections.
The more directions from which we make measurement, the more arrangements of objects we can distinguish.

Analysis of a Point Source Device, 2D model, what do we measure?



A point source device for measuring line integrals of the attenuation coefficient.

2D model, what do we measure?

Beer‘s law: $$\frac{dI}{dr} = -\left(\mu_a(r,\phi) + \frac{1}{r}\right) I$$

First order ordinary differential equation for the intensity I with boundary condition I at $r=r_0>0$ equals I_0.

$$\ln \frac{I(r_\phi,\phi)}{I(r_0,\phi)} = \ln \frac{r_0}{r_\phi} - \int_{a_\phi}^{b_\phi} \mu_a(s,\phi)\, ds$$

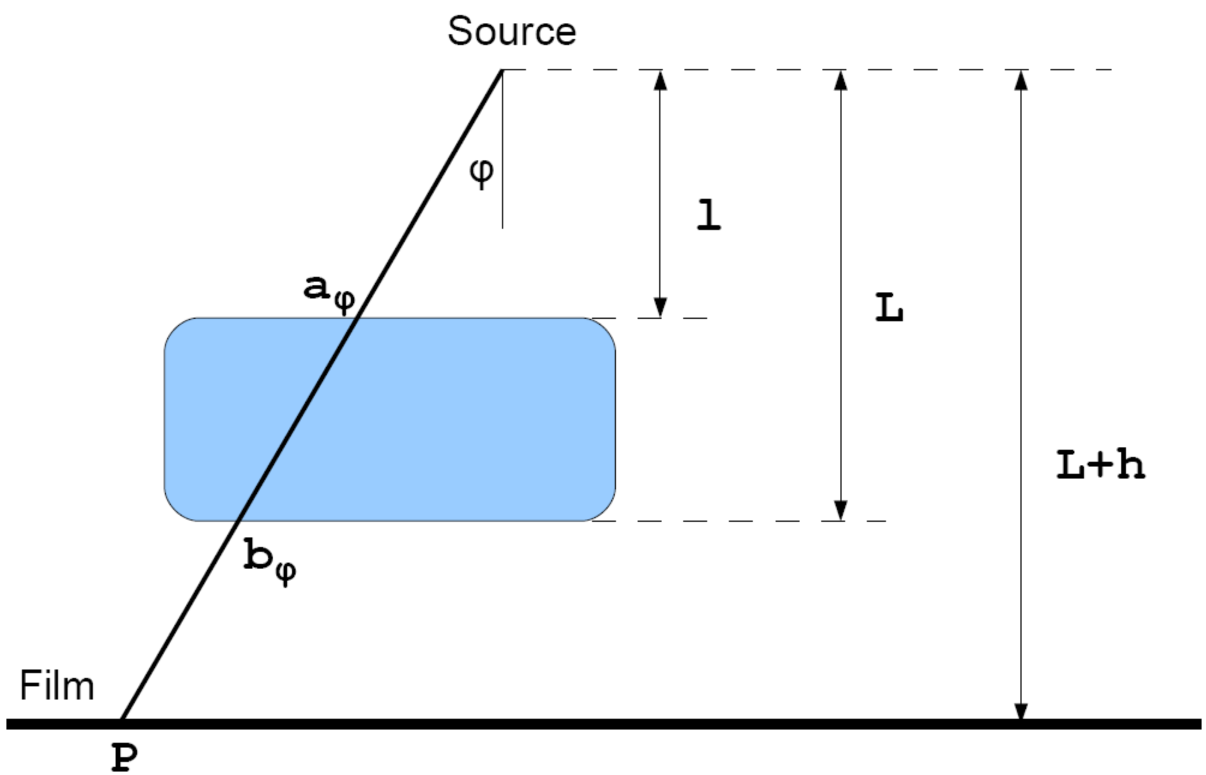
2D model, what do we measure?

$$\ln \frac{I(r_\phi, \phi)}{I(r_0, \phi)} = \ln \frac{r_0}{r_\phi} - \int_{a_\phi}^{b_\phi} \mu_a(s, \phi)\, ds$$

$$a_\phi = \frac{l}{\cos \phi}$$

$$b_\phi = \frac{L}{\cos \phi}$$

$$r_\phi = \frac{L + h}{\cos \phi}$$



Analysis if a Point Source Device

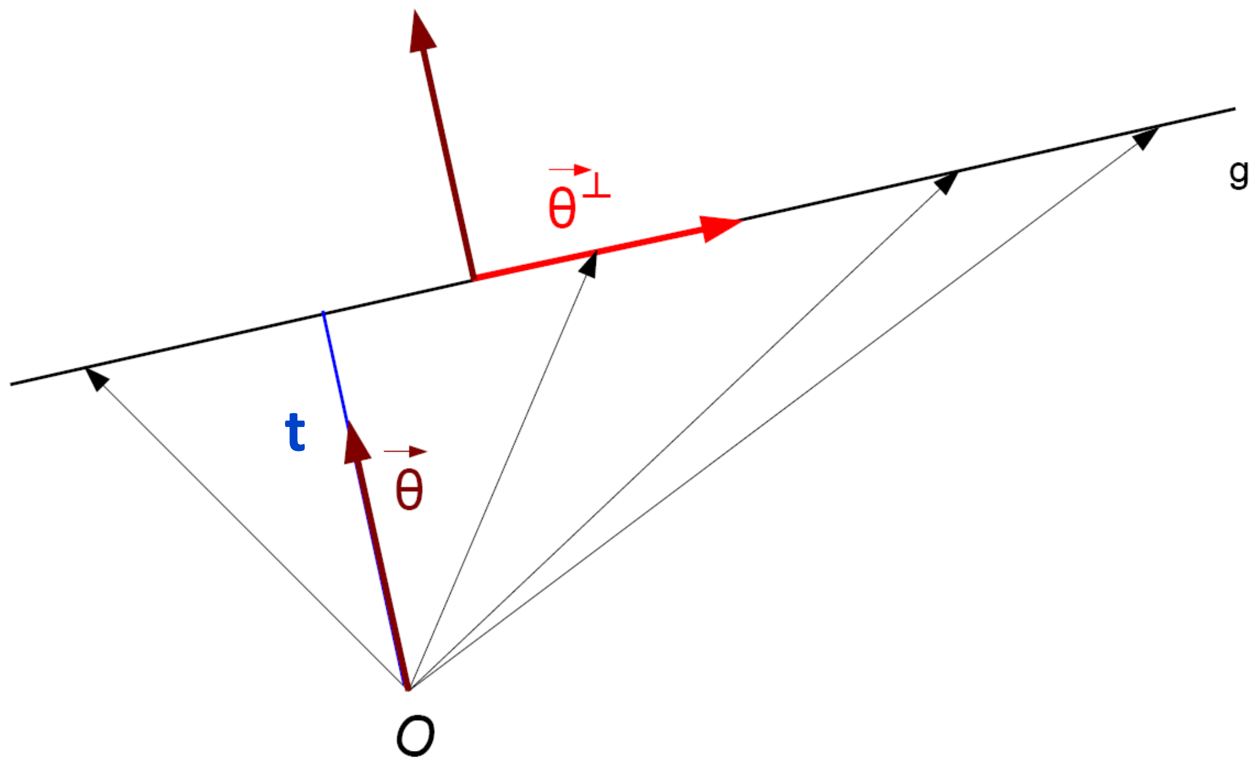
The density of the developed film at a point is proportional to the logarithm of the total energy incident at that point:

density of the film = $\gamma \times \log$ (total energy intensity),
where γ is a constant, we obtain:

$$-\int_{a_\phi}^{b_\phi} \mu_a(s,\, \phi)\, ds = \gamma^{-1}\delta(\phi) - \ln\left[\frac{I_0 \cos^2\phi}{2\pi(L+l)}\right].$$

This formula expresses the measurements as *linear function* of the attunation coefficient.

Oriented lines

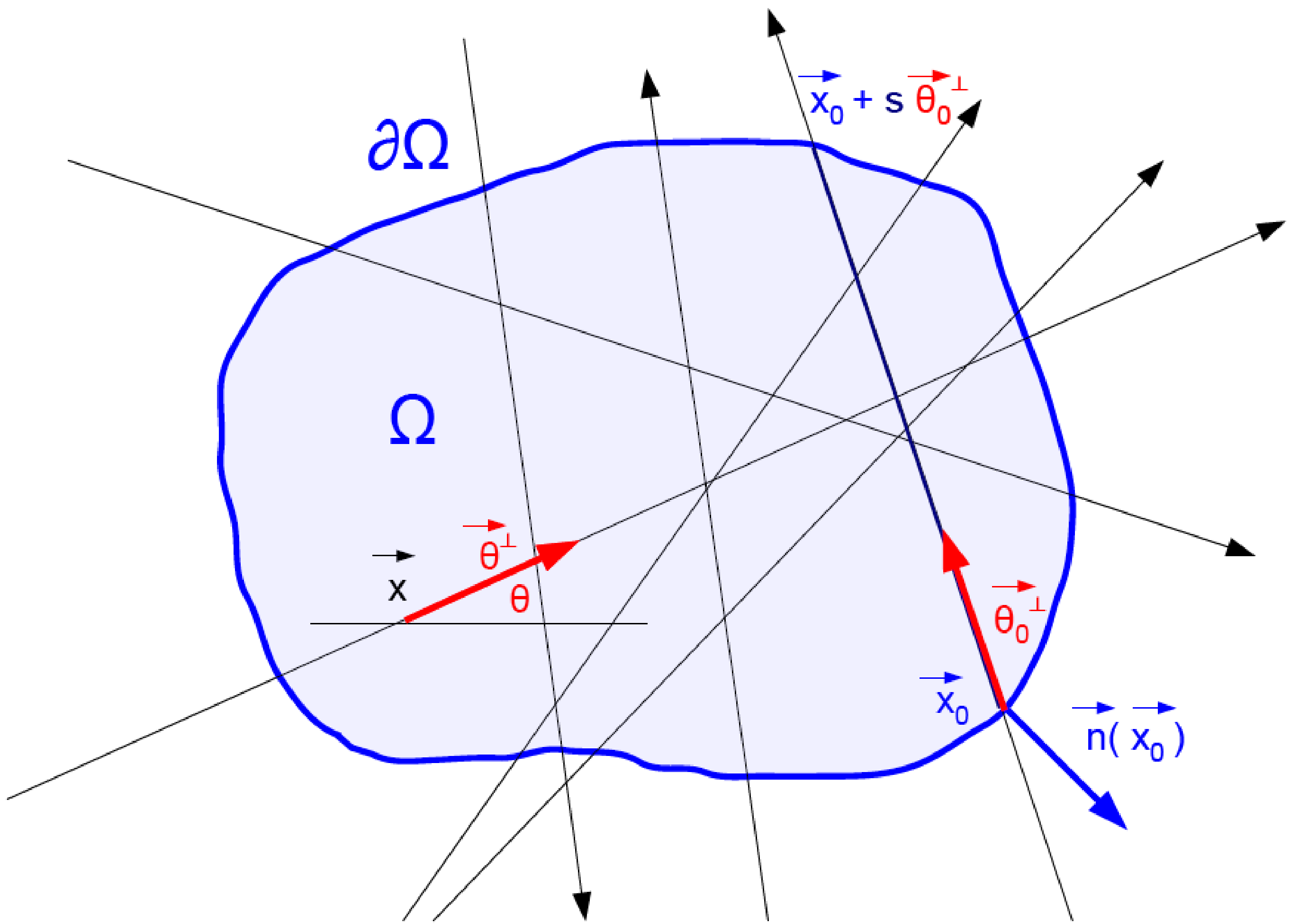


$$g_{t,\vec{\theta}} := \{\vec{x} \in \mathbb{R}^2 : \langle \vec{x}, \vec{\theta}^{\perp} \rangle = t\} = \{\vec{x} = s\vec{\theta}^{\perp} + t\vec{\theta}\}$$

t is the distant of the line from the origin,
s is the parameter of the line.

Radon transform

$$\vec{\theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \vec{\theta}^{\perp} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

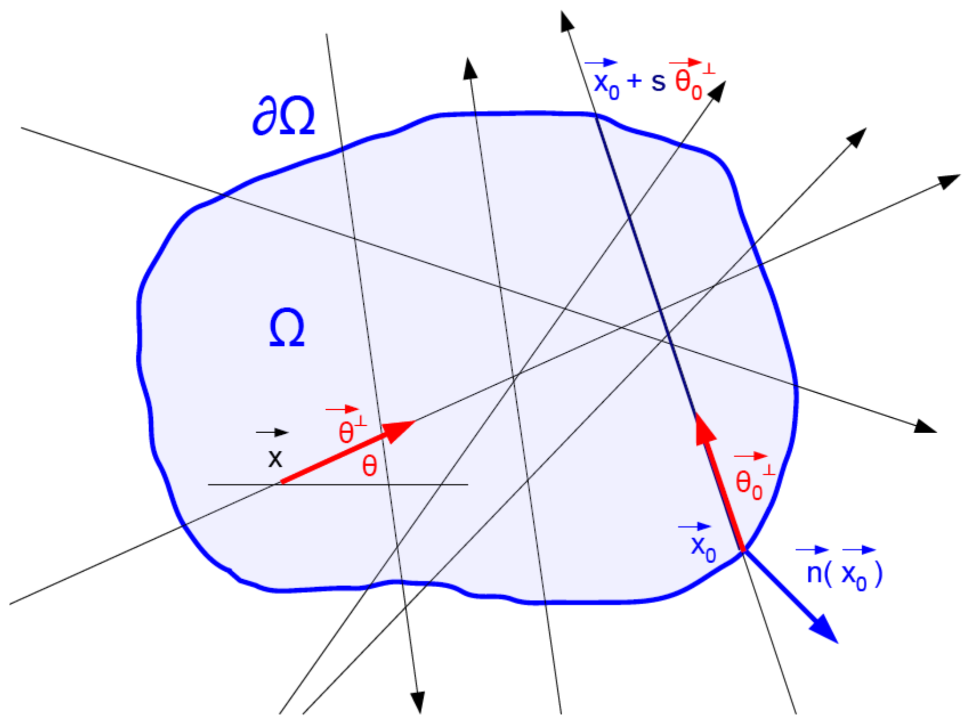


Radon transform

$$\mathcal{R}f(t,\vec{\theta}) = \int_{g_{t,\vec{\theta}}} f\,ds = \int_{-\infty}^{\infty} f(s\vec{\theta}^{\perp} + t\vec{\theta})\,ds$$

$$\vec{\theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\vec{\theta}^{\perp} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$



Radon transform

$$\mathcal{R}f(t,\vec{\theta}) = \int_{g_{t,\vec{\theta}}} f\,ds = \int_{-\infty}^{\infty} f(s\vec{\theta}^{\perp} + t\vec{\theta})\,ds$$

- The Radon transform can be defined, a priori for a function, f whose restriction to each line is *locally integrable* and

$$\int_{-\infty}^{\infty} |f(s\vec{\theta}^{\perp} + t\vec{\theta})|\,ds < \infty, \quad \text{for all} \quad (t,\vec{\theta}) \in \mathbb{R} \times S^1.$$

- This is really two different conditions:

1. The function is regular enough so that restricting it to any line gives a locally integrable function,
2. The function goes to zero rapidly enough for the improper integrals to converge.

In applications functions of interest are usually piecewise continuous and zero outside of some disk.

Ex 2

Properties of the Radon transform

- The Radon transform is linear:

$$\mathcal{R}(af) = a\mathcal{R}f \quad \text{for all} \quad a \in \mathbb{R},$$

$$\mathcal{R}(f+g) = \mathcal{R}f + \mathcal{R}g.$$

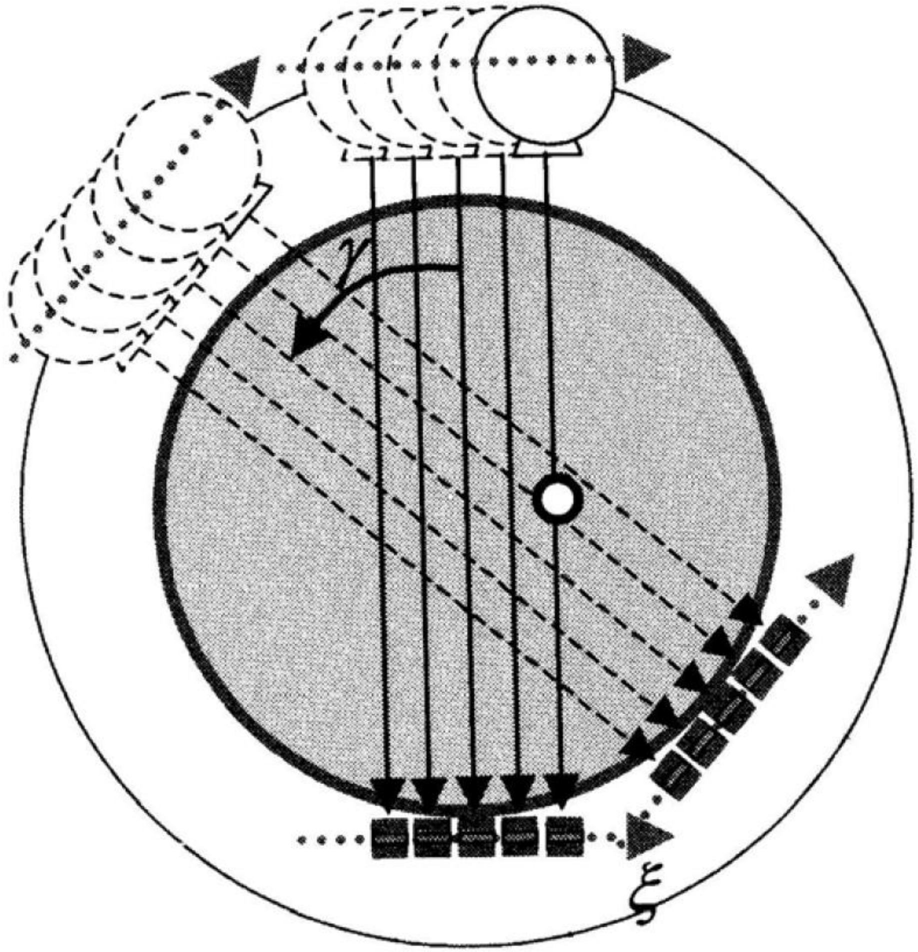
- The Radon transform of f is an even function:

$$\mathcal{R}f(t,\vec{\theta}) = \mathcal{R}f(-t,\, -\vec{\theta}).$$

- The Radon transform is monotone: if f is a non-negative function then

$$\mathcal{R}f(t,\vec{\theta}) \geq 0 \quad \text{for every} \quad (t,\, \vec{\theta}).$$

Pencilgeometry (Nadelstrahlgeometrie)

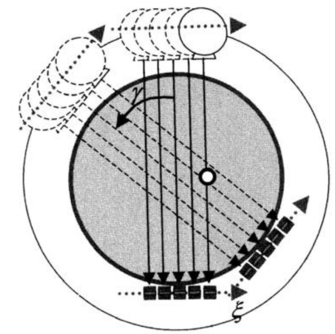


Buzug, Einführung in die Computertomography, Springer Verlag , 2004

Back-projection formula

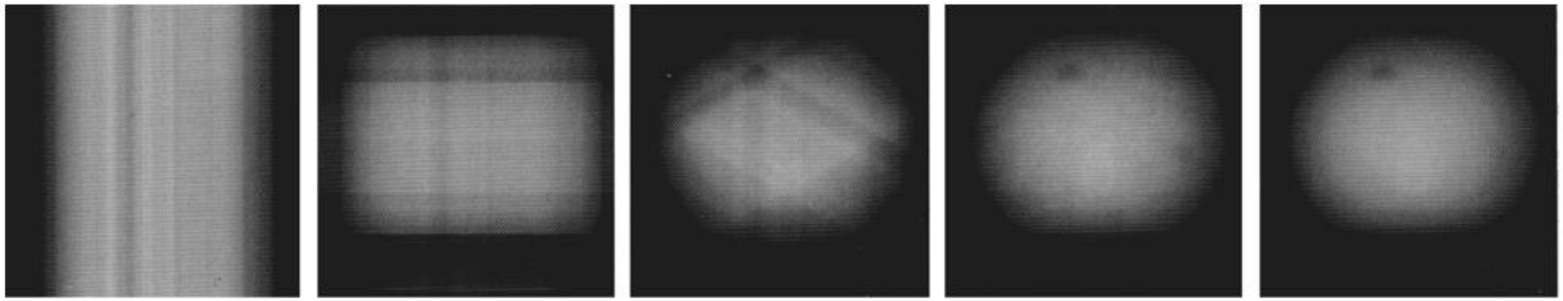
- It is difficult to use the line integrals of a function directly to reconstruct the function:

$$\tilde{f}(\vec{x}) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}f(\langle \vec{x}, \vec{\theta} \rangle, \theta)\, d\theta$$



- Results of the recontruction by back-projection

What is that??



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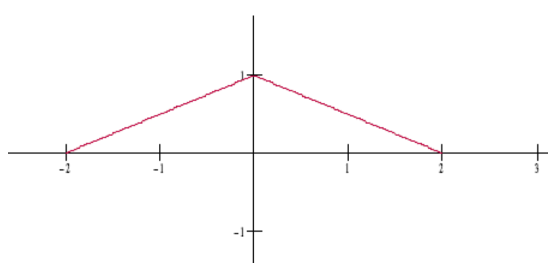
Fourier transform in 1D

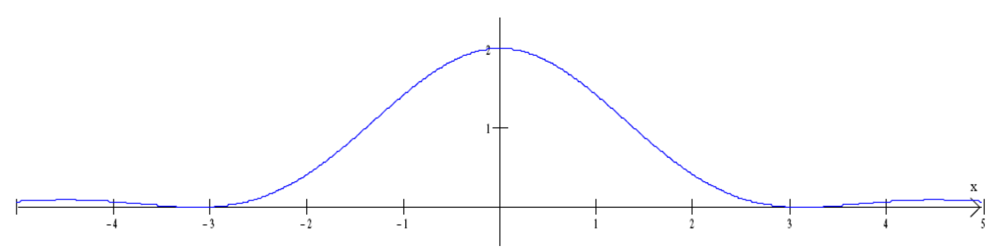
- The Fourier transform of an absolutely integrable function f, defined on the real line, is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx$$

- Suppose that the Fourier transform of f is again an absolutely integrable function then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i\xi x}\, d\xi.$$





Square integrable functions

- A (complex-valued) function f, defined on $\mathbb{R}^n$, is square integrable if

$$||f||_{L^2}^2 = \int_{\mathbb{R}^n} |f(\vec{x})|^2 \, d\vec{x} < \infty.$$

- Examples: The function $f(x) = (1+|x|)^{-\frac{3}{4}}$ is not absolutely integrable but square integrable, the function

$$g(x) = \begin{cases} \frac{1}{\sqrt{|x|}}, & |x| < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

is absolutely integrable but not square integrable.

Ex 3

Fourier transform in nD

- The Fourier transform of an absolutely integrable function is defined by

$$\hat{f}(\vec{\xi}) = \int_{\mathbb{R}^n} f(\vec{x})\, e^{-i\langle\vec{\xi},\vec{x}\rangle}\, d\vec{x}$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1,\ldots,x_n) e^{-ix_1\xi_1} dx_1 \ldots e^{-ix_n\xi_n} dx_n.$$

- Let $f \in L^2(\mathbb{R}^n)$ and define

$$f_R(\vec{x}) = \frac{1}{(2\pi)^n} \int_{|\xi|<R} \hat{f}(\vec{\xi})\, e^{i\langle\vec{x},\vec{\xi}\rangle}\, d\vec{\xi}$$

  then $f = \lim_{R\to\infty} f_R.$

- Parseval formula: If f is square integrable then

$$\int_{\mathbb{R}^n} |f(\vec{x})|^2\, d\vec{x} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{f}(\vec{\xi})|^2\, d\vec{\xi}.$$

Central Slice Theorem

- Let f be an absolutely integrable function. For any real number r and unit vector $\vec{\theta}$, we have the identity

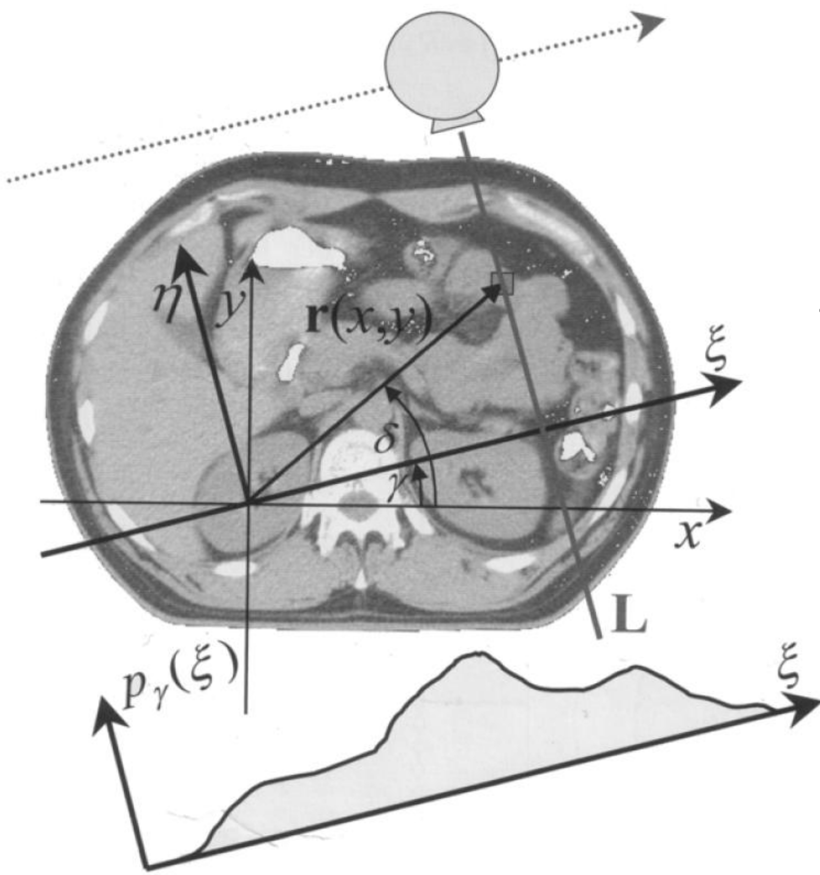
$$\widetilde{\mathcal{R}f}(t,\,\theta) = \int_{-\infty}^{\infty} \mathcal{R}f(t,\,\vec{\theta})\,e^{-itr}\,dt = \hat{f}(r\vec{\theta}).$$

- For a given vector $\vec{\xi} = (\xi_1,\,\xi_2)$ the inner product , $\langle \vec{x},\,\vec{\xi}\rangle$ is constant along any line perpendicular to the direction $\vec{\xi}$. The central slice theorem interprets the compution of the Fourier transform of $\vec{\xi}$ as a two-step process:

1. First, integrate the function along lines perp. to $\vec{\xi}$.
2. Compute the *one-dimensional* Fourier transform of this function of the affine parameter.

Ex 4

Inverse Radon Transform and Central Slice Theorem



1. Choose a line L, determined by the direction $\vec{\theta}$ (Cartesian coord.) or by the angle γ . Then the coorinate axis ξ shows in the same direction.

2. Integrate along all lines perp. to $\vec{\theta}$ (those lines are parallel to the cood. axis η. We obtain the Radon transform $p_\gamma(\xi)$.

3. Compute the *one dimensional* Fourier transform $\hat{p}_\gamma(q)$ of $p_\gamma(\xi)$

4. With u=q cos γ and v=q sin γ, we get F(u,v) = $\hat{p}_\gamma(q)$ and f(x,y) is equal to the *2D* inverse Fourier transform of F(u,v).

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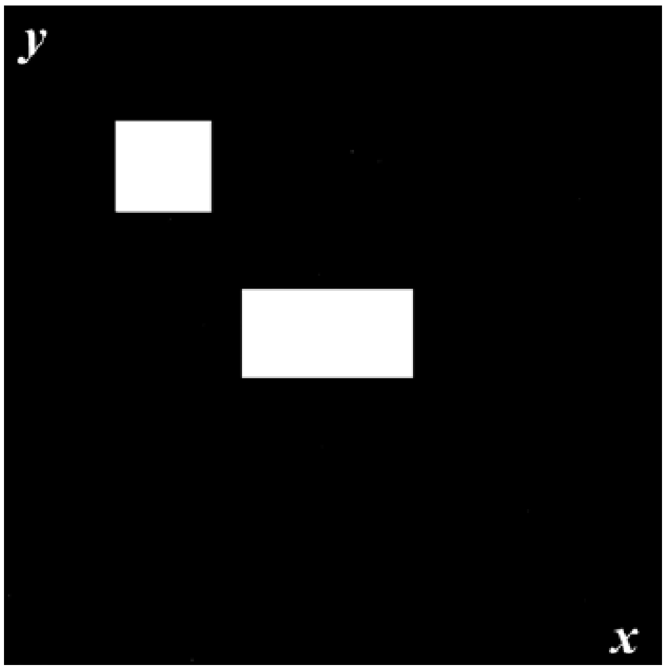
Radon Transform



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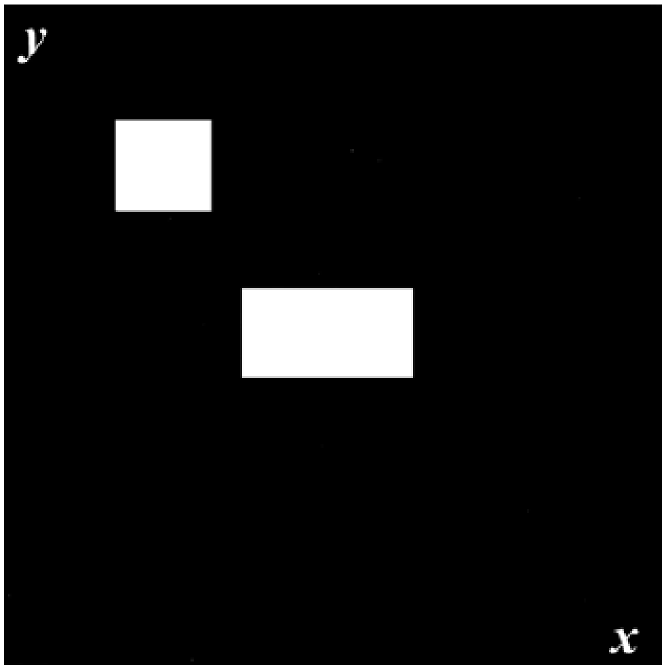
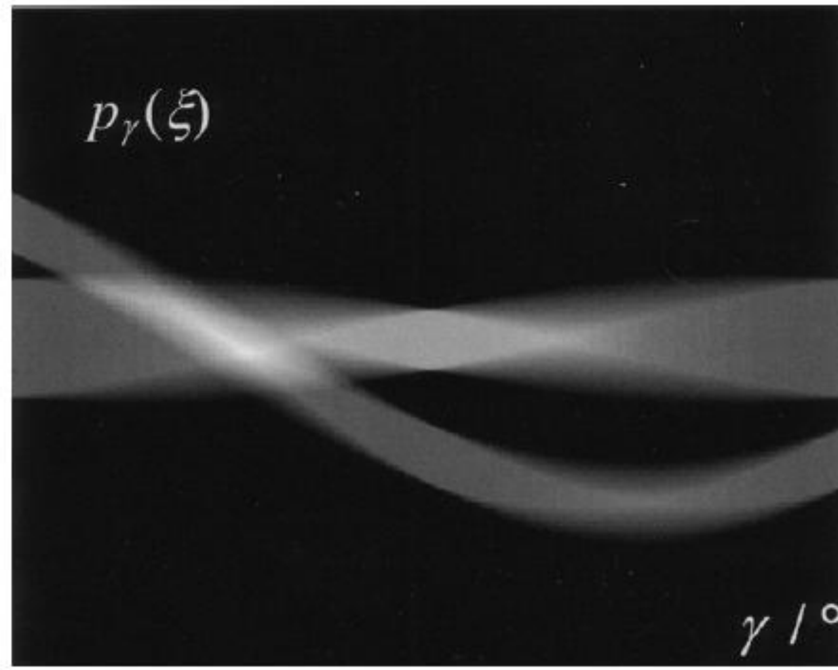
Radon Transform





In Cartesian coordinates.

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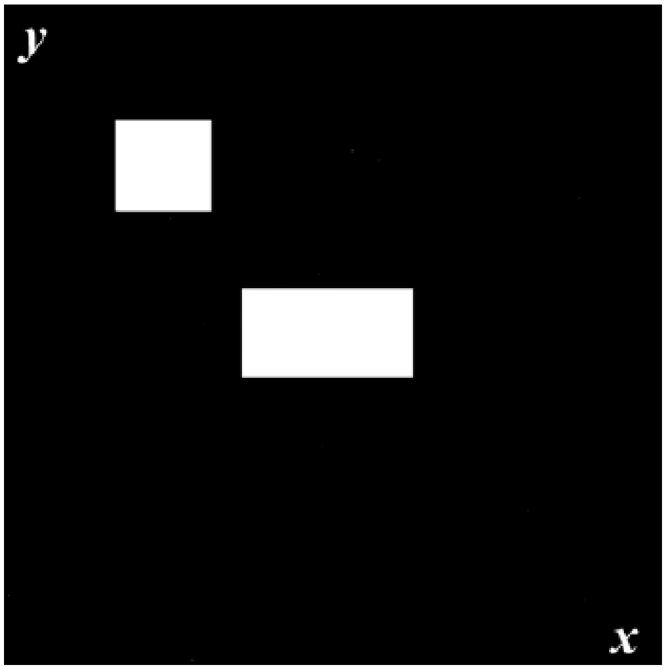
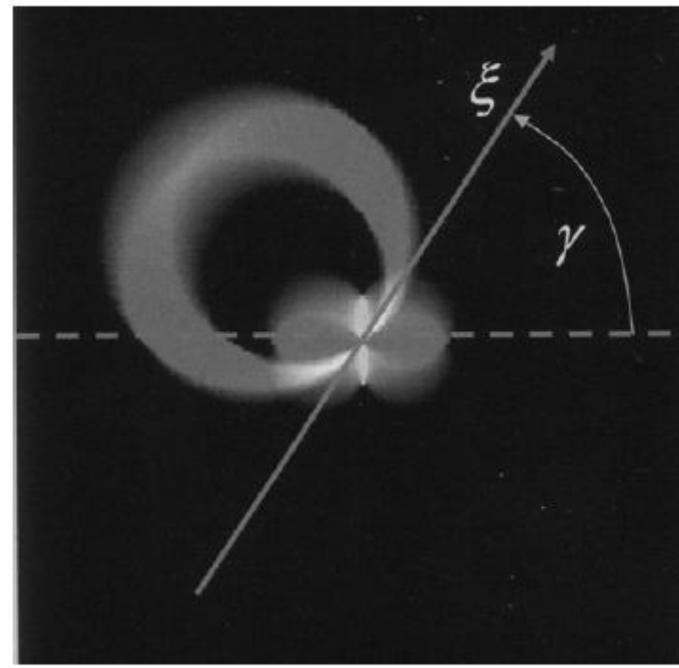
Radon Transform





In Polar coordinates.

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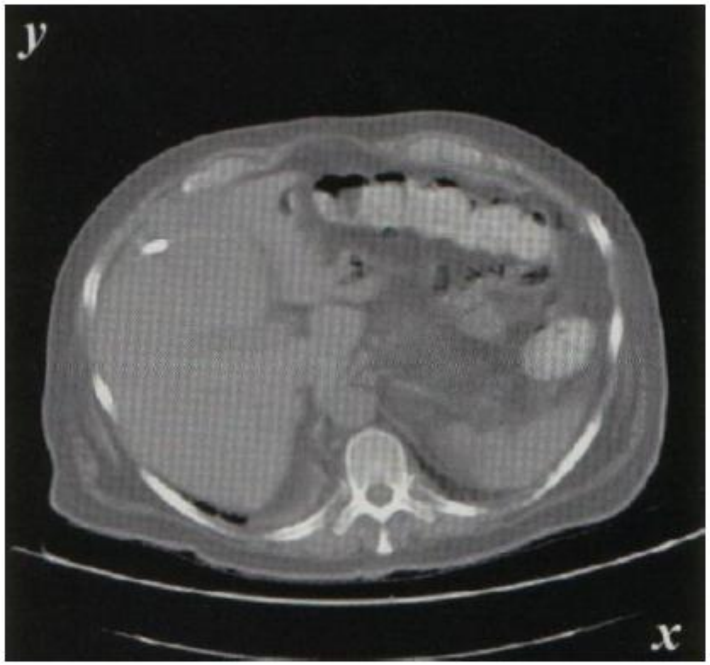
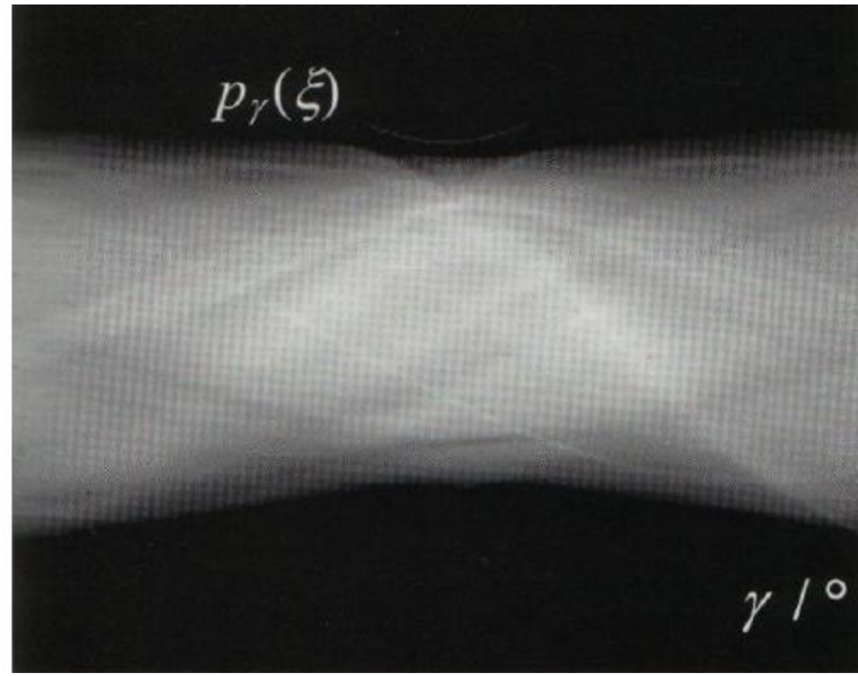
Radon Transform





Abdomen, Radon transform in Cartesian coord.

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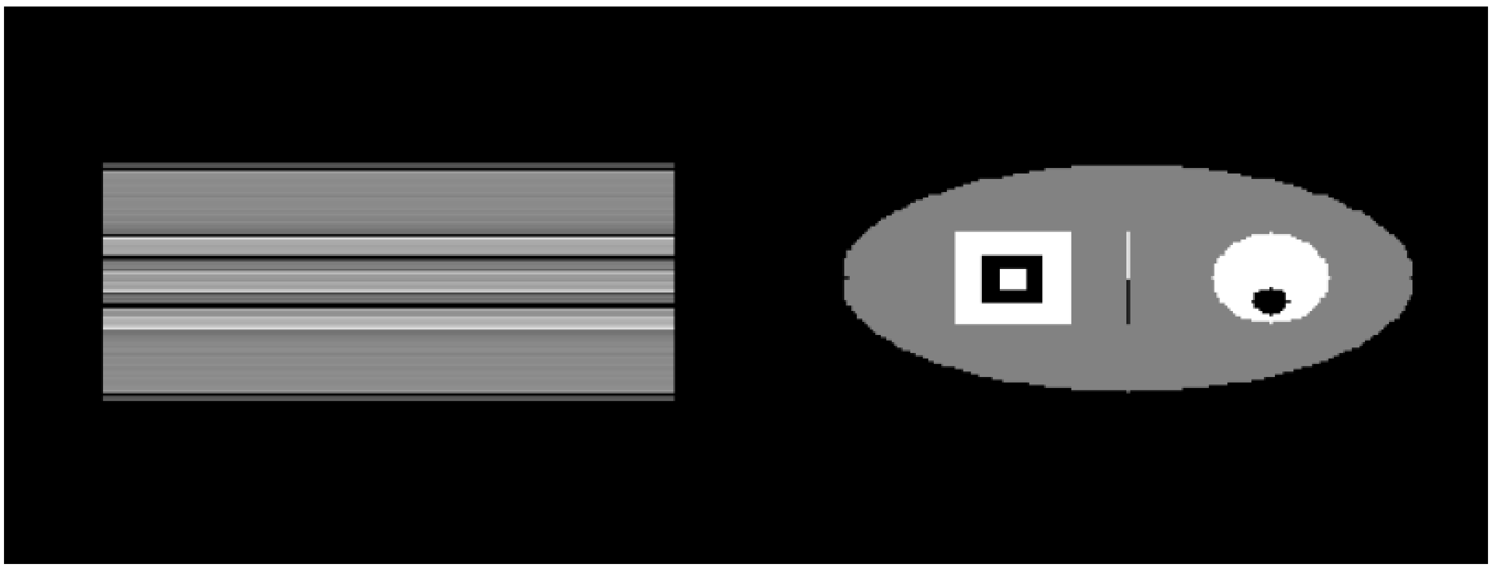
Reconstruction

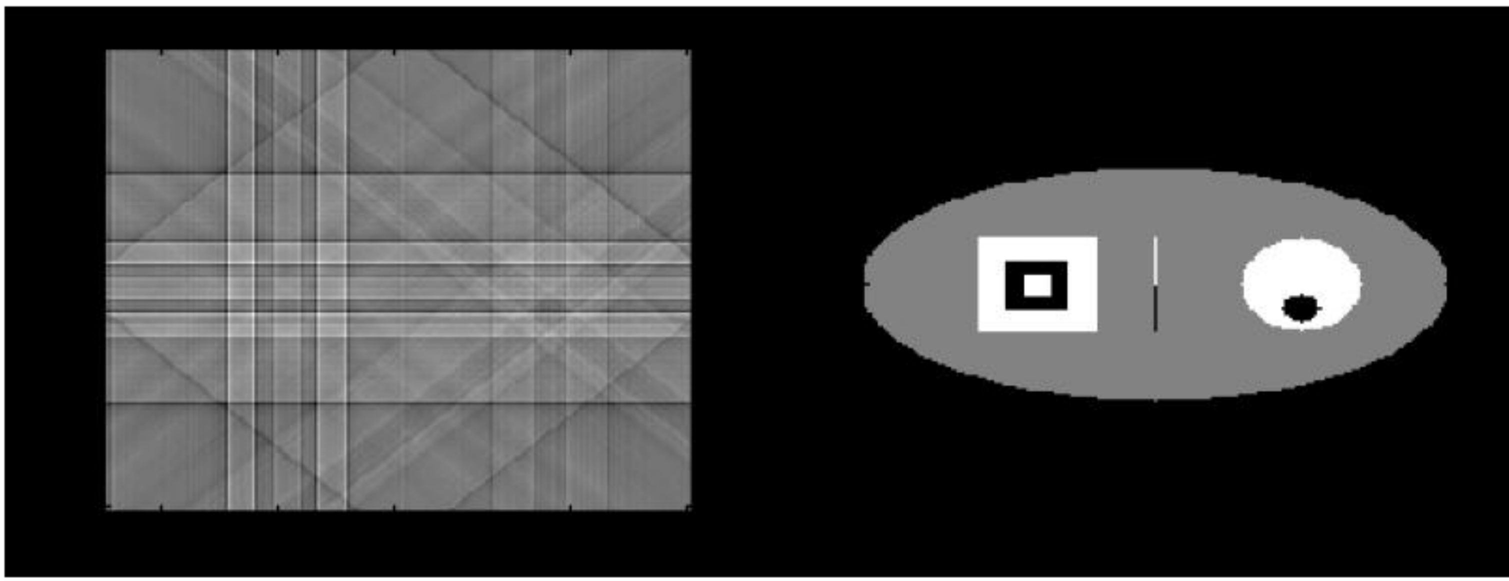
Now, we can try to do some reconstruction by the before mentioned procdure



based on 1 projection.

E. Meyer, www1.am.uni-erlangen.de/~bause/Seminar/seminar.html

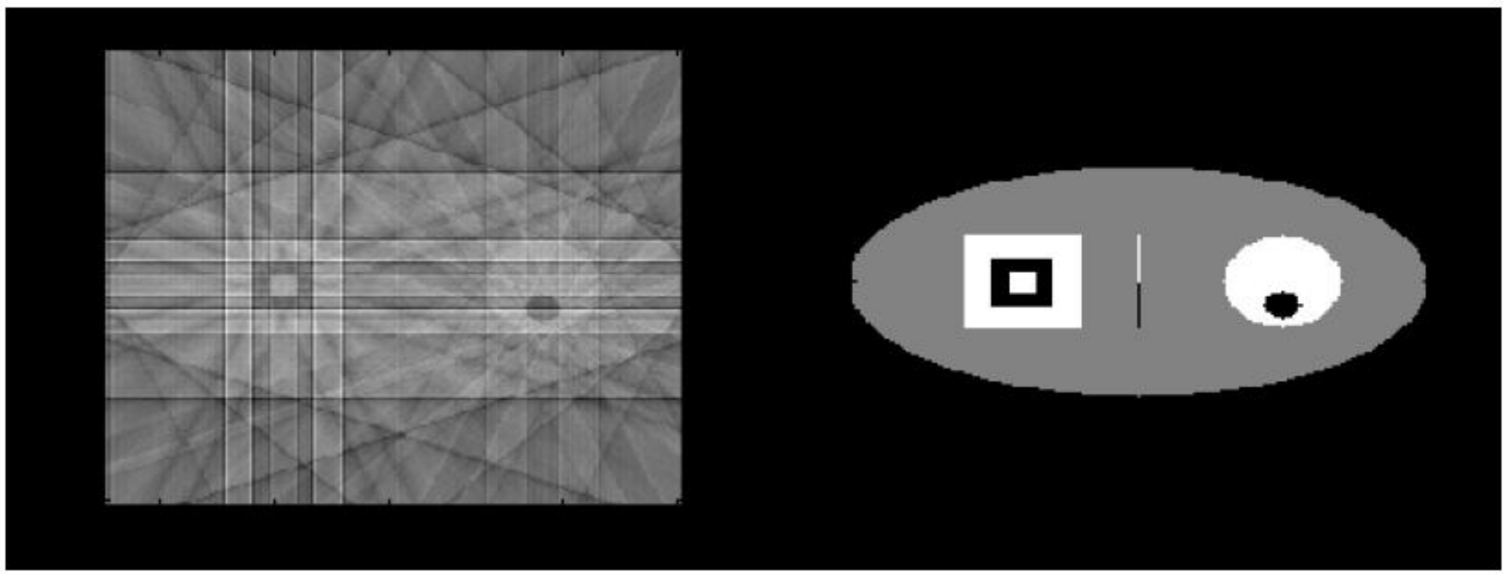
Reconstruction



based on 4 projections.

E. Meyer, www1.am.uni-erlangen.de/~bause/Seminar/seminar.html

Reconstruction



based on 8 projections.

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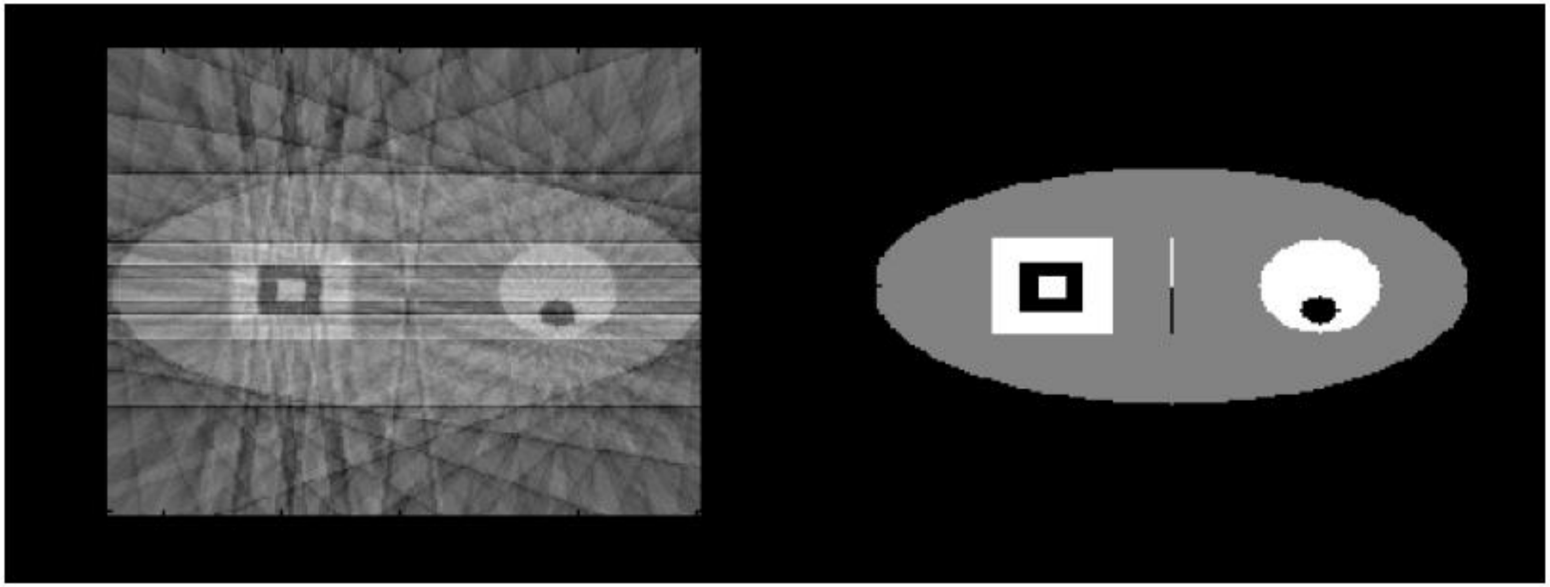
Reconstruction



based on 30 projections.

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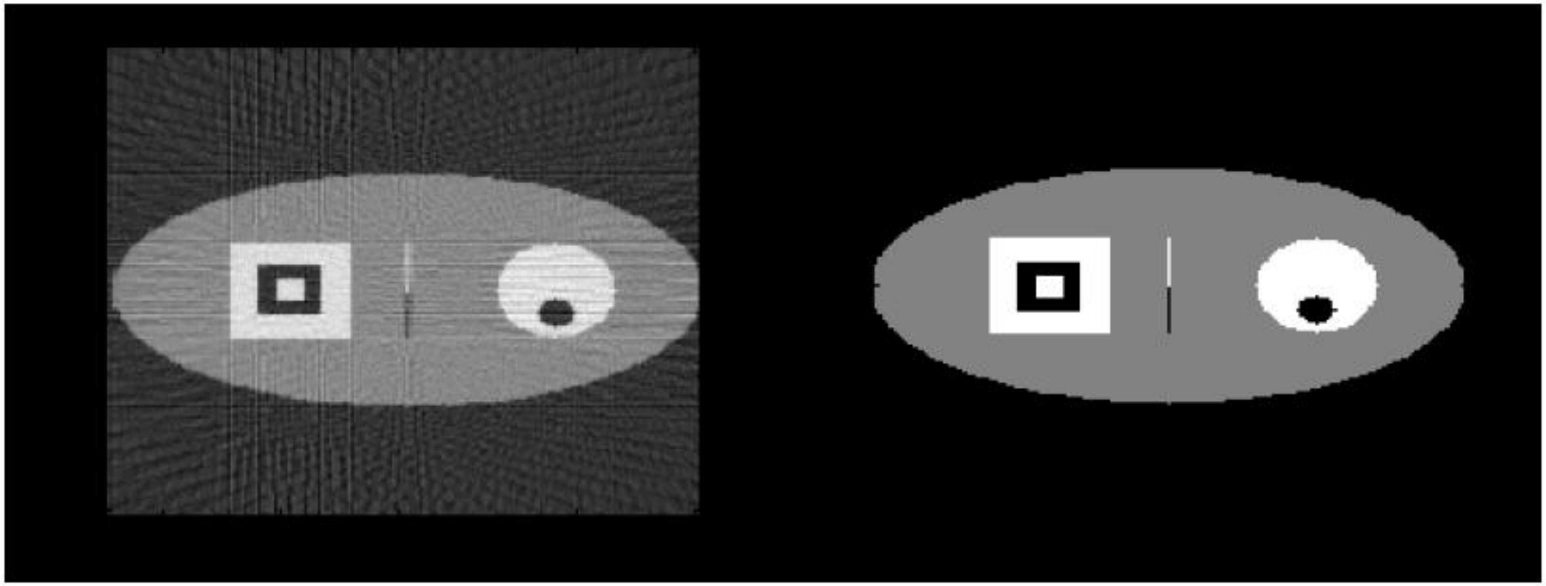
Reconstruction



based on 60 projections.

What is the difference to the back-projection formula?

E. Meyer, www1.am.uni-erlangen.de/~bause/Seminar/seminar.html

Radon Inversion Formula

- If f is an absolutely integrable function and ist Fourier transform is absolutely integrable too, then

$$f(\vec{x}) = \frac{1}{(2\pi)^2} \int_0^{\pi} \int_{-\infty}^{\infty} e^{ir\langle \vec{x}, \vec{\theta} \rangle} \widetilde{\mathcal{R}f}(r, \vec{\theta}) |r| \, dr \, d\vec{\theta}.$$

Radon Inversion Formula

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- Filtered Back-Projection

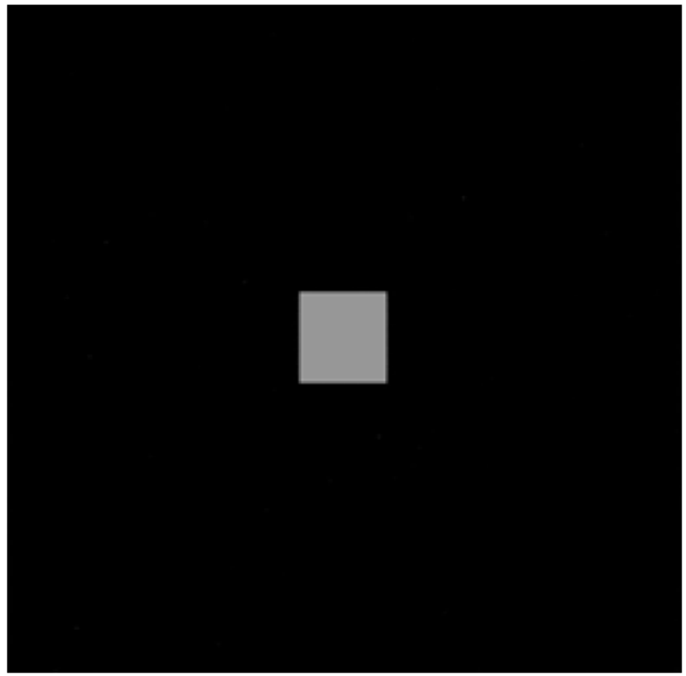
1. The radial integral is interpreted as a *filter* applied to the Radon transform. The filter acts only the affine parameter; is output of the filter is denoted

$$\mathcal{GR}(t, \vec{\theta}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{\mathcal{R}f}(r, \vec{\theta}) e^{irt} |r| \, dr.$$

2. The angular integral is then interpreted as the back-projection of the *filtered* Radon transform.

$$f(\vec{x}) = \frac{1}{2\pi} \int_0^{\pi} (\mathcal{GR})(\langle \vec{x}, \vec{\theta} \rangle, \vec{\theta}) \, d\vec{\theta}.$$

Back-Projection vs. Filtered Back-Projection



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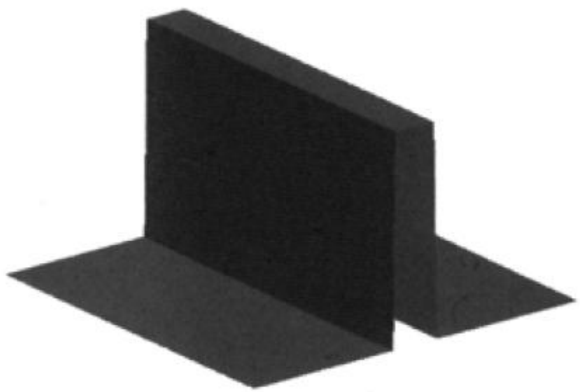
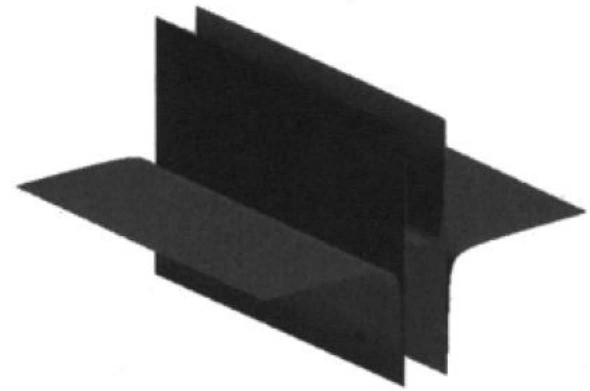
Back-Projection vs.
Filtered Back-Projection



back-projection



filtered back-projection

based on 1 projection

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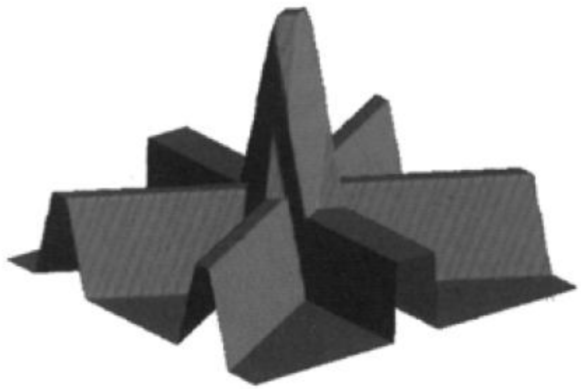
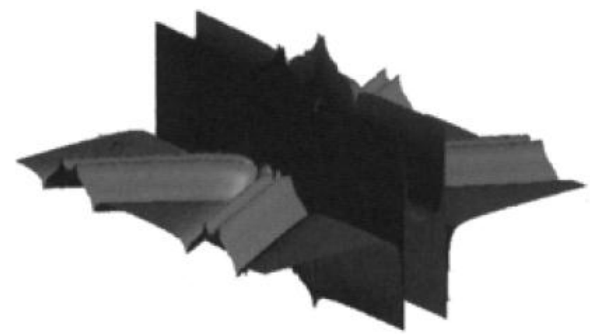
Back-Projection vs. Filtered Back-Projection





back-projection filtered back-projection

based on 3 projections

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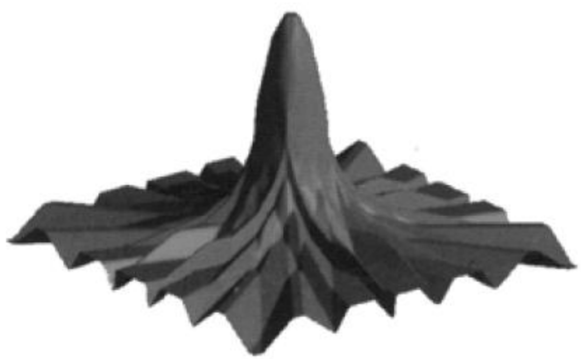
Back-Projection vs. Filtered Back-Projection



back-projection filtered back-projection

based on 10 projections

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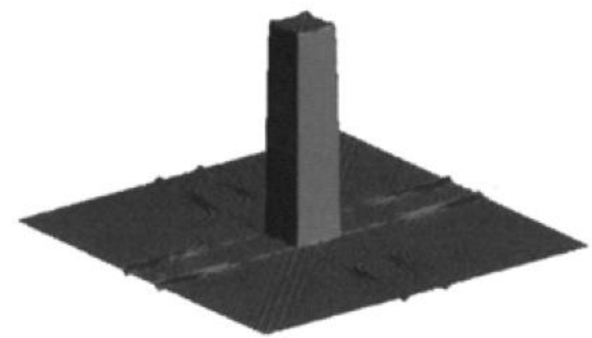
Back-Projection vs. Filtered Back-Projection

back-projection



filtered back-projection

based on 180 projections

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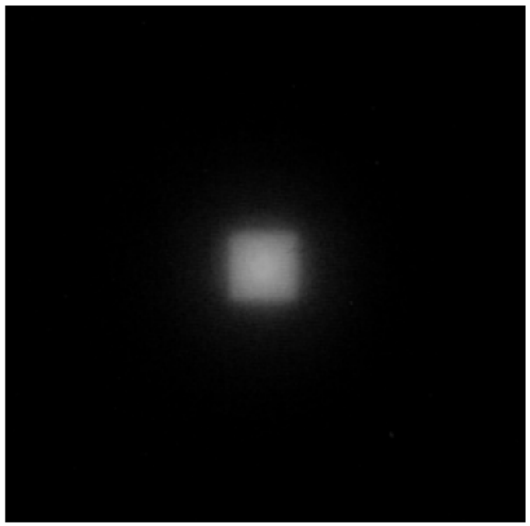
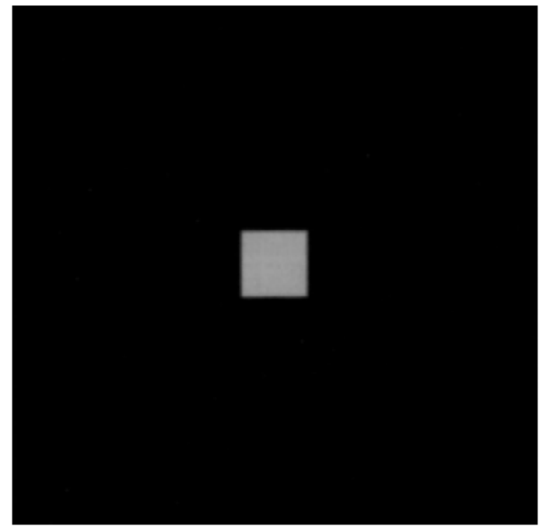
Back-Projection vs. Filtered Back-Projection





back-projection filtered back-projection

based on 180 projections

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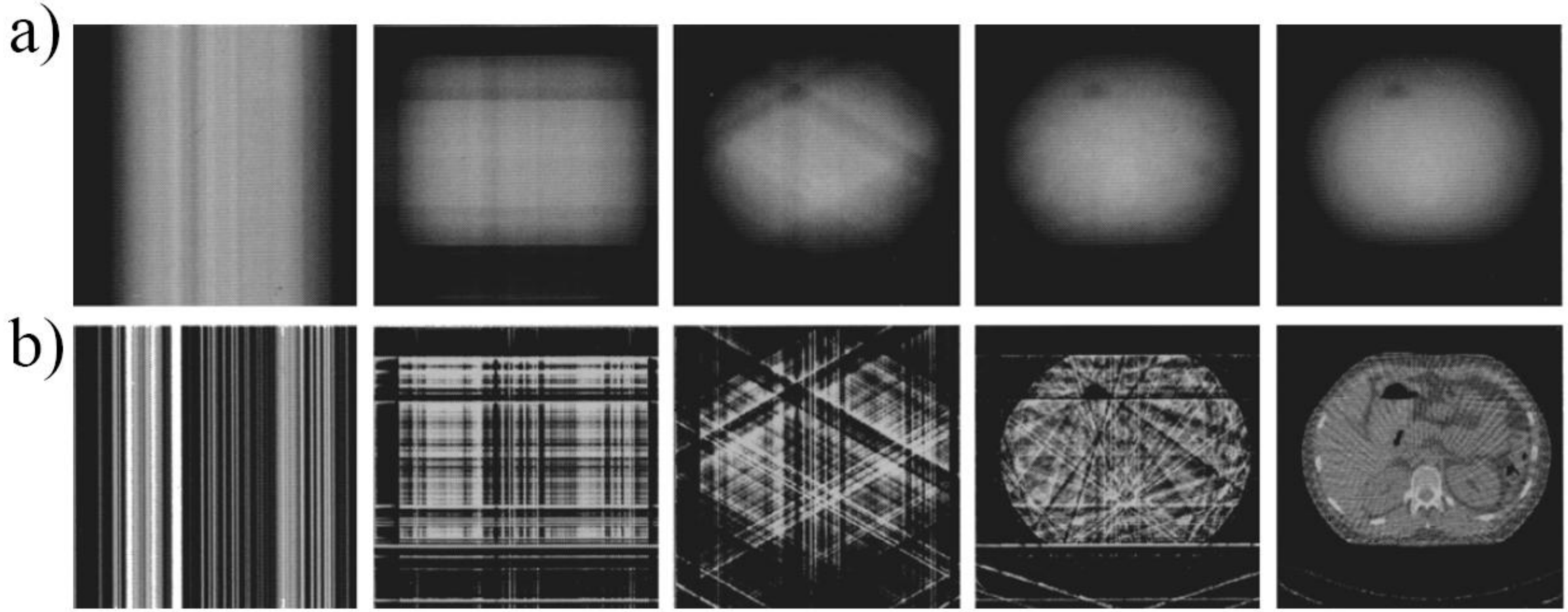
Back-Projection vs. Filtered Back-Projection



a) back-projection and b) filtered back-projection, based on 1, 2, 3, 10, 45 projections resp.

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Different Inversion formulas

- We already had the Radon inversion formula:

$$f(\vec{x}) = \frac{1}{(2\pi)^2} \int_0^{\pi} \int_{-\infty}^{\infty} e^{ir\langle \vec{x}, \vec{\theta} \rangle} \widetilde{\mathcal{R}f}(r, \vec{\theta}) |r| \, dr \, d\vec{\theta}.$$

- We write |r| as sgn(r) r :

$$f(\vec{x}) = \frac{1}{(2\pi)^2} \int_0^{\pi} \int_{-\infty}^{\infty} e^{ir\langle \vec{x}, \vec{\theta} \rangle} \widetilde{\mathcal{R}f}(r, \vec{\theta}) \, \mathrm{sgn}\,(r) \, r \, dr d\vec{\theta}$$

A Different Inversion formula

- We already had the Radon inversion formula:

$$f(\vec{x}) = \frac{1}{(2\pi)^2} \int_0^{\pi} \int_{-\infty}^{\infty} e^{ir\langle \vec{x}, \vec{\theta} \rangle} \widetilde{\mathcal{R}f}(r, \vec{\theta}) \, \mathrm{sgn}\,(r) \, r \, dr d\vec{\theta}$$

$$\frac{1}{i}\mathcal{F}_{t\to r}\left(\frac{d}{dt}(\mathcal{R}f)(t, \vec{\theta})\right)$$

- Where we write |r|as sgn(r) r
- $\mathcal{F}_{t\to r}$ denotes the 1D Fourier transform with respect to t.
- Suppose that g is square integrable on the real line. The Hilbert transform $\mathcal{H}$ of g is defined by

$$\mathcal{H}g = \mathcal{F}^{-1}(\mathrm{sgn}\, \hat{g}).$$

If $\hat{g}$ is also absolutely integrable, then

$$\mathcal{H}g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g} \, \mathrm{sgn}\,(r) \, e^{itr} \, dr.$$

We obtain

$$f(\vec{x}) = \frac{1}{2\pi\, i} \int_0^{\pi} \mathcal{H}(\partial_t \mathcal{R}f)(\langle \vec{x}, \vec{\theta} \rangle, \vec{\theta}) \, d\vec{\theta}.$$

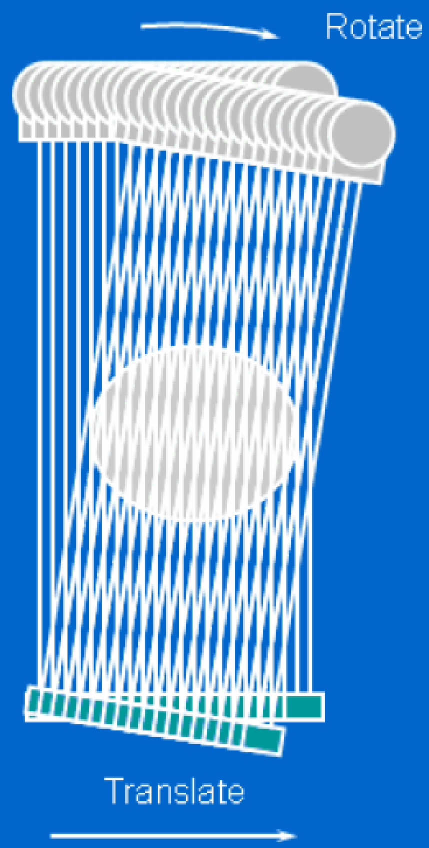
Mathematical Model for CT

- We consider a two-dimensional slice of an three-dimensional object, the physical parameters of interest is the attenuation coefficient f of the two-dimensional slice. According to *Beer's law*, the intensity traveling along a line is attenuated according to the differential equation $$\frac{dI_{(t,\vec{\theta})}}{ds} = -f\, I_{(t,\vec{\theta})},$$ where s is arclength along the line.
- By comparing the intensity of an incident beam of x-rays to that emitted, we *measure* the *Radon transform* of f: $$\mathcal{R}f(t,\, \vec{\theta}) = -\log\left[\frac{I_{0,(t,\vec{\theta})}}{I_{i,(t,\vec{\theta})}}\right].$$
- Using the Radon inversion formula, the attenuation coefficient f is reconstructed from the *measurements* $\mathcal{R}f$.

Scanner geometry

First generation CT scanner

- Single detector
- Translate - rotate acquisition
  - Translates across patient
  - Rotates around patient
- Very slow
  - minutes per slice

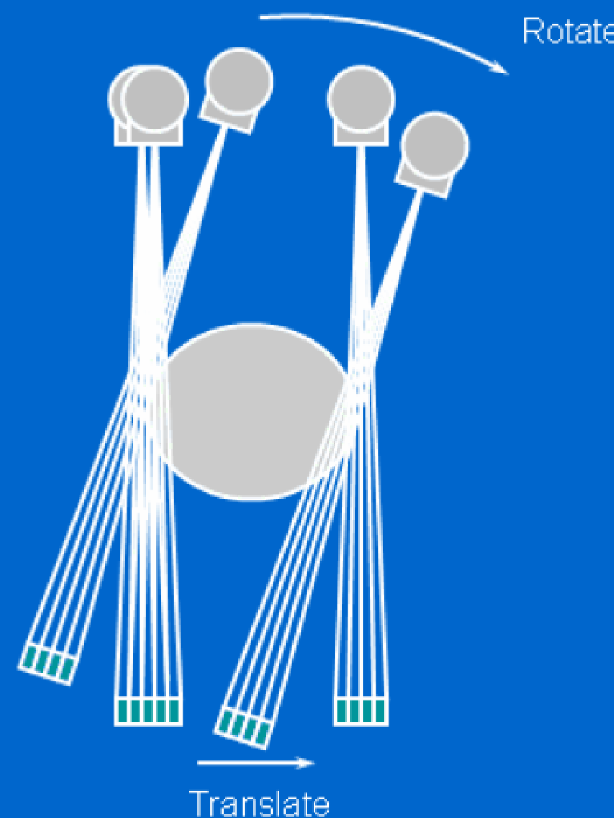


http://www.impactscan.org/slides/impactcourse/basic_principles_of_ct/

Scanner geometry

Second generation CT scanner

- Narrow fan beam (10°)
- Multiple detectors
- Multiple angle acquisition at each position
  - Larger angle rotate
  - Translate still required
- Slow
  - 20s per slice

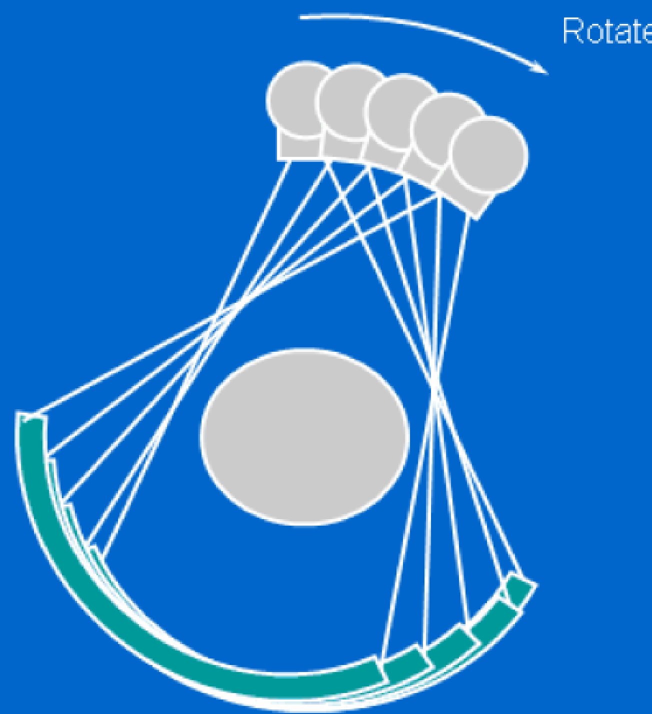


http://www.impactscan.org/slides/impactcourse/basic_principles_of_ct/

Scanner geometry

Third generation CT scanner

- Fan beam
- Multiple (500 - 1000) rotating detectors
- Rotation only
  - no translation required
- Much faster
  - as fast as 0.5 s per rotation
- Most common modern scanner design

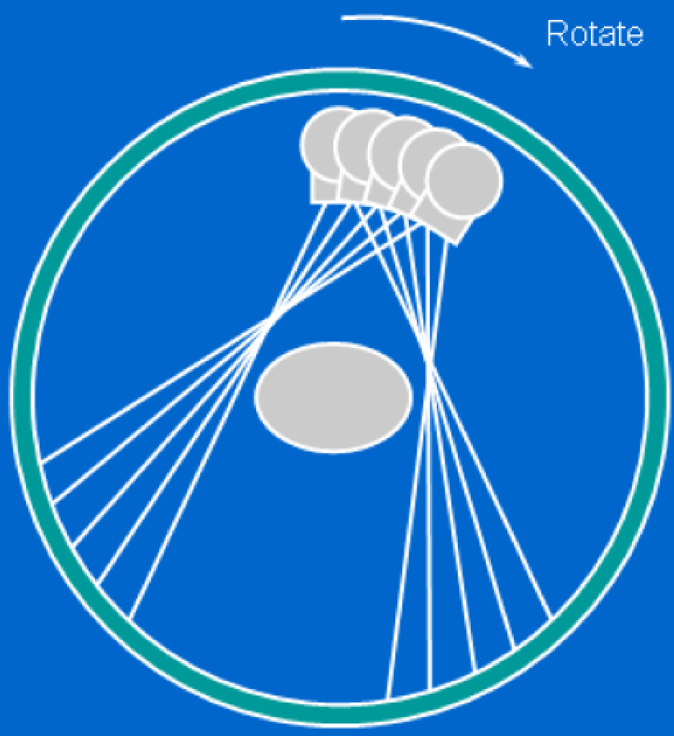


http://www.impactscan.org/slides/impactcourse/basic_principles_of_ct/

Scanner geometry

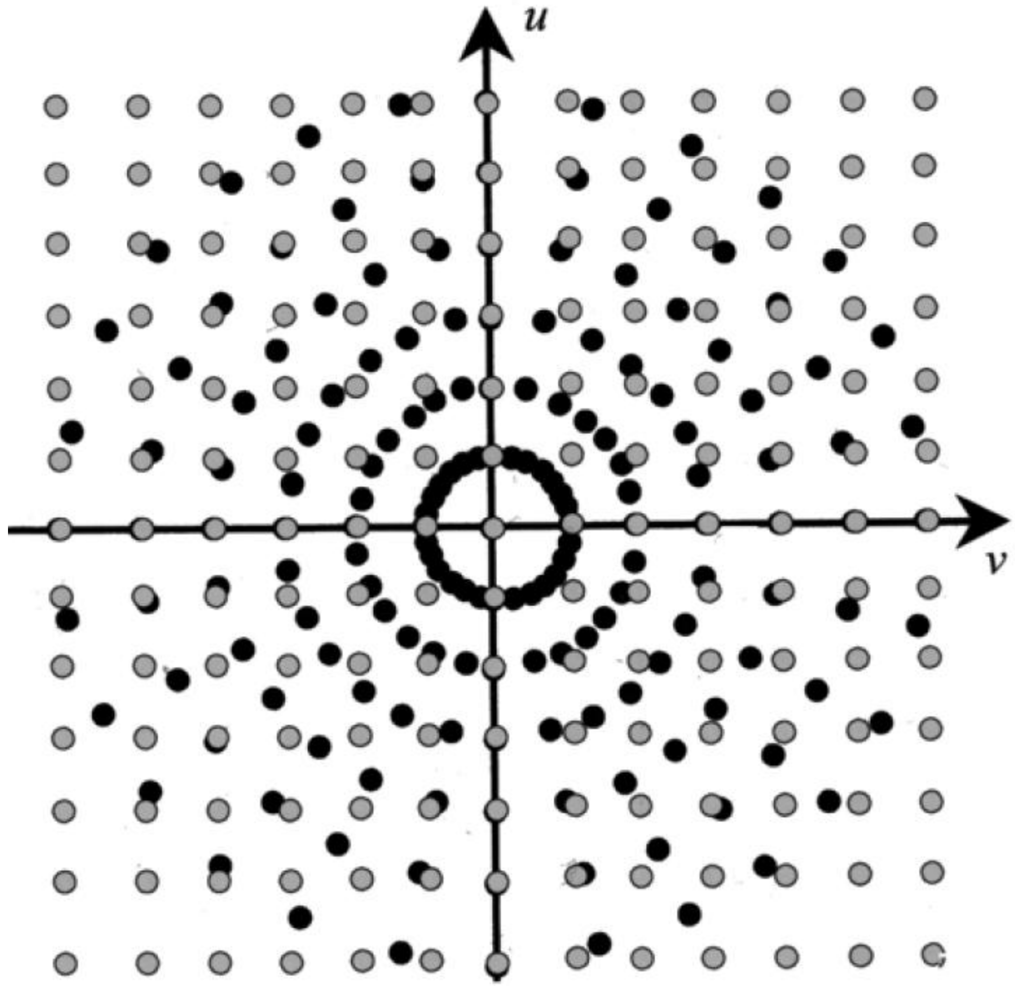
Fourth generation CT scanners

- Fan beam
- Static detectors all round gantry
- Only tube rotates
- Avoids ring artefact problems of 3rd generation scanners



http://www.impactscan.org/slides/impactcourse/basic_principles_of_ct/

Radon transform - Polar grid
Fourier transform – Cartesian grid

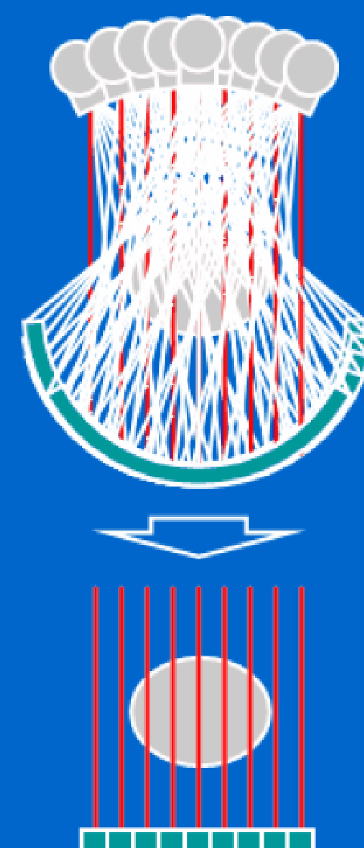


Buzug, Einführung in die Computertomography, Springer Verlag , 2004

Why fan beam?

Re-binning fan beam data

- 3rd generation CT scanners use a fan beam to measure projection data
- To get parallel projections, data from adjacent detectors in subsequent views can be combined



http://www.impactscan.org/slides/impactcourse/basic_principles_of_ct/

Reconstruction Algorithm for a Parallel Beam Machine

- We assume that we can measure *all* the data from a finite set of equally spaced angles. In this case data would be

$$\{\mathcal{R}f(t,\, k\Delta\theta) :\ k = 0, \ldots M, \quad t \in [-L,\, L]\}\,, \quad \Delta\theta = \frac{\pi}{M+1}.$$

- With these data we can apply the central slice theorem to compute angular samples of the two-dimensional Fourier transform of f,

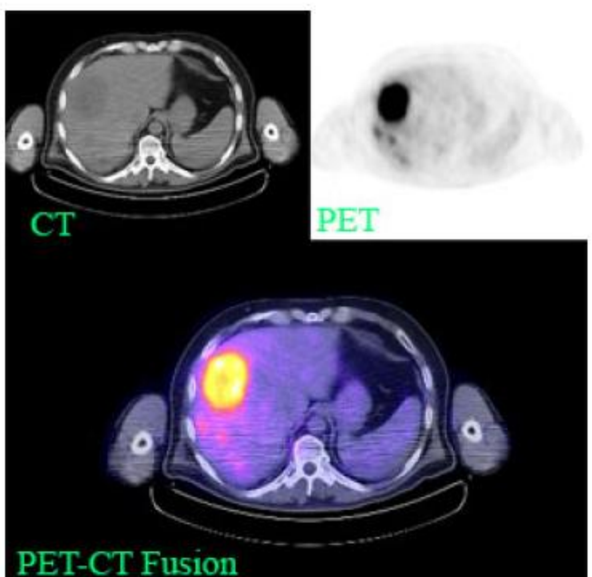
$$\hat{f}(r,\, \vec{\theta}(k\Delta\theta)) = \int_{-\infty}^{\infty} \mathcal{R}f(t,\, \vec{\theta}(k\Delta\theta))e^{-irt}\, dt.$$

- Using the two-dimensional Fourier inversion formula and using a Riemann sum in the angular direction gives

$$f(x,y) \approx \frac{1}{4\pi(M+1)} \sum_{k=0}^{M} \int_{-\infty}^{\infty} \hat{f}(r\, \vec{\theta}(k\Delta\theta))\, e^{ir\langle (x,y), \vec{\theta}(k\Delta\theta)\rangle} |r|\, dr.$$

Concluding remarks

- The model present here is a CT-model, there exist other types of tomographical methods that are based on other mathematical models.
- All mathematical models are based on so-called integral geometry and connected with wave equations.
- Modern tomography even combines different methods:



fusion of CT-scan (grey)

and PET-scan (grey)

PET = Positron Emission Tomography

http://www.sdirad.com/PatientInfo/pt_pet.htm

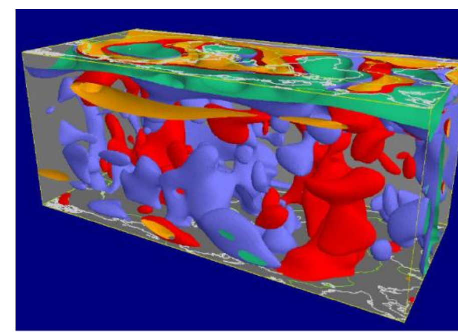
Concluding remarks

Most of the pictures are dealing with medical applications but Computer tomography can be applied to more applications, as for example:

Material sciences

Tomographic visualisation of a metallic foam structure

http://www2.tu-berlin.de/fak3/sem/GB_index.html



Geology

http://www.geo.cornell.edu/geology/classes/Geo101/101images_spring.html

Seismic tomography reveals a more complex interior structure.

Archeology

3D-Computer Tomography of Prehispanic Sound Artifacts.

Supported by the Ethnological Museum Berlin and the St. Gertrauden Hospital, Berlin.

http://www.mixcoacalli.com/wp-content/uploads/2007/09/ct2.jpg

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- Thorsten M. Buzug, *Einführung in die Computertomographie,* Springer Verlag, 2004
- Esther Meyer, *Die Mathematik der Computertomogrphie,* Seminarvortrag, *www1.am.unierlangen.de/~bause/Seminar/seminar.html*
- Other pictures are from *www.impactscan.org/slides/impactcourse/basic_principles_of_ct/*

  www.sdirad.com/PatientInfo/pt_pet.htm