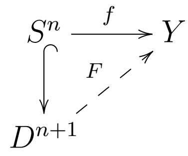
HOMEWORK 12

This should be a combination of some exposition and actual exercises:

Let Y be a path connected CW–complex. Then $\pi_n(Y) = 0$ iff any morphism $f : S^n \to Y$ can be extended to $F : D^{n+1} \to Y$.



This is a rather easy observation. The direction $\Leftarrow$ says that any map from a sphere can be extended to a map F. Then F describes a nullhomotopy of f.

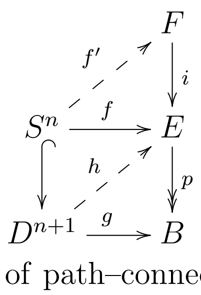
The direction $\Rightarrow$ says that any map $f : S^n \to Y$ is nullhomotopic. This means that there exists a nullhomotopy $H : S^n \times I \to Y$, where $H|_{S^n \times \{1\}} = y$ for some $y \in Y$.

We can imagine that we glue D^{n+1} to the top of the cylinder $S^n \times I$ and we have a map $H \cup \{y\} : S^n \times I \cup D^{n+1} \times \{1\}$. Thus by lifting extension property $H \cup \{y\}$ can be lifted to a map $G : D^{n+1} \times I$. We define $F = G|_{D^{n+1} \times \{y\}}$.

Exercise 1. Prove the following theorem: Let X, A be a pair of CW–complexes and let Y be a path–connected space. Then $f : A \to Y$ can be extended to a map $F : X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells in dimension n. (Hint: use the preceding exercise and extend f to skeletons X^i).

Exercise 2. By presenting a counterexample, disprove the following: Let X, A be a pair of CW–complexes and let Y be a path–connected space. Then $f : A \to Y$ can be extended to a map $F : X \to Y$ if $\pi_n(Y) = 0$ for all n such that $X \setminus A$ has cells in dimension n. (Hint: Assume $(X, A) = (D^{n+1}, S^n)$)

Let us have the following diagram:



where $p : E \to B$ is a fibration of path–connected spaces with fibre F such that $\pi_n F = 0$. We will prove, that there exists a diagonal h that will make everything commute. First, we notice that as $pf : S^n \to B$ can be extended to $g : D^{n+1} \to B$ it must be nullhomotopic and the nullhomotopy is provided by g. Let $b_0 \in B$ be the point in the center of the disc $g(D^{n+1})$. Then we have a homotopy $\hat{g} : S^n \times I \to B$, which is pf on one end and b_0 on the other end.

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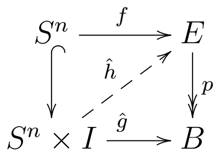
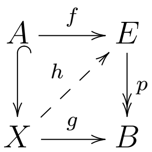
By the lifting property of fibration, we then get a lift $\hat{h}$ in the diagram



We can see that $\hat{h}$ is the homotopy between f and some f', where $pf' = b_0$ and f' is therefore a mapping into the fibre F (or its image $i(F)$). But as $\pi_n(F) = 0$, we can see that f' is nullhomotopic. Let h' be the nullhomotopy of f'. We will interpret this as a map $h' : D^{n+1} \to i(F)$. Now we can glue the disc h' to top of the cylinder $S^n \times I$. Thus we have a map $h = \hat{h} \cup \tilde{h} : D^{n+1}(\sim D^{n+1} \cup S^n \times I) \to E$. This map is the lift we were originally looking for.

Exercise 3. Prove the theorem: Let (X, A) be a CW–pair and $p : E \to B$ a fibration of path connected spaces such that $\pi_n(F) = 0$ whenever there is a cell of dimenstion $n + 1$ in $X \setminus A$. Then there exists a lift in the following diagram:



Exercise 4. Let (X, A) be a CW–pair and $p : E \to B$ a fibration of path connected spaces which is also a weak equivalence. Prove, that there exists a lift in the following diagram: