Note: This tutorial originates in 2017[1].

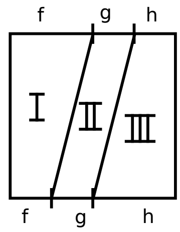
Define the n-th homotopy group of the space X with the base point x_0 as the group of homotopy classes of the continuous maps $(I^n, \partial I^n) \to (X, x_0)$ with the operation given by the prescription:

$$(f+g)(t_1,\dots,t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & 0 \le t_1 \le \frac{1}{2}, \\ g(2t_1 - 1, t_2, \dots, t_n) & \frac{1}{2} \le t_1 \le 1. \end{cases}$$

Denote it $\pi_n(X, x_0)$. Note that a monoid structure induced by concatenation of maps with respect to other components as the first one coincides with above prescription by the *Eckmann-Hilton* argument. To see that $f+g$ is a continuous map again we refer to the well-known *pasting* lemma.

Exercise 1. *Show the operation on $\pi_n(X, x_0)$ is associative.*

Solution. We want to show $(f+g)+k \sim f+(g+k)$. We will find prescription for the homotopy by the following diagram:



In the following notation[2], understand $f(t_1)$ as $f(t_1, t_2, \dots, t_n)$ for all $t_2, \dots, t_n \in I$.

$$h(s,t_1) = \begin{cases} f(\frac{4}{1+s}t_1) & (s,t_1) \in \mathrm{I} & (t_1 \in [0, \frac{1}{4} + s\frac{1}{4}], s \in [0,1]) \\ g(4t_1 - (1+s)) & (s,t_1) \in \mathrm{II} & (t_1 \in [\frac{1}{4} + s\frac{1}{4}, \frac{1}{2} + s\frac{1}{4}], s \in [0,1]) \\ k(\frac{4}{2-s}t_1 - \frac{2+s}{2-s}) & (s,t_1) \in \mathrm{III} & (t_1 \in [\frac{1}{2} + s\frac{1}{4}, 1], s \in [0,1]) \end{cases}$$

□

Exercise 2. *Show that the element given by prescription*

$$(-f)(t_1,\dots,t_n) = f(1-t_1, t_2, \dots, t_n)$$

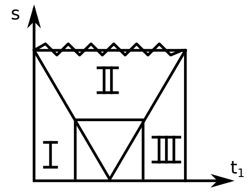
is really the inverse element of f.

Solution. We want to show $f + (-f) \sim$ *const*. The constant will be (function given by) the point $f(2t_1) = f(0)$. Again let us draw a diagram (the square (its boundary) in the middle sign the same value; the wavy line sign one value too).

[1] see `https://is.muni.cz/el/sci/jaro2019/M8130/um/68045051/`

[2] and some other following notations

$$h(s,t_1)=\begin{cases} f(2t_1) & (s,t_1)\in \mathrm{I}\\ f(\frac{1-s}{2})=(-f)(\frac{1+s}{2}) & (s,t_1)\in \mathrm{II}\\ (-f)(1-2t_1)=f(2t_1) & (s,t_1)\in \mathrm{III}, \end{cases}$$



where $\mathrm{II}=\{(s,t_1)\mid s\in[0,1], t_1\in[\frac{1-s}{2},\frac{1+s}{2}]\}$). □

Remark. One can see, that proving by pictures is much more pleasant.

There is a long exact sequence:

$$\cdots\to\pi_{n+1}(X,A,x_0)\xrightarrow{\partial}\pi_n(A,x_0)\xrightarrow{i_*}\pi_n(X,x_0)\xrightarrow{j_*}\pi_n(X,A,x_0)\xrightarrow{\partial}\pi_{n-1}(A,x_0)\to\dots$$

Exercise 3. *Show the exactness of this sequence in* $\pi_n(X,A,x_0)$ *and* $\pi_n(A,x_0)$.

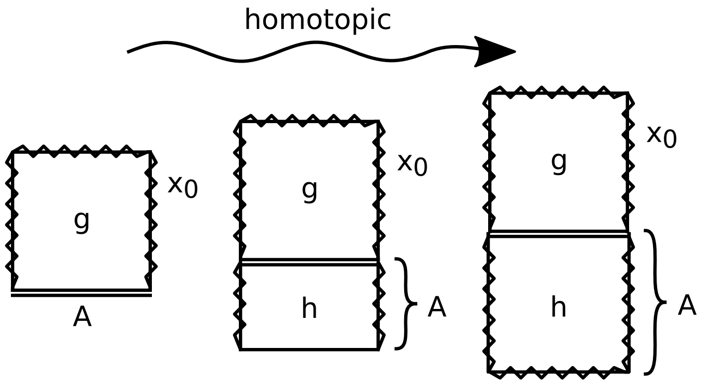
Solution. At first we will show the exactness in $\pi_n(X,A,x_0)$.

Let us show the inclusion "$\operatorname{im} j_*\subseteq\ker\partial$". Take an arbitrary $f\in\pi_n(X,x_0)$, thus $f\colon(I^n,\partial I^n)\to(X,x_0)$. From definition $j_*(f)=j\circ f\colon(I^n,\partial I^n,J^{n-1})\to(X,A,x_0)$, where $J^{n-1}=I^n-I^{n-1}$, and $\partial([f])=[f|_{I^{n-1}}]=\mathit{const}$, since f is constant on whole $\partial I^n\supseteq I^{n-1}$.

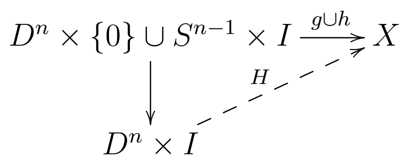
"$\operatorname{im} j_*\supseteq\ker\partial$": Take an arbitrary $g\in\ker\partial\subseteq\pi_n(X,A,x_0)$, thus $g\colon(I^n,\partial I^n,J^{n-1})\to(X,A,x_0)$. Since $g\in\ker\partial$, there is the homotopy $h\colon(I^{n-1},\partial I^{n-1})\times I\longrightarrow(A,x_0)$ such that $h(x,0)=g|_{I^{n-1}}(x)$ and $h(x,1)=\mathit{const}$. Because $h(x,t)\in A$ and $h(x',t)=x_0$ for all $x\in I^{n-1}$, $x\in\partial I^{n-1}$ and $t\in[0,1]$, we can take $f\in\pi_n(X,x_0)$ defined by

$$f(x,t)=\begin{cases} g(x,2t) & \text{for } t\in[0,\frac12]\\ h(x,2t-1) & \text{for } t\in[\frac12,1]. \end{cases}$$

It is not hard to prove that $j_*(f)$ is homotopic to g, see picture below.



Another approach to showing this inclusion is to view homotopy groups $\pi_n(X,A,x_0)$ as the homotopy classes $[(D^n,S^{n-1},s_0),(X,A,x_0)]$ and the connecting homomorphism ∂ as the restriction $|_{S^{n-1}}$. Then we have the following commutative diagram



The homotopy H exists by HEP of the pair (D^n, S^{n-1}). Then we can take $[H(-,1)] \in \pi_n(X, x_0)$ and $j_*([H(-,1)]) = [g]$.

Now, let us show the exactness in $\pi_n(A, x_0)$.

"$\operatorname{im} \partial \supseteq \ker i_*$" Let $f \in \ker i_* \subseteq \pi_n(A, x_0)$ be an arbitrary. Because $i_* f \sim const$, we have homotopy $h\colon (I^n, \partial I^n) \times I \to (X, x_0)$ such that $h(x, 0) = f(x)$ and $h(x, 1) = x_0$. It holds $h \in \pi_{n+1}(X, A, x_0)$, since $h(x, 0) \in A$, $h(x, 1) = x_0$ and $h(x', t) = x_0$ for all $x \in I^n$ and $x' \in \partial I^n$.

"$\operatorname{im} \partial \subseteq \ker i_*$" Let $h \in \pi_{n+1}(X, A, x_0)$ be an arbitrary. Denote $h|_{I^n} = f$. Then h gives the homotopy $i_* f \sim const$ in (X, x_0), since $h(x, 0) = f(x)$, $h(x, 1) = x_0$ and $h(x', t) = x_0$ for all $x' \in \partial I^n$ and $x \in I^n$. □

A map $p\colon E \to B$ is called a fibration if it has the homotopy lifting property for all $(D^n, \emptyset)$:

$$\begin{array}{ccc} D^n \times \{0\} & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ D^n \times I & \longrightarrow & B \end{array}$$

If p is a fibration then it has the homotopy lifting property also for all pairs (X, A) of CW-complexes:

$$\begin{array}{ccc} X \times \{0\} \cup A \times I & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ X \times I & \longrightarrow & B \end{array}$$

Recall that $p\colon E \to B$ is a fibre bundle with fibre F if there are open subsets U_α such that $B = \bigcup_\alpha U_\alpha$ and the following diagram commutes for all U_α:

$$\begin{array}{ccc} U_\alpha \times F & \xrightarrow{\text{homeo.}} & p^{-1}(U_\alpha) \\ & \searrow^{pr_1} & \downarrow p \\ & & U_\alpha \end{array}$$

Typical examples of fibre bundles with fibre S^1 over I are a trivial bundle or Mobius band[3].

Exercise 4. *Show that every fibre bundle is a fibration.*

Solution. At first consider a trivial fibre bundle $E = B \times F$. Take an arbitrary commutative diagram of the form:

$$\begin{array}{ccc} D^n \times \{0\} & \xrightarrow{f} & B \times F \\ \downarrow & & \downarrow \mathrm{pr}_B \\ D^n \times I & \xrightarrow{h} & B \end{array}$$

[3] http://rin.io/intro-to-bundles/

Then $h(-,0) = f$ and we can define $H\colon D^n \times I \to B \times F$ by $H(x,t) = \big(h(x,t), \mathrm{pr}_F(f(x))\big)$. One can see that the diagram commutes with H too.

Now, let $p\colon E \to B$ be an arbitrary fibre bundle with fibre F and $B = \bigcup_\alpha U_\alpha$. We can take I^n instead of D^n and consider a diagram:

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{f} & E \\ \downarrow & & \downarrow p \\ I^n \times I & \xrightarrow{h} & B \end{array}$$

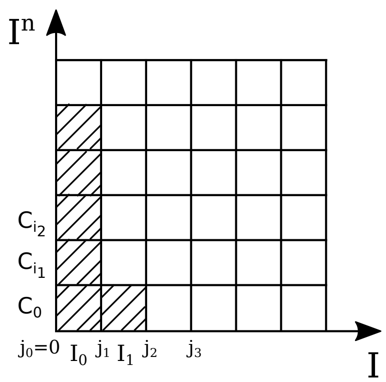
Because I^n is compact, we can divide $I^n \times I$ to finitely many subcubes $C_i \times I_k$ where $I_i = [j_k, j_{k+1}]$ such that $h(C_i \times I_k) \subseteq U_\alpha$ for some α. This is possible due to *Lebesgue's number lemma*. Since $U_\alpha \times F \to U_\alpha$ makes a trivial bundle, we can use the same approach as above for each subcube. Since we know $H|_{C_i \times \{0\}}$, we can find the lift H for all cubes in the first "column" (see the picture below) in the same way as for the trivial case (as we know that each Serre fibration has HLP w.r.t. a pair of CW-complexes)

$$\begin{array}{ccc} C_i \times \{0\} \cup A_i^0 \times I_0 & \xrightarrow{f \cup H|_{A_i^0 \times I_0}} & U_\alpha \times F \\ \downarrow & \overset{H}{\nearrow} & \downarrow pr \\ C_i \times I_0 & \xrightarrow{h} & U_\alpha \end{array}$$

where $A_i^k = \bigcup_{h((C_i \cap C_l) \times I_k) \subseteq U_\alpha, l<i} (C_i \cap C_l)$. Since we know $H|_{C_i \times \{j_1\}}$ now, we can continue with the second "column":

$$\begin{array}{ccc} C_i \times \{j_1\} \cup A_i^1 \times I_1 & \xrightarrow{f \cup H|_{A_i^1 \times I_1}} & U_\alpha \times F \\ \downarrow & \overset{H}{\nearrow} & \downarrow pr \\ C_i \times I_1 & \xrightarrow{h} & U_\alpha \end{array}$$

Thus, we can proceed through all columns in this way until we will get H on the whole $I^n \times I$. The illustration of this situation[4]:



[4]it is drawn as planar, but it should be n-dimensional

□

Exercise 5. *Show the structure of the fibre bundle* $S^n \xrightarrow{p} \mathbb{R}\mathrm{P}^n$.

Solution. The fibre is $S^0 = \{-1, 1\}$, since $x, -x \longmapsto [x]$. Now, we want to find a neighbourhood U of $[x]$ such that $p^{-1}(U) \cong U \times S^0$. Assume that

$$[x] = [1 : x_1 : \cdots : x_n] \in U_0 = \{[y_0 : y_1 : \cdots : y_n] \mid y_0 \neq 0\}$$

then we have the homeomorphism $\varphi\colon U_0 \times S^0 \longrightarrow p^{-1}(U_0) \subseteq S^n$ given by $\varphi([x], 1) = \frac{(1,x_1,\ldots,x_n)}{\|(1,x_1,\ldots,x_n)\|}$ and $\varphi([x], -1) = \frac{(-1,-x_1,\ldots,-x_n)}{\|(1,x_1,\ldots,x_n)\|}$. We can cover the whole $\mathbb{R}\mathrm{P}^n$ by the open subsets $U_i = \{[y_0 : y_1 : \cdots : y_n] \mid y_i \neq 0\}$. □

Exercise 6. *Show the structure of the fibre bundle* $S^{2n+1} \xrightarrow{p} \mathbb{C}\mathrm{P}^n$ *with the fibre* S^1.

Solution. Let us look on the special case $S^1 \hookrightarrow S^3 \to \mathbb{C}\mathrm{P}^1 \cong S^2$ called "Hopf fibration". Realise that we can consider $S^3 \subseteq \mathbb{C}^2$, so we can (locally) define the projection $S^3 \to \mathbb{C}\mathrm{P}^1$ by $(z_1, z_2) \mapsto \frac{z_1}{z_2}$.

In the general case, realize that $S^{2n+1} \subseteq \mathbb{C}^{n+1}$. Take $U_0 = \{[z_0 : z_1 : \cdots : z_n] \mid z_0 \neq 0\} \subseteq \mathbb{C}\mathrm{P}^n$. Then the map $U_0 \times S^1 \to p^{-1}(U_0) \subseteq S^{2n+1}$ is given by

$$([1 : z_1 : \cdots : z_n], e^{it}) \longmapsto \frac{(e^{it}, e^{it}z_1, \ldots, e^{it}z_n)}{\|(e^{it}, e^{it}z_1, \ldots, e^{it}z_n)\|}$$

We can do the same for other U_i from the covering of $\mathbb{C}\mathrm{P}^n$. □