Note: This tutorial originates in 2017[1].

Exercise 53. *Long exact sequence of the fibration (Hopf)* $S^1 \to S^3 \to \mathbb{C}P^1 = S^2$.

Solution. This is an important example of a fibration, it deserves our attention. First we write long exact sequence

$$\pi_3(S^1) \xrightarrow{i_*} \pi_3(S^3) \xrightarrow{j_*} \pi_3(S^2) \xrightarrow{\partial} \pi_2(S^1) \to \pi_2(S^3) \to \pi_2(S^2) \to$$

$$\to \pi_1(S^1) \to \pi_1(S^3) \to \pi_1(S^2) \to \cdots$$

and also for $\mathbb{Z} \to \mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z}$, that is $\pi_n(\mathbb{Z}, 0) \to \pi_n(\mathbb{R}) \to \pi_n(S^1) \to \pi_{n-1}(\mathbb{Z})$ for $n > 0$. Since $\mathbb{Z}$ is discrete, $(S^n, s_0) \to (\mathbb{Z}, 0), s_0$ goes to a base point 0, therefore the map is constant . Hence we get $\pi_n(\mathbb{Z}, 0) = 0$ for $n \geq 1$. Also, $\pi_n(\mathbb{R})$ is zero as well, because $\mathbb{R}$ is homotopy equivalent to point. We get that this whole sequence are zeroes for $n \geq 2$. What we are left with is

$$0 \to \pi_1(S^1) \to \pi_0(\mathbb{Z}) \to \pi_0(\mathbb{R}) = 0,$$

so $\pi_1(S^1) \cong \mathbb{Z}$. Continue now with the updated long exact sequence:

$$0 \to \pi_3(S^3) \to \pi_3(S^2) \to 0 \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) = \mathbb{Z} \to \pi_1(S^3) \cdots ,$$

and $\pi_1(S^3) = 0$, because $\pi_k(S^n) = 0$ for $k < n$ considering the map $S^k \to S^n$ that can be deformed into cellular map and so it is not surjective. So we get

$$\pi_3(S^3) \cong \pi_3(S^2), \quad \pi_2(S^2) \cong \pi_1(S^1) = \mathbb{Z}.$$

It implies that $\pi_2(S^2) \cong \mathbb{Z}$.

Let us remark that if $\pi_3(S^3) = \mathbb{Z}$, then $\pi_3(S^2) = \mathbb{Z}$. Later we will prove that

$$\pi_n(S^n) \cong H_n(S^n) \cong \mathbb{Z}.$$

□

Remark. *(Story time) Eduard Čech was the first who defined higher homotopy groups (1932, Höherdimensionale Homotopiegruppen) but the community didn't support the study of these groups as they were considered not interesting. Were they mistaken? The rest of this remark is left as an exercise for the reader.*

Remark. *(For geometers) For G Lie group and H its subgroup we have a fibre bundle $H \hookrightarrow G \to G/H$.*

As an example consider orthonormal group $O(n)$, we have inclusions $O(1) \subseteq O(2) \subseteq \cdots O(n)$. Then

$$O(n-k) \to O(n) \to O(n)/O(n-k) = V_{n,k}$$

[1] see `https://is.muni.cz/el/sci/jaro2019/M8130/um/68045051/`

is a fibre bundle, we call $V_{n,k}$ Stiefel manifold, k-tuples of orthonormal vectors in $\mathbb{R}^n$.

Also, we can take $O(k) \to O(n)/O(n-k) \to O(n)/(O(k) \times O(n-k))$, Grassmannian manifold (k-dimensional subspaces in $\mathbb{R}^n$):

$$O(k) \to V_{n,k} \to G_{n,k},$$

is also a fibre bundle.

Exercise 54. *Long exact sequence of the fibration $F \to E \to B$ ends with*

$$\pi_1(B) \to \pi_0(F, x_0) \to \pi_0(E, x_0) \to \pi_0(B).$$

Show exactness in $\pi_0(E)$.

Solution. We showed $\pi_n(B) \cong \pi_n(E, F)$ for $n \geq 1$ which gives us the exactness of the fibration sequence from the exactness of the sequence of the pair (E, F) till $\pi_0(E, x_0)$.

Denote $S^0 = \{-1, 1\}$ and consider the composition

$$(\{-1, 1\}, -1) \to (F, x_0) = (p^{-1}(b_0), x_0) \to (E, x_0) \to (B, p(x_0) = b_0),$$

where -1 goes to x_0 and b_0 and 1 goes to $F = p^{-1}(b_0)$ so we get the constant map.

For the other part of the exactness we will use homotopy lifting property.

Consider $f : (\{-1, 1\}, -1) \to (E, x_0)$ such that pf is homotopic to the constant map into b_0 and denote the homotopy by H. It means that $pf(1)$ is connected with b_0 by a curve. Have a diagram

$$\begin{array}{ccc} S^0 \times \{0\} \cup \{-1\} \times I & \xrightarrow{f \cup \mathrm{const}} & E \\ \big\downarrow & \overset{G}{\nearrow} & \big\downarrow p \\ S^0 \times I & \xrightarrow{H} & B \end{array}$$

and remark that $\operatorname{im} G(-, 1) \subseteq F$ due to the fact $H(-, 1) = b_0$. Thus a desired preimage of $[f]$ in i_* is $G(-, 1)$. □

Exercise 55. *Covering: $G \to X \to X/G$, with action of G on X properly discontinuous, where X is path connected.*

Take

$$\pi_1(G, 1) \to \pi_1(X) \to \pi_1(X/G) \to \pi_0(G) \to \pi_0(X),$$

that is

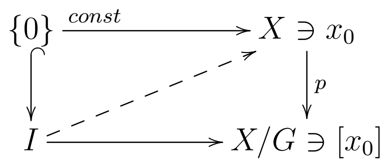
$$0 \to \pi_1(X) \to \pi_1(X/G) \xrightarrow{\partial} \pi_0(G) \to 0.$$

Show that ∂ is a group homomorphism.

Solution. First note that $\pi_0(G)$ is G taken as a set. We have the following piece of a long exact sequence of homotopy groups (define $i\colon G \to X$ as $i(g) = g \cdot x_0$ and identify g with $g \cdot x_0 \in X$ – this is possible due to the **properly discontinuous** action):

$$\begin{array}{ccccc} \pi_1(X) & \xrightarrow{j_*} & \pi_1(X, G, x_0) & \xrightarrow{\bar{\partial}} & \pi_0(G) \\ & & \cong \big\downarrow p_* & \overset{\partial}{\nearrow} & \\ & & \pi_1(X/G, [x_0]) & & \end{array}$$

Let's recall how the isomorphism p_* is defined for this particular case:



(1)

A group structure on $\pi_1(X, G, x_0)$ is transferred by the isomorphism p_*. Take general elements $[\omega], [\tau] \in \pi_1(X/G, [x_0])$ and lift them to elements $[\overline{\omega}], [\overline{\tau}] \in \pi_1(X, G, x_0)$ using the diagram (1). Since $\omega(1) = \tau(1) = [x_0]$ it must be $\overline{\omega}(1), \overline{\tau}(1) \in G$.For simplicity, we denote $\overline{\omega}(1) = g_1 \cdot x_0$ and $\overline{\tau}(1) = g_2 \cdot x_0$. Consider the following representative of some element in $\pi_1(X, G, x_0)$:

$$\overline{\omega} \cdot \overline{\tau}(t) = \begin{cases} \overline{\omega}(2t) & t \in [0, \frac{1}{2}], \\ g_1 \cdot \overline{\tau}(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

The above map is continuous by the pasting lemma and a definition of G-action. We define $[\overline{\omega}] \cdot [\overline{\tau}] := [\overline{\omega} \cdot \overline{\tau}]$. It is easy to see that $\overline{\omega} \cdot \overline{\tau}$ satisfies the diagram (1) for the bottom arrow $\omega * \tau$. As p_* is an isomorphism we obtain $p_*([\overline{\omega}] \cdot [\overline{\tau}]) = p_*([\overline{\omega} \cdot \overline{\tau}]) = [\omega * \tau] = [\omega] * [\tau] = p_*([\overline{\omega}]) * p_*([\overline{\tau}])$. It remains to show that $\overline{\partial}$ is a homomorphism what is quite straightforward:

$$\overline{\partial}([\overline{\omega}] \cdot [\overline{\tau}]) = \overline{\partial}([\overline{\omega} \cdot \overline{\tau}]) = (\overline{\omega} \cdot \overline{\tau})(1) = g_1(g_2 \cdot x_0) = \overline{\omega}(1) \cdot \overline{\tau}(1) = \overline{\partial}([\overline{\omega}]) \cdot \overline{\partial}([\overline{\tau}]).$$

□

Exercise 56. *Van Kampen theorem - Applications.*
Klein bottle K. Model as a square with identified sides as seen in Fig 1,2,3.

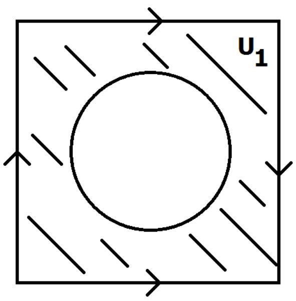


Fig 1

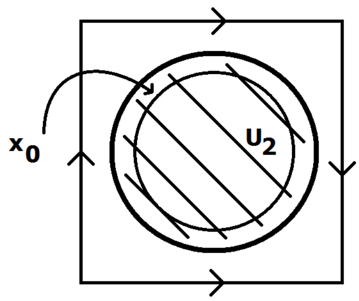


Fig 2

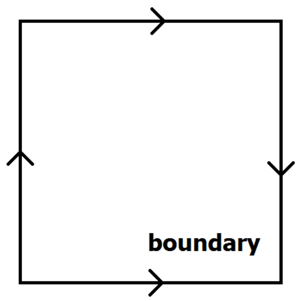


Fig 3

We denote open sets U_1, U_2 (disc) and point x_0 as in figure 1 and 2. (some of the notation in the solution is established in the theorem)

Solution. Set U_1 is homotopy equivalent to the boundary (Fig 3), and this boundary is in fact a wedge of two circles, so $U_1 \simeq S^1 \vee S^1$.

We can compute: $\pi_1(U_2, x_0) = \{1\}$ by contractibility, $\pi_1(U_1, x_0) = \pi_1(S^1 \vee S^1, x_0) =$ free group on two generators α, β as was already shown in lecture.

Then $\pi_1(U_1, x_0) * \pi_1(U_2, x_0) =$ free group on two generators α, β, also $\pi_1(U_1 \cap U_2, x_0) = \mathbb{Z}$.

Now, the intersection $U_1 \cap U_2 \simeq (S^1, x_0)$ and we take generator ω, $i_{2,1*}(\omega^{-1}) = 1$ in $\pi_1(U_2, x_0)$, $i_{1,2*}(\omega) = \alpha\beta\alpha^{-1}\beta$.

So, kernel of φ (the map from the theorem) is generated by element $\alpha\beta\alpha^{-1}\beta$
$\pi_1(K)$ is the group with two generators α, β and one relation $\alpha\beta\alpha^{-1}\beta = 1$ □