Note: This tutorial originates in 2017[1].

Exercise 57. *Recall definitions of n-connectedness and n-equivalence. Prove the following lemma: An inclusion $A \hookrightarrow X$ is n-equivalence if and only if (X, A) is n-connected.*

Solution. "$\Leftarrow$" Take long exact sequence:

$$\to \pi_n(A, x_0) \xrightarrow{f_n} \pi_n(X, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0) \xrightarrow{f_{n-1}} \pi_{n-1}(X) \to \pi_{n-1}(X, A) \to$$

and use the assumption that $\pi_i(X, A, x_0) = 0$ for $i \leq n$. Then we get that f_n is epimorphism and f_{n-1} is isomorphism.

"$\Rightarrow$" Reasoning is the same as in the other direction, the only thing we need to realize is $\pi_0(A, x_0) \xrightarrow{\cong} \pi_0(X, x_0)$. □

Exercise 58. *Show $\pi_k(S^\infty) = 0$ for all k, where S^∞ is $\operatorname{colim} S^n$.*

Solution. We have $S^1 \subset S^2 \subset S^3 \subset \cdots \subset S^n \subset \cdots \subset S^\infty$. Take element in $\pi_k(S^\infty)$, that is $f\colon S^k \to S^\infty$. We know that $f(S^k)$ is compact in S^∞. Consider other CW-complex structure than $e^0 \cup e^k$ for S^k that is $S^k = \bigcup_{i=0}^k e_1^i \cup e_2^i$ (two hemispheres). Then the following holds: $S^\infty = \bigcup_{i=0}^\infty e_1^i \cup e_2^i$. So $f(S^k) \subseteq (S^\infty)^{(N)} = S^N$ for some N where $(S^\infty)^{(N)}$ is N-skeleton of S^∞. Now,

$$f\colon S^k \to S^N \to S^{N+1} \hookrightarrow S^\infty,$$

so the composition $S^k \to S^{N+1}$ is a map that is not onto and therefore f is homotopic to constant map. (map into a disc is homotopic to constant map, disc is contractible) Then we have $[f] = 0$. Thus we have proved $\pi_k(S^\infty) = 0$. One can conclude that S^∞ is contractible by the Whitehead theorem as $S^\infty \to *$ is a weak equivalence. □

Remark. *Another approach is to define the map $T\colon S^\infty \to S^\infty$ by the assignment*

$$T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$

and show that $T \sim \mathrm{id}$ via the affine homotopy

$$H(x, t) = \frac{x(1-t) + tT(x)}{\|x(1-t) + tT(x)\|}$$

($x(1-t) + tT(x) \neq 0$ – realize that location of leading non-zero element for id *and T is different). Similarly, there is an affine homotopy $T \sim C$ where $C(x) = (1, 0, 0, \dots)$ is a constant map. Thus we obtain* $\mathrm{id} = \pi_k(\mathrm{id}) = \pi_k(T) = \pi_k(C) = 0\colon \pi_k(S^\infty) \to \pi_k(S^\infty)$ *what implies $\pi_k(S^\infty) = 0$.*

[1] see `https://is.muni.cz/el/sci/jaro2019/M8130/um/68045051/`

Remark. *In general, the functor $\pi_n\colon \mathrm{Top}_* \to \mathrm{Ab}$ for $n > 1$ doesn't preserve colimits. Consider diagram $D^n \leftarrow S^{n-1} \to D^n$ where both maps are inclusions of a boundary. A pushout of the diagram is S^n as we identified two hemispheres on their boundaries (equator). However pushout of the induced diagram $0 = \pi_n(D^n) \leftarrow \pi_n(S^{n-1}) \to \pi_n(D^n) = 0$ is the trivial group which doesn't agree with $\pi_n(S^n) = \mathbb{Z}$.*

Remark. *A solution of the previous exercise can be generalized to the following statement. Let X be a topological space such that $X = \bigcup_{i\geq 0} X_i$ satisfying $X_0 \subseteq X_1 \subseteq \dots$ where X_i are Hausdorff. Then the functor π_n commutes with colimit*

$$\varinjlim \pi_n(X_i, x_0) \cong \pi_n(X, x_0).$$

Sketch of proof: Every continuous map from a compact space to X factors through some X_i. As the target space X_i is Hausdorff such map is continuous as well as original one.

Exercise 59. *Compute homotopy groups of $\mathbb{R}P^\infty$.*

Solution. Surprisingly use previous exercise: We can view $\mathbb{R}P^\infty$ as lines going through origin in S^∞, or...just take $S^\infty/_{\mathbb{Z}/2}$, where the action is $x \mapsto -x$. So we work with the following fibration (we don't write the distinguished points as they are not needed) $\mathbb{Z}/2 \to S^\infty \to \mathbb{R}P^\infty$ and the long exact sequence

$$\pi_n(\mathbb{Z}/2) \to \pi_n(S^\infty) \to \pi_n(\mathbb{R}P^\infty) \xrightarrow{\partial} \pi_{n-1}(\mathbb{Z}/2) \to \pi_{n-1}(S^\infty),$$

where for all $n \geq 2$ we have all zeroes, for $n = 1$ consider $0 \to \pi_1(\mathbb{R}P^\infty) \xrightarrow{\partial} \pi_0(\mathbb{Z}/2) \to \pi_0(S^\infty)$. Since $\pi_0(S^\infty) = 0$ and $\pi_0(\mathbb{Z}/2) = \mathbb{Z}/2$, we get that the homomorphism ∂ (it is homomorphism, really, we did it in previous tutorial, but it's still a homomorphism independently on whether we did it or not) is an isomorphism of groups. By connectivity we also know the π_0 group. So the final results are:

$$\pi_n(\mathbb{R}P^\infty) = 0 \text{ for } n \geq 2,\ \pi_1(\mathbb{R}P^\infty) = \mathbb{Z}/2,\ \pi_0(\mathbb{R}P^\infty) = 0.$$

□

Exercise 60. *Show that the spaces $S^2 \times \mathbb{R}P^\infty$ and $\mathbb{R}P^2$ have the same homotopy groups but they are not homotopy equivalent.*

Solution. Here, also use previous exercise. It is known that $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$. With this we can compute

$$\pi_0(S^2 \times \mathbb{R}P^\infty) = 0,\ \pi_1(S^2 \times \mathbb{R}P^\infty) = \pi_1(S^2) \times \pi_1(\mathbb{R}P^\infty) = 0 \times \mathbb{Z}/2 \cong \mathbb{Z}/2,$$

$$\text{for } n \geq 2\ \pi_n(S^2 \times \mathbb{R}P^\infty) = \pi_n(S^2) \times 0 \cong \pi_n(S^2).$$

Now consider $\mathbb{R}P^2$ as $S^2/_{\mathbb{Z}/2}$, work with the fibration (scheme as follows)

$$\begin{array}{lccccc} & \mathbb{Z}/2 \longrightarrow & S^2 \longrightarrow & \mathbb{R}P^2 & & \\ & \pi_n(\mathbb{Z}/2) \longrightarrow & \pi_n(S^2) \longrightarrow & \pi_n(\mathbb{R}P^2) \longrightarrow & \pi_{n-1}(\mathbb{Z}/2) & \\ n \geq 2 & 0 & \pi_n(S^2) & \cong \pi_n(\mathbb{R}P^2) & 0 & \\ n = 1 & & 0 & \pi_1(\mathbb{R}P^2) & \cong \mathbb{Z}/2 & 0 \end{array}$$

and $\pi_0(\mathbb{R}P^2) = 0$. Thus we showed that these two spaces have the same homotopy groups. Marvellous. How to show, that they are not homotopy equivalent? Use cohomology group! That's right. It is well known (or we should already know) that

$$H^*(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2[\alpha]/\langle \alpha^3 \rangle, \alpha, \gamma \in H^1 \text{ and}$$

$$H^*(S^2 \times \mathbb{R}P^\infty; \mathbb{Z}/2) = H^*(S^2; \mathbb{Z}/2) \otimes H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[\beta]/\langle \beta^2 \rangle \otimes \mathbb{Z}/2[\gamma], \beta \in H^2.$$

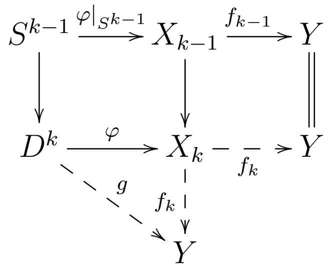
The former space obviously has no non-zero elements of order 4, while the latter has a non-zero element of order 4. This is impossible for homotopy equivalent spaces. We are done.

□

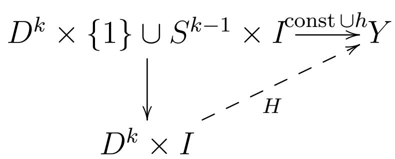
Exercise 61. *Extension lemma: Let (X, A) be a pair of CW-complexes, Y a path-connected space with $\pi_{n-1}(Y) = 0$ whenever there is a cell of dimension n in $X - A$. Then every map $f\colon A \to Y$ can be extended to a map $F\colon X \to Y$.*

Solution. Set $X_{-1} = A, X_0 = X^{(0)} \cup A, X_k = X^{(k)} \cup A$, and $f = f_{-1}\colon X_{-1} \to Y, f_0\colon X_0 \to Y$ which extends f_{-1} to 0-cells in an arbitrary way.

We have $f_{k-1}\colon X_{k-1} \to Y$ together with $\pi_{k-1}(Y) = 0$ for some $k \geq 1$ and we want to extend f_{k-1} to $g\colon D^k \cup X_{k-1} \to Y$ for any k-cell in $X - A$ such that the following diagram commutes:



The class $[g|_{S^{k-1}}] = [f_{k-1} \circ \varphi|_{S^{k-1}}]$ is in $\pi_{k-1}(Y) = 0$ so it is homotopic to a constant map via $h\colon S^{k-1} \times I \to Y$ such that $h(-, 1) = \text{const}$. Now, use HEP to complete the diagram:



Put $g = H(-, 0)$ and do the same procedure for the remaining k-cells in $X - A$. We extended f_{k-1} to $f_k\colon X_k \to Y$ and we proceed to infinity *(and beyond)*, as we always do. □

Exercise 62. *Compare $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$, when distinguished points (are / are not) path connected. Use proof with pillows.*

Solution. First, the case where $X = S^1 \sqcup S^2$ and $x_0 \in S^1$ and $x_1 \in S^2$. Then $\pi_1(X, x_0) = \pi_1(S^1) = \mathbb{Z}$ but $\pi_1(X, x_1) = \pi_1(S^2) = 0$. Thus if distinguished points are not path connected, homotopy groups can be different.

Now consider a curve $\omega\colon I \to X$ connecting x_0 and x_1 i.e. $\omega(0) = x_0$ and $\omega(1) = x_1$. Define a map $\pi_n(X, x_0) \to \pi_n(X, x_1)$ by the assignment $[f] \mapsto [\omega \cdot f]$ where $g = \omega \cdot f$ is determined by Figure 1.

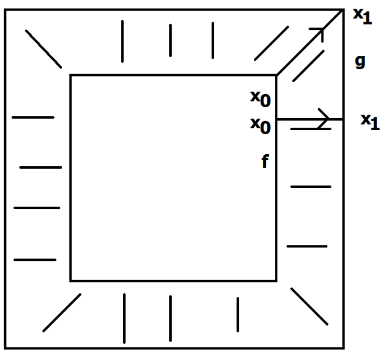


Figure 1

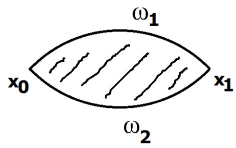


Figure 2

We verify that the map is well-defined and an isomorphism.

- A homotopy invariance. If $f_1 \sim f_2$ then $\omega \cdot f_1 \sim \omega \cdot f_2$. Similarly, Figure 2 shows that $\omega_1 \sim \omega_2 \Rightarrow \omega_1 \cdot f \sim \omega_2 \cdot f$ for curves ω_1, ω_2 connecting points x_0 and x_1. You can imagine this situation by adding the third dimension to Figure 1 which represents a time component of a homotopy.
- A group homomorphism i.e. $\omega \cdot (f + g) \sim (\omega \cdot f) + (\omega \cdot g)$. This is proved on the

following picture.

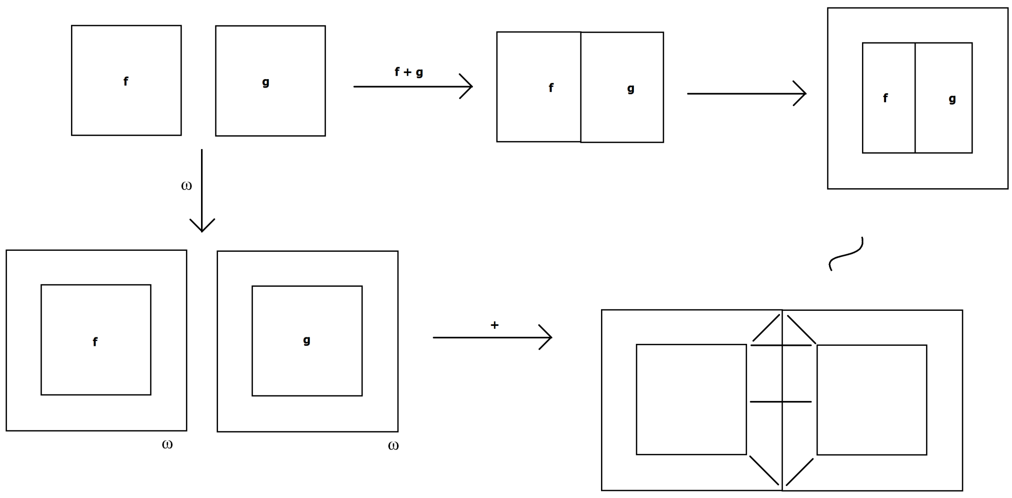


Figure 3

- A bijection i.e. $\omega^{-1} \cdot (\omega \cdot f) \sim (\omega^{-1} * \omega) \cdot f \sim f \sim \omega \cdot (\omega^{-1} \cdot f) \sim (\omega * \omega^{-1}) \cdot f$. This follows from transitivity of the action.

We get that if x_0, x_1 are in the same path component then $\pi_n(X, x_0) \to \pi_n(X, x_1)$ is an isomorphism. In particular, if X is simply connected, then every curve induces the same isomorphism. $\square$